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Assessing Deprivation with an Ordinal Variable: Theory and Application to Sanitation Deprivation in Bangladesh

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Abstract

The challenges associated with poverty measurement within an axiomatic framework, especially with cardinal variables, have received due attention during the last four decades. However, there is a dearth of literature studying how to meaningfully assess poverty with ordinal variables, capturing the *depth* of deprivations. In this paper, we first propose a class of additively decomposable ordinal poverty measures and provide an axiomatic characterisation using a set of basic foundational properties. Then, in a novel effort, we introduce a set of properties operationalising prioritarianism in the form of different degrees of *poverty aversion* in the ordinal context and characterise corresponding subclasses of measures. Moreover, for all the characterised classes and subclasses of measures, we develop stochastic dominance

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conditions, some of which are in themselves novel contributions to the stochastic dominance literature. Finally, we demonstrate the efficacy of our methods using an empirical illustration scrutinising the change in sanitation deprivation in Bangladesh.

Keywords: Ordinal variables, poverty measurement, precedence to poorer people, Hammond transfer, degree of poverty aversion, stochastic dominance.

JEL Codes: I3, I32, D63, O1

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1. Introduction

Around four decades ago, in an influential article titled 'Poverty: An Ordinal Approach to Measurement', Nobel laureate Amartya Sen proposed an axiomatically derived poverty measure to avoid some shortcomings of the traditionally used headcount ratio (Sen, 1976). Sen's approach was ordinal in the sense that his poverty measures assigned an ordinal-rank weight to each poor person's income, an otherwise cardinal variable. Since then, this seminal article has influenced a well-developed literature on poverty measurement involving cardinal variables within an axiomatic framework (Thon, 1979; Clark et al., 1981; Chakravarty, 1983; Foster et al., 1984; Foster and Shorrocks, 1988a,b; Ravallion, 1994; Shorrocks, 1995).

Distances between the values of cardinally measurable variables are meaningful. By contrast, ordinal variables merely consist of ordered categories and the cardinal distances between these categories are hard to interpret when numerals are assigned to them according to their order or rank.[1] And yet, the practice of using ordinal variables has been on the rise, in both developed and developing countries alike, due to the recent surge of interest in studying deprivation in non-monetary indicators, which are often ordinal in nature (e.g. access to basic facilities of different quality).[2] Moreover, there may be instances where ordinal categories of

[1]Based on the classification of measurement scales by Stevens (1946), whenever numeral scales are assigned to different ordered categories of an ordinal variable according to the orders or ranks of these categories, any 'order-preserving' or monotonic transformation should leave the scale form invariant. See Roberts (1979) for further in-depth discussions. In this paper, by ordinal variables we simply refer to variables with ordered categories, where numeral scales may not have necessarily been assigned to the categories.

[2]For example, as part of the first Sustainable Development Goals, the United Nations has set the target to not only eradicate *extreme monetary poverty*, but also to reduce *poverty in all its dimensions* by 2030. See

an otherwise cardinally measurable variable could have more policy relevance. For example, in some cases we might want to focus on ordered categories of income, nutritional status, or years of education completed, rather than these indicators' cardinal values.

How should poverty be meaningfully assessed with an ordinal variable? One straightforward way may be to dichotomise the population into a group of deprived and a group of non-deprived people, and then use the *headcount ratio*. However, this index is widely accused of ignoring the depth of deprivations (Foster and Sen, 1997). In our illustration in Section 5, for instance, in Sylhet province of Bangladesh, between 2007 and 2011, the proportion of population with inadequate sanitation facilities went down from around 70% to nearly 63%; whereas, during the same period, the proportion of people with the worst form of sanitation deprivation ('open defecation') increased significantly, from around 2% to more than 12% (see Table 3).

How can one reasonably capture the depth of deprivations for an ordinal variable? One approach may be to use an aggregate poverty measure that is sensitive to depth of deprivations. Alternatively one could simply consider the relative frequencies of the population experiencing deprivation within each ordered category, separately. The latter approach may seem reasonable especially when the variable under consideration has only a few deprivation categories; also when the required number of population-comparisons is relatively small. Otherwise it may become cumbersome. Even with only four deprivation categories used in our illustration, an analysis of deprivation dynamics across six provinces of Bangladesh over three years would involve comparing a staggering number of 72 data points. Nonetheless,

http://www.un.org/sustainabledevelopment/poverty/ (accessed in April 2017).

and in order to avoid a multitude of comparisons, one may choose to prioritise and focus on those experiencing the most severe category of deprivation; but this route comes at the expense of ignoring deprivations in other categories. Thus, in this paper, we pursue the former approach of using an aggregate poverty measure that is sensitive to all depths of deprivations within a variable.

The challenges associated with measuring well-being and inequality using an ordinal variable in an axiomatic framework have received due attention during the last few decades (e.g., Mendelson, 1987; Allison and Foster, 2004; Apouey, 2007; Abul Naga and Yalcin, 2008; Zheng, 2011; Kobus and Milos, 2012; Permanyer and D'Ambrosio, 2015; Kobus, 2015; Lazar and Silber, 2013; Yalonetzky, 2013; Gravel et al., 2015). Yet, when assessing poverty, such efforts have not been sufficiently thorough. Bennett and Hatzimasoura (2011), in a rare attempt, showed that indeed we can measure poverty with ordinal variables sensibly, but implicitly ruled out entire classes of well-suited measures (as shown by Yalonetzky, 2012). Moreover, their assessment of depth-sensitivity was restricted to the ordinal version of Pigou-Dalton transfers, thereby missing many other options including the burgeoning use of Hammond transfers (e.g. see Ebert, 2007; Gravel et al., 2015; Cowell et al., 2017; Gravel et al., 2018; Oui-Yang, 2018).[3]

Our paper contributes theoretically to the poverty measurement literature in three ways.

[3] We refer to the unidimensional context here. The issue of ordinality has certainly been examined thoroughly in the context of multidimensional poverty measurement (Alkire and Foster, 2011; Bossert et al., 2013; Dhongde et al., 2016; Bosmans et al., 2017). However, even in the multidimensional context, ordinal variables are often dichotomised in empirical applications (see, Alkire and Foster, 2011; Bossert et al., 2013; Dhongde et al., 2016), thereby ignoring the depth of deprivations within indicators.

First, we axiomatically characterise a class of ordinal poverty measures under a minimal set of desirable properties. Our class consists of measures that are weighted sums of population proportions in deprivation categories, where these weights are referred to as *ordering weights* because their values depend on the order of the categories. Our proposed measures are sensitive to the depth of deprivations (unlike the headcount ratio), additively decomposable, and bounded between zero and one.

Second, an adequately designed poverty measure should also ensure that policy makers have proper incentive to prioritise those poorer among the poor in the design of poverty alleviation policies so that *the poorest are not left behind*.[4] In a novel attempt, we operationalise the concept of *precedence to poorer people* by incorporating a *new* form of *degree of poverty aversion* in the ordinal context, reflecting a prioritarian point of view rather than an egalitarian one. Although grounded on prioritarianism, our new form of poverty aversion encompasses, as limiting cases, both previous attempts at sensitising ordinal poverty indices to the depth of deprivations (e.g., Bennett and Hatzimasoura, 2011; Yalonetzky, 2012) as well as current burgeoning approaches to distributional sensitivity in ordinal frameworks based on Hammond transfers (Hammond, 1976; Gravel et al., 2015). We define a range of properties based on this new form of degree of poverty aversion and characterise the corresponding subclasses of ordinal poverty measures. Within our framework, different degrees of poverty aversion merely require setting different restrictions on the ordering weights, preserving the measures' additive decomposability property.

Third, since each of our classes and subclasses admits a large number of poverty measures,

[4] Poverty measures may affect the incentives of policy makers during poverty alleviation (Zheng, 1997).

we develop related *stochastic dominance conditions* whose fulfilment guarantees the robustness of poverty comparisons to alternative functional forms and measurement parameters. Although some of these conditions turn out to be the ordinal-variable analogue of existing dominance conditions for cardinal variables (Foster and Shorrocks, 1988b), several others are novel methodological contributions to the literature on stochastic dominance with ordinal variables (to the best of our knowledge).

To demonstrate the efficacy of our approach, we present an empirical illustration studying the evolution of sanitation deprivation in Bangladesh using Demographic Health Survey datasets. Interestingly, our measures are able to discern the instances where the improvements in overall sanitation deprivation did not necessarily include the poorest. Furthermore, we apply the stochastic dominance conditions to test the robustness of poverty comparisons over time.

The rest of the paper proceeds as follows. After providing the notation, we present and then axiomatically characterise the class of depth sensitive ordinal poverty measures in Section 2. Section 3 introduces the concept of precedence to poorer people, states the properties related to the degrees of poverty aversion, and characterises the subclass of relevant poverty indices. Then section 4 develops stochastic dominance conditions for the characterised class and subclasses of poverty measures. Section 5 provides an empirical illustration analysing sanitation deprivation in Bangladesh. Section 6 concludes.

2. A class of depth-sensitive poverty measures for an ordinal variable

Suppose, there is a *social planner* whose objective is to assess a hypothetical society's poverty in some well-being dimension, which is measured with a set of ordered categories. For instance, self-reported health status may only include response categories, such as 'good health', 'fair health', 'poor health', and 'very poor health'. Similarly, there are also instances where the ordinal categories of an otherwise cardinal variable, such as the *years of schooling completed*, have more policy relevance.

Formally, suppose, there is a fixed set of $S \in \mathbb{N} \setminus \{1\}$ ordered categories $c_1, \ldots, c_S$, where $\mathbb{N}$ is the set of positive integers. The ordered categories are such that $c_{s-1} \succ_D c_s$ for all $s = 2, \ldots, S$, where $\succ_D$, which reads as "is more deprived than", is a binary and transitive relation whereby category c_{s-1} represents a worse-off situation than category c_s. Thus, c_S is the category reflecting least deprivation and c_1 is the state reflecting highest deprivation. Suppose, for example, a society's well-being is assessed by the education dimension and the observed ordered categories are 'no education', 'primary education', 'secondary education', and 'higher education', such that 'no education' $\succ_D$ 'primary education' $\succ_D$ 'secondary education' $\succ_D$ 'higher education'. Then, $c_4 =$ 'higher education' and $c_1 =$ 'no education'. We denote the set of all S categories by $\mathbf{C} = \{c_1, c_2, \ldots, c_S\}$ and the set of all categories excluding the category of least deprivation c_S by $\mathbf{C}_{-S} = \mathbf{C} \setminus \{c_S\}$.

Each individual in society must experience only one of the S categories. We denote the proportion of population experiencing category c_s by p_s for all $s = 1, \ldots, S$. Clearly, $p_s \geq 0$ for all s and $\sum_{s=1}^{S} p_s = 1$. The proportions of population in the society is summarised by the vector: $\mathbf{p} = (p_1, \ldots, p_S)$. Note that $\mathbf{p}$ is nothing but the discrete probability (or relative frequency) distribution of the society's population across the S categories. We denote the set of all possible discrete probability distributions over S categories by $\mathbb{P}$.

It is customary in poverty measurement to define a poverty threshold for identifying the poor and the non-poor populations (Sen, 1976). Suppose, the social planner decides that category c_k for any $1 \leq k < S$ and $k \in \mathbb{N}$ be the *poverty threshold*, so that people experiencing categories $c_1, \ldots, c_k$ are identified as *poor*; whereas, people experiencing categories $c_{k+1}, \ldots, c_S$ are identified as *non-poor*. We assume that at least one category reflects the absence of poverty, as this restriction is both intrinsically reasonable and is required for stating certain properties in Section 2.1. When $k = 1$, only category c_1 reflects poverty and, in this case, p_1 is the proportion of the population identified as poor. For any $c_k \in \mathbf{C}_{-S}$, we denote the *proportion of poor population*, also known as the *headcount ratio*, by $H(\mathbf{p}, c_k) = \sum_{s=1}^{k} p_s$.

We define a *poverty measure* $P(\mathbf{p}, c_k)$ as $P : \mathbb{P} \times \mathbf{C}_{-S} \to \mathbb{R}_+$. In words, a poverty measure is a mapping from the set of probability distributions and the set of poverty thresholds to the real line (note that the set of categories remains fixed). Especially, we propose the following class $\mathcal{P}$ of ordinal poverty measures that are amicable to empirical applications:

$$P(\mathbf{p}, c_k) = \sum_{s=1}^{S} p_s \omega_s \tag{1}$$

where $\omega_1 = 1$, $\omega_{s-1} > \omega_s > 0$ for all $s = 2, ..., k$ whenever $k \geq 2$, and $\omega_s = 0$ for all $s > k$.

A poverty measure for ordinal variables in our proposed class is a weighted sum of the population proportions in $\mathbf{p}$, where the weights (i.e. ω_s's) are *non-negative* for all categories, *strictly positive* for the deprived categories, and *unity* for the most deprived category. We refer to weights ω_s's as *ordering weights* and to $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_S)$ as the *ordering weighting vector*.[5] The ordering weights increase with deprived categories representing higher levels of

[5] Kobus and Milos (2012, Theorem 3) also showed that a subgroup decomposable inequality measure for ordinal variables that is sensitive to spreads away from the median is a monotonic transformation of the *weighted sum of population proportions*. However, our ordering weights are significantly different both in

deprivation. In practice, the ordering weights may take various forms. For example, Bennett and Hatzimasoura (2011) make the ordering weight for each deprivation category depend on the latter's relative deprivation rank. Category s is assigned an ordering weight equal to $\omega_s = [(k - s + 1)/k]^\theta$ for all $s = 1, \ldots, k$ and for some $\theta > 0$. Thus, the least deprived category c_k receives an ordering weight of $\omega_k = 1/k^\theta$; whereas, the most deprived category c_1 receives an ordering weight of $\omega_1 = 1$.

The class of poverty measures in Equation 1 bears several policy-relevant features. First, unlike the headcount ratio, the poverty measures in our class are *sensitive to the depth of deprivations* as they assign larger weights to the more deprived categories. Thus, unlike the headcount ratio, the proposed measures are sensitive to changes in deprivation status among the poor even when they do not become non-poor owing to those changes. Second, the proposed poverty measures are *additively decomposable*, which has two crucial policy implications. One is that the society's overall poverty measure may be expressed as a population-weighted average of the population subgroups' poverty measures, whenever the entire population is divided into mutually exclusive and collectively exhaustive population subgroups. The other is that additively decomposable measures are convenient for cross-sectional and inter-temporal econometric analysis as well as impact evaluation exercises. Third, the poverty measures are conveniently *normalised* between zero and one. They are equal to zero only in a society where nobody is poor; whereas, they are equal to one only whenever everybody in the society experiences the worst possible deprivation category

terms of their restrictions and interpretation vis-a-vis those involved in ordinal inequality measurement. Moreover, our SCD property (see below) imposes a more stringent restriction on the permissible functional transformations.

c_1. Fourth, the poverty measure boils down to the *headcount ratio* either when the poverty threshold is represented by the most deprived category or whenever the underlying ordinal variable has merely two categories.

2.1. Axiomatic characterisation

We now provide an axiomatic characterisation of the class of measures proposed in Equation 1. In other words, certain reasonable assumptions lead to this particular class of measures. As in many other areas of wellbeing measurement, the policy maker needs to make certain assumptions to perform interpersonal comparisons and meaningfully aggregate the information available on the ordered deprivation categories in order to obtain a *numerical poverty measure*.

We present these assumptions in terms of properties or axioms. By relying on discrete probability distributions, it is already implicitly assumed that the proposed measures in $\mathcal{P}$ satisfy *anonymity* and *population principle*. Anonymity requires that merely shuffling the individual deprivation levels within a society should not alter the society's poverty level; whereas the population principle requires that a mere duplication of each individual's deprivation level within a society should not affect the society's poverty level. The population principle allows comparing poverty levels of societies with different population sizes. The policy maker makes the following four additional assumptions.

The first property is *ordinal monotonicity*, which requires that if the living standard of a poor person improves so that the person moves to a category of less deprivation, then societal poverty should be lower. Formally, the property requires that if a poor person or a group of poor people moves from a category c_t reflecting poverty (i.e. $t \leq k$) to a less deprived category c_u (i.e. $c_t \succ_D c_u$), while the deprivation levels of everybody else in the society remain unchanged, then poverty should fall. In terms of probability distributions, we set the requirement that if a fraction of the poor population is moved from category c_t to category c_u, while the proportion of population in other categories remain unchanged, then poverty should fall:

Ordinal Monotonicity (OMN) For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$ and for any $c_k \in \mathbf{C}_{-S}$, if $q_t < p_t$ for some $t \leq k$ and $t < u$ but $p_s = q_s \; \forall s \neq \{t, u\}$, then $P(\mathbf{q}, c_k) < P(\mathbf{p}, c_k)$.

The second property is *single-category deprivation.* The property requires that whenever there is only one category reflecting poverty (i.e. c_1), then the poverty measure should be equal to the headcount ratio $H(\mathbf{p}, c_1) = p_1$. In other words, we assume that whenever there is only one category reflecting poverty and the others reflect an absence of poverty, then the headcount ratio becomes a sufficient statistic for the assessment of poverty. In fact, in this situation, any functional transformation of the headcount ratio would not add any meaningful information to the poverty assessment while being inferior in terms of intuitive interpretation.

Single-Category Deprivation (SCD) For any $\mathbf{p} \in \mathbb{P}$ and $c_1 \in \mathbf{C}_{-S}$, $P(\mathbf{p}, c_1) = p_1$.

The third property, *focus*, is essential for a poverty measure. It requires that, *ceteris paribus*, a change in a non-poor person's situation should not alter societal poverty evaluation as long as the non-poor person remains in that status. In terms of probability distributions, we set the requirement that as long as the proportion of poor population within each of the k categories reflecting poverty remains unchanged, the level of poverty should be the same. Note that the proportions of non-poor people may remain unchanged or may be different across the $S - k$ categories *not* reflecting poverty, but this should not matter for poverty evaluation.

Focus (FOC) For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$ and for any $c_k \in \mathbf{C}_{-S}$, if $q_s = p_s \ \forall s \leq k$, then $P(\mathbf{q}, c_k) = P(\mathbf{p}, c_k)$.

Finally, the social planner may be interested in exploring the relationship between the overall poverty evaluation and the subgroup poverty evaluation, where population subgroups can be geographical regions, ethnic groups, etc. Suppose society is partitioned into $M \in \mathbb{N} \backslash \{1\}$ mutually exclusive and collectively exhaustive population subgroups. We denote the population share in subgroup m by π_m, such that $\pi_m \geq 0 \ \forall m = 1, \ldots, M$ and $\sum_{m=1}^{M} \pi_m = 1$. We further denote the probability distribution across S categories within subgroup m by $\mathbf{p}^m = (p_1^m, \ldots, p_S^m) \in \mathbb{P}$ for every $m = 1, \ldots, M$, such that $\mathbf{p} = \sum_{m=1}^{M} \pi_m \mathbf{p}^m$. The final property, *subgroup decomposability*, requires overall societal poverty to be expressible as a population-weighted average of subgroup poverty levels:

Subgroup Decomposability (SUD) For any $M \in \mathbb{N}/\{1\}$, for any $\mathbf{p} \in \mathbb{P}$ such that $\mathbf{p} = \sum_{m=1}^{M} \pi_m \mathbf{p}^m$ where (i) $\mathbf{p}^m \in \mathbb{P} \ \forall m = 1, \ldots, M$, (ii) $\pi_m \geq 0$, and (iii) $\sum_{m=1}^{M} \pi_m = 1$, and for any $c_k \in \mathbf{C}_{-S}$,

$$P(\mathbf{p}, c_k) = \sum_{m=1}^{M} \pi_m P(\mathbf{p}^m, c_k).$$

These four properties lead to the class $\mathcal{P}$ of poverty measures in Equation 1, which we present in Theorem 2.1:

Theorem 2.1 A poverty measure $P \in \mathcal{P}$ satisfies properties OMN, SCD, FOC and SUD if and only if

$$P(\mathbf{p}, c_k) = \sum_{s=1}^{S} p_s \omega_s$$

where $\omega_1 = 1$, $\omega_{s-1} > \omega_s > 0$ for all $s = 2, ..., k$ whenever $k \geq 2$, and $\omega_s = 0$ for all $s > k$.

Proof. See Appendix A1. ■

A poverty measure satisfying these four stated properties turns out to be a weighted sum of population proportions. Note that the normalisation behaviour of our measures between zero and one, especially owing to the restrictions on ω's, is not an axiomatic assumption, but a logical conclusion from the foundational properties. Each poverty measure in class $\mathcal{P}$ may have the following alternative interpretation: when a policy maker only observes the population's relative frequencies across ordered deprivation categories, then, based on the four assumptions, the policy maker assigns particular deprivation values in the form of ω_s's to these individuals. Each poverty measure in our proposed class is an average of these assigned deprivation values.[6]

[6]For an alternative version of the axiomatic characterisation of the measures in class $\mathcal{P}$ using anonymity and population principle alongside the aforementioned properties, see Seth and Yalonetzky (2018).

3. Precedence to the poorer among the poor

Poverty alleviation is a gradual process, where it is imperative to ensure that the poorest of the poor are not *left behind*. Although all poverty measures in Equation (1) are sensitive to the depth of deprivations, not all of them ensure that the poorest among the poor population receive precedence over the less poor population during a poverty alleviation process. For that purpose, we introduce an intuitive concept of giving *precedence to poorer people* in the *ordinal* framework, in tune with the *prioritarian view* which holds that 'benefiting people matters more the worse off these people are' (Parfit, 1997, p. 213).[7] Furthermore, our notion of precedence is presented in a general framework where the *degree of precedence* to poorer people can vary between a minimum and a maximum.[8]

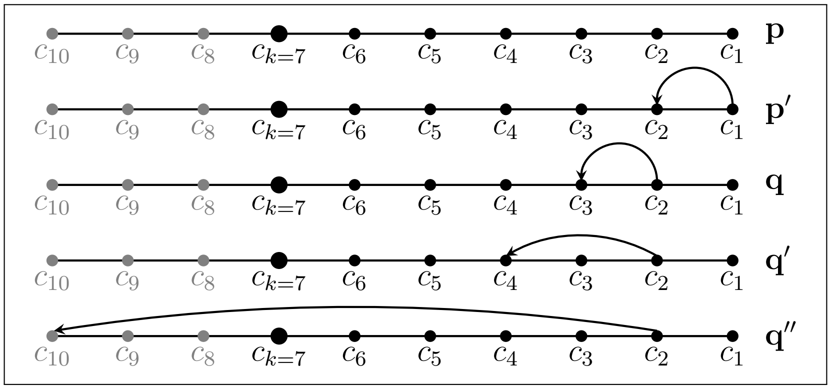
Let us introduce the concept using the example in Figure 1. Suppose, there are ten ordered deprivation categories, where c_1 is the most deprived category and c_{10} is the least deprived category. Denoted by black circles, categories $c_1, \ldots, c_7$ reflect poverty; whereas the other three categories, denoted by gray circles, do not. Thus, c_7 is the poverty threshold category, which is highlighted by a large black circle in each distribution. The original distribution is $\mathbf{p}$, where each individual experiences one of the ten categories.

Suppose the policy maker has the following two competing options: either (a) obtain distri-

[7]For an application of the prioritarian concept to the multidimensional context, see Bosmans et al. (2017). For further discussion, comparing the prioritarian view with the egalitarian view see Fleurbaey (2015, p. 208).

[8]The concept is analogous to the degree of poverty or inequality aversion in the monetary poverty measurement literature (Clark et al., 1981; Chakravarty, 1983; Foster et al., 1984), but not technically identical.

Figure 1: Precedence to poorer and the degree of precedence



bution $\mathbf{p}'$ from $\mathbf{p}$ by assisting a fraction $\epsilon \in (0,1)$ of poor people to move from category c_1 to category c_2 ; or (b) obtain distribution $\mathbf{q}$ from $\mathbf{p}$ by assisting a fraction ϵ of poor people to move from category c_2 to category c_3. Which option should lead to a larger reduction in poverty? One way of giving precedence to poorer people is to require that the move from $\mathbf{p}$ to $\mathbf{p}'$ should lead to a larger reduction in poverty than the move from $\mathbf{p}$ to $\mathbf{q}$. It is a minimal criterion for giving precedence to poorer people which we call *minimal precedence to poorer people* (PRE-M). This property requires that, *ceteris paribus*, moving a fraction of poorer people to an adjacent less deprived category leads to a larger reduction in poverty than moving a fraction of less poor people to a respectively adjacent less deprived category.[9]

The PRE-M property presents a minimal criterion for giving precedence to poorer people. Yet what happens when the policy maker faces the alternatives of improving the situation of a fraction of poorer people by one category and improving the situation of a similar fraction

[9]We have defined only the strict versions of these properties, requiring poverty to be strictly lower in the aftermath of specific pro-poorest distributional change. Consequently the ensuing results impose strict inequality restrictions on weights. However, these strict restrictions may be relaxed with alternative versions if the latter only require that poverty does not rise due to the same pro-poorest distributional change.

of less poor people by several categories? To ensure that the policy maker still chooses to improve the situation of the poorer people in these cases, we introduce the property of *greatest precedence to poorer people* (PRE-G). This property requires that, *ceteris paribus*, moving a fraction ϵ of poorer people to a less deprived category leads to a larger reduction in poverty than moving a fraction ϵ of less poor people to *any number* of less deprived categories. Note here that the improvement is not restricted to a particular number of adjacent categories. For example, if PRE-G held, a move from $\mathbf{p}$ to $\mathbf{p}'$ in Figure 1 should lead to a larger reduction in poverty than even a move from $\mathbf{p}$ to $\mathbf{q}''$.

Conceptually, the PRE-G property is analogous to the notion of a Hammond transfer (Hammond, 1976; Gravel et al., 2015), which essentially involves, simultaneously, an improvement in a poor person's situation and a deterioration of a less poor person's situation, such that their deprivation ranks are not reversed (in the case of poverty measurement). Also, an ordinal poverty measure satisfying property PRE-G also satisfies property PRE-M, but the reverse is not true. A policy maker supporting property PRE-G over property PRE-M should be considered more poverty averse.

We could also consider intermediate forms of preference between the minimal (PRE-M) and the greatest (PRE-G) forms of precedence. For example, instead of the greatest forms of precedence, the policymaker may be satisfied with requiring, that, *ceteris paribus*, moving a fraction ϵ of poorer people to an adjacent less deprived category leads to a larger reduction in poverty than moving a fraction ϵ of less poor people to, say, two adjacent less deprived categories. In such case, a move from $\mathbf{p}$ to $\mathbf{p}'$ in Figure 1 should lead to a larger reduction in poverty than a move from $\mathbf{p}$ to $\mathbf{q}'$. We refer to this case as *precedence to poorer people of* *order two* (PRE-2).

Likewise, we may obtain the policy maker's preferred degree α of giving precedence to poorer people. Thus, we introduce the general property of *precedence to poorer people of order* α, which requires that, *ceteris paribus*, moving a fraction ϵ of poorer people to an adjacent less deprived category leads to a larger reduction in poverty than moving a fraction ϵ of less poor people up to an $\alpha\,(\geq 1)$ number of adjacent less deprived categories. A formal general statement of the property, which includes PRE-M and PRE-G as *limiting cases*, is the following:

Precedence to Poorer People of Order $\boldsymbol{\alpha}$ (PRE-$\boldsymbol{\alpha}$) For any $\mathbf{p}, \mathbf{p}', \mathbf{q}' \in \mathbb{P}$, for any $k \geq 2$, for any $c_k \in \mathbf{C}_{-S}$, and for some $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, for some $s < t \leq k < S$ and for some $\epsilon \in (0, 1)$, if (i) $\mathbf{p}'$ is obtained from $\mathbf{p}$ such that $p'_s = p_s - \epsilon$ while $p'_u = p_u\ \forall u \neq \{s, s+1\}$, and (ii) $\mathbf{q}'$ is obtained from $\mathbf{p}$ such that $q'_t = p_t - \epsilon$ while $q'_u = p_u\ \forall u \neq \{t, \min\{t+\alpha, S\}\}$, then $P(\mathbf{p}', c_k) < P(\mathbf{q}', c_k)$.

Note that PRE-1 is essentially the PRE-M property. This is the case where the social planner is least poverty averse. As the value of α increases, the social planner's poverty aversion rises. In this framework, the social planner's poverty aversion is highest at $\alpha = k - 1$. We shall subsequently show that PRE-α for $\alpha = k - 1$ leads to the same subclass of ordinal poverty measures as the PRE-G property. The PRE-α property imposes further restrictions on the class of measures in Theorem 2.1. In Theorem 3.1, we present the subclass of measures $\mathcal{P}_\alpha$ that satisfy the general PRE-α property:

Theorem 3.1 For any $k \geq 2$ and for some $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, a poverty measure $P \in \mathcal{P}$ additionally satisfies property PRE-α if and only if:

a. $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+\alpha}$ $\forall s = 2, \ldots, k-\alpha$ and $\omega_{s-1} > 2\omega_s$ $\forall s = k-\alpha+1, \ldots, k$ whenever $\alpha \leq k-2$.

b. $\omega_{s-1} > 2\omega_s$ $\forall s = 2, \ldots, k$ whenever $\alpha = k-1$.

Proof. See Appendix A2. ■

In a novel effort, Theorem 3.1 presents various subclasses of indices based on the degree of poverty aversion α, which we denote as $\mathcal{P}_\alpha$. In order to give precedence to poorer people, additional restrictions must be imposed on the ordering weights. Corollary 3.1 presents the limiting case of $\mathcal{P}_1$, featuring the least poverty averse social planner:

Corollary 3.1 For any $k \geq 2$, a poverty measure $P \in \mathcal{P}$ additionally satisfies property PRE-M (i.e. PRE-1) if and only if $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+1}$ $\forall s = 2, ..., k-1$ and $\omega_{k-1} > 2\omega_k$.

Proof. The result follows directly from Theorem 3.1 by setting $\alpha = 1$. ■

To give precedence to poorer people in the spirit of property PRE-M, the ordering weights must be such that the difference $\omega_{s-1} - \omega_s$ is larger than the subsequent difference $\omega_s - \omega_{s+1}$, in addition to the restrictions imposed by Theorem 2.1. Suppose, we summarise the ordering weights by: $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_S)$. Let us consider an example involving five categories and two ordering weight vectors: $\boldsymbol{\omega}' = (1, 0.8, 0.5, 0, 0)$ and $\boldsymbol{\omega}'' = (1, 0.5, 0.2, 0, 0)$, where $k = 3$. The ordering weights in $\boldsymbol{\omega}'$ fulfill all properties presented in Theorem 2.1, but the largest reduction in poverty is obtained whenever a poor person moves from the least poor category to the adjacent non-poor category. By contrast, ordering weights in $\boldsymbol{\omega}''$ require that the largest reduction in poverty be obtained whenever a poor person moves from the poorest category to the adjacent second poorest category. Thus, unlike the ordering weights in $\boldsymbol{\omega}'$, the ordering weights in $\boldsymbol{\omega}''$ make sure that poorer people receive precedence.

Next we present the subclass of poverty measures satisfying property PRE-G, i.e. $\mathcal{P}_G$:

Proposition 3.1 For any $k \geq 2$, a poverty measure $P \in \mathcal{P}$ additionally satisfies property PRE-G if and only if $\omega_{s-1} > 2\omega_s \ \forall s = 2, \ldots, k$.

Proof. See Appendix A3. ■

The additional restriction on the ordering weights in Proposition 3.1 effectively prioritises the improvement in a poorer person's situation over improvement of any extent in a less poor person's situation. Let us consider an example involving five categories and two ordering weight vectors: $\boldsymbol{\omega}' = (1, 0.6, 0.3, 0.1, 0)$ and $\boldsymbol{\omega}'' = (1, 0.48, 0.23, 0.1, 0)$, where $k = 4$. Clearly, both sets of weights in $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ satisfy the restriction in Corollary 3.1 that $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+1}$ for all $s = 2, \ldots, k$. However, the ordering weights in $\boldsymbol{\omega}'$ do not satisfy the restriction in Proposition 3.1, since $\omega'_1 < 2\omega'_2$; whereas the ordering weights in $\boldsymbol{\omega}''$ do satisfy the restriction in Proposition 3.1 as $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$.

An interesting feature of the set of weights satisfying property PRE-G is that, for $k \geq 3$, any deprivation category up to the third least detrimental deprivation category (i.e. $k-2$), should receive a weight greater than the sum of weights assigned to all categories reflecting lesser deprivation, i.e. $\omega_s > \sum_{\ell=s+1}^{k} \omega_\ell$ for all $s = 1, \ldots, k-2$.

Finally, it is worth pointing out that, remarkably, the subclasses $\mathcal{P}_G$ (Proposition 3.1) and $\mathcal{P}_{k-1}$ (Theorem 3.1 when $\alpha = k-1$) are identical; even though the distributional changes involved in the PRE-α property are only specific cases of those involved in PRE-G. Besides being of interest in itself, this perfect overlap between the subclasses of indices will prove useful in Section 4 because by deriving the dominance conditions for the subclasses $\mathcal{P}_\alpha$, we will also obtain the relevant dominance conditions for subclass $\mathcal{P}_G$.[10]

4. Poverty dominance conditions

Stochastic dominance conditions come in handy whenever we want to ascertain the robustness of a poverty ranking of distributions to alternative reasonable comparison criteria, e.g. selection of poverty lines, choice of different functional forms, etc. (Atkinson, 1987; Foster and Shorrocks, 1988b; Fields, 2001). Moreover, often stochastic dominance conditions reduce an intractable problem of probing the robustness of a comparison across an infinite domain of alternative criteria to a finite set of distributional tests (Levy, 2006). In Sections 2 and 3, we introduced the class of poverty measures $\mathcal{P}$ and its subclasses $\mathcal{P}_\alpha$. The main parameters for these measures are the set of ordering weights $\{\omega_1, \ldots, \omega_k\}$ and the poverty threshold

[10] Noteworthy is the expected resemblance between the weighting restriction identified in Proposition 3.1 and that in the class of welfare functions for ordinal variables derived in Gravel et al. (2015, Lemma 2). The latter characterises welfare functions that increase both when someone moves to a better category (so-called increments) and in the aftermath of Hammond transfers. Setting $\alpha_k = 0$ and changing the inequality sign in Gravel et al. (2015, Lemma 2) to interpret their α functions (not to be confused with our α parameter) as ordering weights for poverty measurement yields the weight restriction in Proposition 3.1.

category c_k. It is thus natural to inquiry into the circumstances under which poverty comparisons based on ordinal variables are robust to alternative ordering weights as well as alternative poverty threshold categories. In this section, first we introduce the first-order dominance conditions relevant to $\mathcal{P}$, followed by the second-order dominance conditions for $\mathcal{P}_\alpha$ for all α.

4.1. Poverty dominance conditions for all measures in $\mathcal{P}$

Theorem 4.1 provides poverty dominance conditions that are relevant to all measures in class $\mathcal{P}$, but for a *given poverty threshold category* $c_k \in \mathbf{C_{-S}}$:

Theorem 4.1 For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$ and for some $k \geq 1$, $P(\mathbf{p}, c_k) < P(\mathbf{q}, c_k)\ \forall P \in \mathcal{P}$ for a given $c_k \in \mathbf{C_{-S}}$ if and only if $\sum_{\ell=1}^{s}(p_\ell - q_\ell) \leq 0\ \forall s \leq k$ with at least one strict inequality.

Proof. See Appendix A4. ∎

Once a particular poverty threshold category $c_k \in \mathbf{C_{-S}}$ is chosen, Theorem 4.1 states that poverty in distribution $\mathbf{p} \in \mathbb{P}$ is strictly lower than that in distribution $\mathbf{q} \in \mathbb{P}$ for all measures in $\mathcal{P}$ if and only if all partial sums of the probabilities up to category c_k in $\mathbf{p}$ are nowhere higher and at least once strictly lower than the respective partial sums in $\mathbf{q}$. The result in Theorem 4.1 may also be presented in terms of the headcount ratio $H(\mathbf{p}, c_k)$ as follows: the poverty comparison for a particular poverty threshold category c_k is robust to all poverty measures in $\mathcal{P}$ if and only if $H(\mathbf{p}, c_s) \leq H(\mathbf{q}, c_s)$ for all $s \leq k$ and $H(\mathbf{p}, c_s) < H(\mathbf{q}, c_s)$ for at least one $s \leq k$.

Corollary 4.1 extends the first-order dominance condition obtained in Theorem 4.1 to any measure $P \in \mathcal{P}$ and *all poverty threshold categories* in $\mathbf{C_{-S}}$:

Corollary 4.1 For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$ and for some $k \geq 1$, $P(\mathbf{p}, c_k) < P(\mathbf{q}, c_k)$ for any $P \in \mathcal{P}$ and for all $c_k \in \mathbf{C_{-S}}$ if and only if $\sum_{\ell=1}^{s}(p_\ell - q_\ell) \leq 0 \; \forall s = 2, \ldots, S-1$ and $p_1 < q_1$.

Proof. The sufficiency part is straightforward and follows from Equation A8. We prove the necessary condition as follows. First, consider $k = 1$. Then, $P(\mathbf{p}, c_1) < P(\mathbf{q}, c_1)$ only if $p_1 < q_1$. Subsequently, the requirement that $\sum_{\ell=1}^{s} p_\ell \leq \sum_{\ell=1}^{s} q_\ell \; \forall s = 2, \ldots, S-1$ follows from Theorem 4.1. ■

Interestingly, in terms of headcount ratios, poverty in distribution $\mathbf{p}$ is lower than poverty in distribution $\mathbf{q}$ for any $P \in \mathcal{P}$ and for all possible poverty threshold categories if and only if $H(\mathbf{p}, c_s) \leq H(\mathbf{q}, c_s)$ for all $s \leq k$ and $H(\mathbf{p}, c_1) < H(\mathbf{q}, c_1)$. Thus, the results in Theorem 4.1 and Corollary 4.1 are the ordinal versions of the headcount-ratio orderings for continuous variables derived by Foster and Shorrocks (1988b).

4.2. Poverty dominance conditions for all measures in $\mathcal{P}_\alpha$

We now present poverty dominance conditions that are relevant to all measures in subclass $\mathcal{P}_\alpha$ for some $\alpha \in [1, k-1]$. We refer to the poverty dominance for a particular value of α as PRE-α dominance. For the presentation of our results in this section, we define the following additional notation. First, by $\lfloor b \rfloor$ for any $b \in \mathbb{R}_{++}$, we denote the largest possible non-negative integer that is not greater than b (for example, if $b = 5.2$ then $\lfloor b \rfloor = 5$). Secondly, for some $k \geq 2$, we denote the r^{th} ordinal weighting vector by $\boldsymbol{\omega}^r = (\omega_1^r, \ldots, \omega_S^r)$ for all $r = 1, \ldots, k$. In Theorem 4.2, we show that one needs to evaluate poverty orderings across two distributions at k distinct ordering weighting vectors to test PRE-α dominance and we explicitly derive the weights ω_s^r for all $s = 1, \ldots, S$ and for all $r = 1, \ldots, k$.

Theorem 4.2 presents the PRE-α dominance conditions for a *given poverty threshold category* $c_k \in \mathbf{C_{-s}}$ and $k \geq 2$:

Theorem 4.2 For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, for some $k \geq 2$, and for some $\alpha \in [1, k-1] \subseteq \mathbb{N}$, $P(\mathbf{p}, c_k) < P(\mathbf{q}, c_k)$ $\forall P \in \mathcal{P}_\alpha$ for a given $c_k \in \mathbf{C_{-s}}$ if and only if, with at least one strict inequality, $\sum_{s=1}^{S} \omega_s^r (p_s - q_s) \leq 0$ $\forall r = 1, \ldots, k$ such that:

$$\omega_s^r = \begin{cases} 0 & \text{for } s > r \text{ and } r = 1, \ldots, k \\ 1 & \text{for } s = 1 \text{ and } r = 1, \ldots, k \\ 2^{1-s} & \text{for } s = 2, \ldots, r \text{ and } r = 2, \ldots, \alpha + 1 \\ \dfrac{2^{r-s}}{\sum\limits_{j=0}^{\overline{r}} (-1)^j \binom{r-\alpha j-1}{j} 2^{r-(\alpha+1)j-1}} & \text{for } s = r - \alpha, \ldots, r \text{ and } r = \alpha + 2, \ldots, k \\ \dfrac{\sum\limits_{j=0}^{\overline{s}} (-1)^j \binom{r-s-\alpha j}{j} 2^{r-s-(\alpha+1)j}}{\sum\limits_{j=0}^{\overline{r}} (-1)^j \binom{r-\alpha j-1}{j} 2^{r-(\alpha+1)j-1}} & \text{for } s = 2, \ldots, r - \alpha - 1 \text{ and } r = \alpha + 3, \ldots, k \end{cases}, \quad (2)$$

where $\overline{r} = \left\lfloor \dfrac{r-1}{\alpha+1} \right\rfloor$ and $\overline{s} = \left\lfloor \dfrac{r-s}{\alpha+1} \right\rfloor$.

Proof. See Appendix A5. ■

For a given $c_k \in \mathbf{C_{-s}}$ and for a given $\alpha \in [1, k-1]$, according to Theorem 4.2, poverty in distribution $\mathbf{p}$ is lower than that in distribution $\mathbf{q}$ for all $P \in \mathcal{P}_\alpha$ (i.e., $\mathbf{p}$ PRE-α dominates $\mathbf{q}$) if and only if $\sum_{s=1}^{S} \omega_s^r p_s \leq \sum_{s=1}^{S} \omega_s^r q_s$ $\forall r = 1, \ldots, k$ with at least one strict inequality, where the values for all ω_s^r's are determined by Equation 2. The first condition within Equation

2 requires that $\omega_s^r = 0$ whenever $s > r$; whereas, the second condition requires, based on Theorem 2.1, that $\omega_1^r = 1$ for all $r = 1, \ldots, k$. For example, for $k = 3$, $\boldsymbol{\omega}^1 = (\omega_1^1, \omega_2^1, \omega_3^1) = (1, 0, 0)$. The third condition within Equation 2 requires that for values of r ranging between 2 and $\alpha + 1$ for a given value of α, $\omega_s^r = 2^{1-s}$ for all $s = 1, \ldots, r$, due to the restriction $\omega_{s-1} > 2\omega_s \ \forall s = 2, \ldots, r$ whenever $\alpha = r - 1$ or $r = \alpha + 1$ as in part b. of Theorem 3.1. The fourth and fifth conditions within Equation 2 follow from the second condition and the first condition, respectively, of part a. of Theorem 3.1.

For any chosen value of α, k different conditions must be tested for concluding dominance in a poverty comparison.[11] Let us provide an example with $S = 6$ and $k = 5$. In this case, α may take four different values: $\alpha = 1, 2, 3, 4$. Moreover, since $S > k$, $\omega_6^k = 0$ for all k. For each value of α, the condition $\sum_{s=1}^{6} \omega_s^r p_s \leq \sum_{s=1}^{6} \omega_s^r q_s$, with one strict inequality, must be satisfied for the five ordinal weighting vectors $\boldsymbol{\omega}^r$ corresponding to $r = 1, \ldots, 5$, as presented in Table 1. Each column of the table presents the five ordering weighting vectors for a given value of α; whereas, the k^{th} row of the table reports the k^{th} ordering weights vector for different values of α. The corresponding values of $\bar{r}$ are also reported where they are relevant. For instance, if $\alpha = 2$, then poverty comparisons should be checked at the following five ordering weighting vectors: $\boldsymbol{\omega}^1 = (1, 0, 0, 0, 0, 0)$, $\boldsymbol{\omega}^2 = \left(1, \frac{1}{2}, 0, 0, 0, 0\right)$, $\boldsymbol{\omega}^3 = \left(1, \frac{1}{2}, \frac{1}{4}, 0, 0, 0\right)$, $\boldsymbol{\omega}^4 = \left(1, \frac{4}{7}, \frac{2}{7}, \frac{1}{7}, 0, 0\right)$, and $\boldsymbol{\omega}^5 = \left(1, \frac{7}{12}, \frac{4}{12}, \frac{2}{12}, \frac{1}{12}, 0\right)$.

We also provide simplified versions of the dominance conditions presented in Theorem 4.2 for the two extreme cases of minimal precedence (i.e. $\alpha = 1$) as in Corollary 3.1 and of

[11] This approach to dominance testing is analogous to that used in robustness tests for composite indices with respect to alternative weights (see, Seth and McGillivray, 2018; Foster et al., 2012).

Table 1: The ordering weighting vectors for conducting dominance tests for different α's when $S = 6$ and $k = 5$

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
$\boldsymbol{\omega}^1$	$(1, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0)$
$\boldsymbol{\omega}^2$	$\left(1, \frac{1}{2}, 0, 0, 0, 0\right)$	$\left(1, \frac{1}{2}, 0, 0, 0, 0\right)$	$\left(1, \frac{1}{2}, 0, 0, 0, 0\right)$	$\left(1, \frac{1}{2}, 0, 0, 0, 0\right)$
$\boldsymbol{\omega}^3$	$\left(1, \frac{2}{3}, \frac{1}{3}, 0, 0, 0\right)$; $\bar{r} = 1$	$\left(1, \frac{1}{2}, \frac{1}{4}, 0, 0, 0\right)$	$\left(1, \frac{1}{2}, \frac{1}{4}, 0, 0, 0\right)$	$\left(1, \frac{1}{2}, \frac{1}{4}, 0, 0, 0\right)$
$\boldsymbol{\omega}^4$	$\left(1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}, 0, 0\right)$; $\bar{r} = 1$	$\left(1, \frac{4}{7}, \frac{2}{7}, \frac{1}{7}, 0, 0\right)$; $\bar{r} = 1$	$\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, 0\right)$	$\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, 0\right)$
$\boldsymbol{\omega}^5$	$\left(1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0\right)$; $\bar{r} = 2$	$\left(1, \frac{7}{12}, \frac{4}{12}, \frac{2}{12}, \frac{1}{12}, 0\right)$; $\bar{r} = 1$	$\left(1, \frac{8}{15}, \frac{4}{15}, \frac{2}{15}, \frac{1}{15}, 0\right)$; $\bar{r} = 1$	$\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 0\right)$

greatest precedence (i.e. $\alpha = k - 1$) as in Proposition 3.1. Whenever $\alpha = 1$, we must test k restrictions on the linear combinations of partial sums of probabilities as in Corollary 4.2:

Corollary 4.2 For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$ and for some $k \geq 2$, $P(\mathbf{p}, c_k) < P(\mathbf{q}, c_k)$ $\forall P \in \mathcal{P}_1$ for a given $c_k \in \mathbf{C_{-s}}$ if and only if $\sum_{\ell=1}^{s} \sum_{j=1}^{\ell} (p_j - q_j) \leq 0$ $\forall s \leq k$ with at least one strict inequality.

Proof. See Appendix A6. ■

The dominance condition presented in Corollary 4.2 is the ordinal version of the "P_2" poverty ordering as in Foster and Shorrocks (1988b). Corollary 4.3, on the other hand, presents the dominance conditions for greatest precedence (i.e. $\alpha = k - 1$):

Corollary 4.3 For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$ and for some $k \geq 2$, $P(\mathbf{p}, c_k) < P(\mathbf{q}, c_k)$ $\forall P \in \mathcal{P}_G$ for a given $c_k \in \mathbf{C_{-s}}$ if and only if $\sum_{\ell=1}^{s} 2^{1-\ell}(p_\ell - q_\ell) \leq 0$ $\forall s \leq k$ with at least one strict inequality.

Proof. It is straightforward to verify from Theorem 4.2 by setting $\alpha = k - 1$. ■

So far, we have presented the dominance conditions for a given c_k. Finally, Corollary 4.4 provides the second-order dominance conditions relevant to any measure $P \in \mathcal{P}_\alpha$ for some $\alpha \in [1, k-1]$, but for *all* $c_k \in \mathbf{C}_{-\mathbf{S}}$ such that $k \geq 2$:

Corollary 4.4 For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, for some $k \geq 2$, and for some $\alpha \in [1, k-1] \subseteq \mathbb{N}$, $P(\mathbf{p}, c_k) < P(\mathbf{q}, c_k)\ \forall P \in \mathcal{P}_\alpha$ for all $c_k \in \mathbf{C}_{-\mathbf{S}}$ if and only if (a) $p_1 \leq q_1$ and $2p_1 + p_2 \leq 2q_1 + q_2$ with at least one strict inequality and (b) $\sum_{s=1}^{S} \omega_s^r (p_s - q_s) \leq 0\ \forall r = 3, \ldots, S-1$, where $\boldsymbol{\omega}^r = (\omega_1^r, \ldots, \omega_S^r)\ \forall r = 3, \ldots, S-1$ are obtained from Equation 2 of Theorem 4.2.

Proof. First consider the case when $k = 2$. Then from Equation 2 of Theorem 4.2, it follows that $p_1 \leq q_1$ and $2p_1 + p_2 \leq 2q_1 + q_2$, with at least one strict inequality. Therefore, conditional (a) is both necessary and sufficient. Whenever, $k \geq 3$, part (b) follows from Theorem 4.2 by setting $k = S - 1$. ■

In summary, the robustness of poverty comparisons for various classes and subclasses of ordinal measures introduced in Sections 2.1 and 3 can be assessed with a battery of dominance tests based on the theorems and corollaries presented in this section.

5. Empirical illustration: Sanitation deprivation in Bangladesh

We now present an empirical illustration in order to showcase the efficacy of our proposed measurement method. In the current global development context, both the United Nations through the Sustainable Development Goals[12] and the World Bank through their Report of the Commission on Global Poverty (World Bank, 2017) have acknowledged the need for

[12] Available at https://sustainabledevelopment.un.org/sdgs.

assessing, monitoring, and alleviating poverty in multiple dimensions besides the monetary dimension. In practice, most non-income dimensions are assessed by ordinal variables. In this section, we show how our measurement tools may be applied to analyse inter-temporal sanitation deprivation in Bangladesh.

For our analysis, we use the nationally representative Demographic Health Survey (DHS) datasets of Bangladesh for the years 2007, 2011, and 2014. We compute the point estimates and standard errors incorporating the sampling weights as well as respecting the survey design.[13] Excluding the non-usual residents, we were able to consider the information on 50,215 individuals from 10,398 households in the 2007 survey, 79,483 individuals from 17,139 households in the 2011 survey, and 77,680 individuals from 17,299 households in the 2014 survey.

Table 2: The five ordered categories of access to sanitation facilities

Category	Description
Open defecation	Human faeces disposed of in fields, forests, bushes, open bodies of water, beaches or other open spaces or disposed of with solid waste
Unimproved	Pit latrines without a slab or platform, hanging latrines and bucket latrines
Limited	Sanitation facilities of an otherwise acceptable type shared between two or more households
Basic unsafe	A basic sanitation facility which is not shared with other households, but excreta are not disposed safely, such as flushed but not disposed to piped sewer system, septic tank or pit latrine
Improved	Sanitation facility which is not shared with other households and where excreta are safely disposed in situ or treated off-site and includes flush/pour flush to piped sewer system, septic tank or pit latrine, ventilated improved pit latrine, composting toilet or pit latrine with a slab

[13]See NIPORT et al. (2009, 2013, 2016) for details about the survey design.

One target of the United Nations' sixth Sustainable Development Goal (whose aim is to 'ensure availability and sustainable management of water and sanitation for all') is 'by 2030, [to] achieve access to adequate and equitable sanitation and hygiene for all and end open defecation.' In order to hit the target, the Joint Monitoring Programme (JMP) of the World Health Organisation and the UNICEF proposes using 'a *service ladder approach* to benchmark and track progress across countries at different stages of development", building on the existing datasets.[14] We pursue this service ladder approach and apply our ordinal poverty measures to study the improvement in sanitation deprivation in Bangladesh. We classify households' *access to sanitation* in the five ordered categories presented in Table 2. The five categories are ordered as 'open defecation' $\succ_D$ 'unimproved' $\succ_D$ 'limited' $\succ_D$ 'basic unsafe' $\succ_D$ 'improved'. We consider all persons living in a household deprived in access to sanitation if the household experiences any category other than 'improved'.

Table 3: Change in population distribution across sanitation categories in Bangladesh

	Bangladesh			Dhaka			Rajshahi			Sylhet		
	2007	2011	2014	2007	2011	2014	2007	2011	2014	2007	2011	2014
Open defecation	7.5	4.2	3.3	7.5	4.0	2.2	13.8	3.9	3.2	2.1	12.5	9.4
	(0.8)	(0.3)	(0.5)	(1.4)	(0.7)	(0.8)	(2.3)	(0.8)	(0.9)	(0.4)	(1.5)	(1.3)
Unimproved	47.1	38.3	25.7	44.3	35.9	22.6	45.3	36.4	28.3	57.2	34.3	23.0
	(1.1)	(0.9)	(1.3)	(1.9)	(1.7)	(2.8)	(2.6)	(3.1)	(2.7)	(3.3)	(2.0)	(2.6)
Limited	13.4	16.7	20.9	14.4	18.0	26.1	14.7	20.7	20.2	10.1	17.7	22.3
	(0.5)	(0.6)	(0.8)	(1.1)	(1.5)	(2.0)	(1.2)	(1.4)	(1.3)	(1.9)	(0.9)	(1.8)
Basic	3.5	4.3	2.3	8.6	10.5	5.4	0.2	0.2	0.3	0.6	0.1	0.3
	(0.4)	(0.5)	(0.4)	(1.0)	(1.6)	(1.0)	(0.1)	(0.1)	(0.1)	(0.2)	(0.1)	(0.2)
Not deprived	28.5	36.6	47.8	25.2	31.6	43.7	26.0	38.8	48.0	30.1	35.4	45.0
	(1.0)	(0.9)	(1.1)	(1.9)	(1.9)	(2.5)	(1.9)	(2.3)	(2.3)	(2.3)	(1.9)	(1.7)

Sources: Authors' own computations. Standard errors are reported in parentheses.

[14] The JMP document titled *WASH Post-2015: Proposed indicators for drinking water, sanitation and hygiene* was accessed in April 2017 at https://www.wssinfo.org.

Table 3 shows how the estimated population shares in different deprivation categories have evolved over time in Bangladesh. Clearly, the estimated percentage in the 'improved' category has gradually increased (statistically significantly) from 28.5% in 2007 to 36.6% in 2011 to 47.8% in 2014. Thus, the proportion of the population in deprived categories has gone down over the same period. Changes within the deprived categories are however mixed. Although the estimated population shares in the two most deprived categories ('open defecation' and 'unimproved') have decreased (statistically significantly) systematically between 2007 and 2014, the population shares in the other two deprivation categories have not.

Table 4: Change in sanitation deprivation by ordinal poverty measures in Bangladesh and its divisions

	H			P_I			P_M			P_G		
	2007	2011	2014	2007	2011	2014	2007	2011	2014	2007	2011	2014
Barisal	66.1	60.5	46.8	47.7	44.0	32.6	35.0	32.6	23.4	25.2	23.5	16.7
	(2.7)	(2.1)	(3.3)	(1.9)	(1.7)	(2.9)	(1.4)	(1.4)	(2.4)	(1.1)	(1.1)	(1.8)
Chittagong	67.1	59.2	44.9	47.0	38.9	29.2	34.8	27.3	20.2	26.2	19.6	14.9
	(2.9)	(2.1)	(3.1)	(2.8)	(1.7)	(2.9)	(2.7)	(1.5)	(2.7)	(2.6)	(1.2)	(2.5)
Dhaka	74.8	68.4	56.3	50.1	42.6	33.5	36.6	29.4	21.8	27.8	21.6	15.4
	(1.9)	(1.9)	(2.5)	(1.7)	(1.3)	(1.8)	(1.6)	(1.1)	(1.6)	(1.4)	(1.0)	(1.3)
Khulna	69.0	61.4	50.3	48.9	41.8	33.0	35.7	29.4	22.7	25.9	20.9	16.1
	(1.8)	(1.6)	(2.3)	(1.4)	(1.2)	(1.7)	(1.1)	(1.0)	(1.4)	(1.0)	(0.7)	(1.0)
Rajshahi	74.0	61.2	52.0	55.2	41.6	34.7	43.0	29.6	24.2	34.1	21.6	17.6
	(1.9)	(2.3)	(2.3)	(1.8)	(1.9)	(1.8)	(1.9)	(1.6)	(1.5)	(1.9)	(1.3)	(1.3)
Sylhet	69.9	64.6	55.0	50.1	47.1	37.9	36.8	36.2	27.9	26.5	28.9	21.9
	(2.3)	(1.9)	(1.7)	(1.9)	(1.6)	(1.7)	(1.5)	(1.4)	(1.6)	(1.1)	(1.4)	(1.4)
Bangladesh	71.5	63.4	52.2	50.4	42.3	33.5	37.5	30.1	23.1	28.5	22.2	16.8
	(1.0)	(0.9)	(1.1)	(0.9)	(0.7)	(0.9)	(0.9)	(0.6)	(0.8)	(0.8)	(0.5)	(0.7)

Sources: Authors' own computations. Standard errors are reported in parentheses.

Has this estimated reduction pattern been replicated within all divisions? Table 3 also presents the changes in the discrete probability distributions of three divisions: Dhaka, Rajshahi, and Sylhet.[15] The estimated population shares in the 'improved' category have increased (statistically significantly) gradually in all three regions (Table 3), and so the shares of deprived population have gone down. We need, however, to point out two crucial aspects.

First, let us compare the reduction patterns in Dhaka and Rajshahi. The population share in the 'improved' category is higher in Rajshahi in 2011 and 2014 and statistically indistinguishable in 2007, implying that sanitation deprivation in Dhaka is never lower than sanitation deprivation in Rajshahi. However, the estimated population shares in the two most deprived categories ('open defecation' and 'unimproved') are higher in Rajshahi than in Dhaka in 2007 and 2014 and statistically indistinguishable in 2011. Second, like Dhaka and Rajshahi in Table 3, sanitation deprivation in Sylhet has also improved gradually. However, the estimated population share in the poorest category ('open defecation') is significantly higher in 2011 and in 2014 than in 2007. A simple headcount measure, which only captures the proportion of the overall deprived population, would always overlook these substantial differences.

Table 4 presents four different poverty measures for Bangladesh and for its six divisions (as per the pre-2010 administrative map). We assume the poverty threshold category to be 'basic unsafe'. The first poverty measure is the headcount ratio (H), which, in this context, is the population share experiencing any one of the four deprivation categories. The second measure is P_I, such that $P_I \in \mathcal{P} \setminus \{\mathcal{P}_\alpha\}$ for $\alpha \geq 1$, and is defined by the ordering

[15] A new division called Rangpur was formed in 2010, which was a part of the Rajshahi Division. The new division did not exist during the 2007 DHS, so we had to combine this division with the Rajshahi division in the 2011 and 2014 DHS in order to preserve comparability over time. Likewise, our selected surveys were not affected by the carving out of northern Dhaka to form the Mymesingh division in 2015.

weights $\boldsymbol{\omega}^I = (1, 0.75, 0.5, 0.25, 0)$. The third measure is $P_M \in \mathcal{P}_1 \setminus \{\mathcal{P}_\alpha\}$ for $\alpha \geq 2$ with ordering weights $\boldsymbol{\omega}^M = (1, 0.75^2, 0.5^2, 0.25^2, 0)$, i.e. respecting the restrictions in Corollary 3.1, but not respecting, for instance, the restrictions in Proposition 3.1 or the restrictions in Theorem 3.1 for $\alpha \geq 2$; whereas, the fourth measure is $P_G \in \mathcal{P}_G$ with ordering weights $\boldsymbol{\omega}^G = (1, 0.4, 0.15, 0.05, 0)$, i.e. respecting the restrictions in Proposition 3.1. Note that measures P_M and P_G give precedence to those who are in the poorer categories. All four measures lie between zero and one, but we have multiplied them by one hundred so that they lie between zero (lowest deprivation) and 100 (highest deprivation).

Comparisons of these measures provide useful insights, especially into the two crucial aspects that we have presented in Table 3. The headcount ratio estimate in Dhaka is statistically indistinguishable from the headcount ratio estimate in Rajshahi for 2007, despite deprivation in the two poorest categories being higher in Rajshahi. However, this crucial aspect is captured by the latter three measures, which show statistically significantly higher poverty estimates in Rajshahi than in Dhaka. Similarly, the headcount ratio estimate is higher in Dhaka than in Rajshahi for 2011, but the difference vanishes when poverty is assessed by the other three ordinal measures.

Since most of the information in Table 4 points to experiences of poverty reduction, we further conduct pair-wise dominance tests in order to verify whether the comparisons over time are robust to all measures in class $\mathcal{P}$, and in sub-classes $\mathcal{P}_1$ and $\mathcal{P}_G$. Note, in this case, that $S = 5$ and $k = 4$. The dominance test for $\mathcal{P}$ is based on Theorem 4.1, the dominance test for $\mathcal{P}_1$ is based on Corollary 4.2 (i.e. $\alpha = 1$), and the dominance test for $\mathcal{P}_G$ is based on Corollary 4.3 (i.e. $\alpha = k - 1 = 3$). Checking pair-wise dominance for $\mathcal{P}_1$ requires comparing poverty levels at the following four extreme points: $(1,0,0,0,0)$, $(1,\frac{1}{2},0,0,0)$, $(1,\frac{2}{3},\frac{1}{3},0,0)$ and $(1,\frac{3}{4},\frac{2}{4},\frac{1}{4},0)$. Meanwhile, checking pair-wise dominance for $\mathcal{P}_G$ requires comparing poverty levels at the following four extreme points: $(1,0,0,0,0)$, $(1,\frac{1}{2},0,0,0)$, $(1,\frac{1}{2},\frac{1}{4},0,0)$ and $(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},0)$.

Table 5: Pair-wise dominance tests for changes in poverty over time in Bangladesh and its divisions

	2007–2011			2011–2014		
	$\mathcal{P}$	$\mathcal{P}_1$	$\mathcal{P}_G$	$\mathcal{P}$	$\mathcal{P}_1$	$\mathcal{P}_G$
Barisal	No	No	No	Yes	Yes	Yes
Chittagong	Yes**	Yes**	Yes**	No	No	No
Dhaka	Yes**	Yes**	Yes**	Yes**	Yes**	Yes**
Khulna	Yes*	Yes*	Yes*	Yes	Yes	Yes
Rajshahi	Yes***	Yes***	Yes***	Yes	Yes	Yes
Sylhet	No	No	No	Yes*	Yes*	Yes*
Bangladesh	Yes***	Yes***	Yes***	Yes*	Yes*	Yes*

Sources: Authors' own computations. A "Yes" implies reduction of the poverty levels within each region over time evaluated at the four extreme weights; a "No" implies otherwise. The levels of statistical significance for the dominance tests are: *** for 1%, ** for 5% and * for 10%.

Table 5 presents the pair-wise dominance tests over time for Bangladesh and its divisions. Each cell of the table reports whether a *reduction* or *increase* in poverty within each region over time in Table 4 is robust (or not) to alternative ordering weighting vectors within the class (i.e., $\mathcal{P}$) and subclasses (i.e., $\mathcal{P}_1$ and $\mathcal{P}_2$) to which the ordering weighting vectors used in Table 4 belong.[16] For example, it is evident from Table 4 that, according to P_M, poverty

[16]We implemented a standard intersection-union test (IUT) whose alternative hypothesis is that poverty levels at the four extreme points are jointly lower in year-region "A" (e.g. Bangladesh in 2011) vis-à-vis "B" (e.g. Bangladesh in 2007). We reject the null (i.e. at least one poverty level in "A" is equal or higher than in "B") in favour of this alternative only if every poverty level is lower in "A" than in "B" with a given significance level α. As explained by Berger (1997, p. 226), the IUT's overall significance level is also α and no correction for multiple comparisons (e.g. Bonferroni, etc.) is required.

in Bangladesh has fallen from 37.5 points in 2007 to 30.1 points in 2011. So we could ask: is this reduction between 2007 and 2011 robust to all $P \in \mathcal{P}_1$? The corresponding dominance test reported in Table 5 shows that the reduction is certainly robust with 1% significance level. If a cell in Table 5 reports either "No" or "Yes", but without appropriate statistical significance level, then we conclude that the relevant comparison in Table 4 is not robust. For instance, poverty reduction in Dhaka has been robust for the whole class $\mathcal{P}$ at the 5% level of significance throughout both time periods (2007-2011 and 2011-2014); whereas, the poverty reduction for neither Barisal nor Chittagong for the period 2011-2014 reported in Table 4 can be claimed to be robust for any of the three (sub)classes of indices considered.

6. Concluding remarks

There is little doubt that poverty is a multidimensional concept and the current global development agenda correctly seeks to 'reduce poverty in all its dimensions'. To meet this target, it is indeed important to assess poverty from a multidimensional perspective. However, one should not discount the potential interest in evaluating the impact of a targeted program in reducing deprivation in a single dimension such as educational or health outcomes and access to public services, which may often be assessed by an ordinal variable with multiple ordered deprivation categories. The frequently used headcount ratio, in this case, is ineffective as it overlooks the depth of deprivations, i.e. any changes within the ordered deprivation categories.

Our paper has thus posed the question: 'How should we assess poverty when variables are ordinal?' Implicitly, the companion question is 'Can we meaningfully assess poverty beyond the headcount ratio when we have an ordinal variable?' Drawing on six reasonable axiomatic properties, our answer is: 'Poverty can be measured with ordinal variables through weighted averages of the discrete probabilities corresponding to the ordered categories.' We refer to these weights as ordering weights, which need to satisfy a specific set of restrictions in order to ensure the social poverty indices fulfil these key properties. Our axiomatically characterised class of social poverty indices has certain desirable features, such as additive decomposability and being bounded between zero (when none experiences any deprivation) and one (when everyone experiences the most deprived category).

In contrast to previous attempts in the literature on poverty measurement with ordinal variables, we have gone fruitfully further in the direction of operationalising different concepts of 'precedence to the poorer people among the poor', which ensures that the policymaker has an incentive to assist the poorer over the less poor. We have shown that it is possible to devise reasonable poverty measures that prioritise welfare improvements among the most deprived when variables are ordinal. We have axiomatically characterised a set of subclasses of ordinal poverty measures based on different notions of precedence to those poorer among the poor. Each subclass is defined by an additional restriction on the admissible ordering weights. The precedence-sensitive measures have proven useful in the illustration pertaining to sanitation deprivation in Bangladesh by highlighting those provinces where the overall headcount improvement did not come about through reductions in the proportion of the population in the most deprived categories.

Since several poverty measures are admissible within each characterised class and subclass, we have also developed stochastic dominance conditions for each subclass of poverty measures.

Their fulfilment guarantees that all measures within a given class (or subclass) rank the same pair of distributions robustly. While some of these conditions represent the ordinal-variable analog of existing conditions for continuous variables in the poverty dominance literature (Foster and Shorrocks, 1988b); others are, to the best of our knowledge, themselves a novel methodological contribution to the literature on stochastic dominance with ordinal variables.

There has been a recent surge in the literature on multidimensional poverty measurement, especially within the counting framework. In this framework, however, it is still a usual practice to dichotomise deprivations within each dimension when using existing counting measures, ignoring the depth of deprivation across ordered categories. Future research could focus on the development of counting measures that incorporate both the depth of deprivations within dimensions and notions of precedence to the poorest among the poor.

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Appendices

Appendix A1. Proof of Theorem 2.1

It is straightforward to check that each poverty measure in Equation 1 satisfies the four properties: OMN, SCD, FOC and SUD.

We now prove the necessary part, namely, if a poverty measure satisfies these four properties, then it takes the functional form in Equation 1. First, we define $S \in \mathbb{N} \setminus \{1\}$ basis vectors: $\mathbf{e}^s = (e_1^s, \ldots, e_S^s) \in \mathbb{P}\ \forall s = 1, \ldots, S$, such that $e_s^s = 1\ \forall s$ and $e_s^t = 0\ \forall s \neq t$. For a given $c_k \in \mathbf{C}_{-S}$, denote $P(\mathbf{e}^s, c_k) = \omega_s$ for some $\omega_s \in \mathbb{R}_+$ for every $s = 1, \ldots, S$.

Next, suppose a society with probability distribution $\mathbf{p} = (p_1, \ldots, p_S) \in \mathbb{P}$ is divided into $M = S$ mutually exclusive and collectively exhaustive sub-groups, such that the probability distribution of each subgroup is $\mathbf{p}^s = (p_1^s, \ldots, p_S^s) = \mathbf{e}^s$ for every $s = 1, \ldots, S\ (= M)$. As each subgroup contains the population in each category, naturally, the population share of each subgroup is $\pi_s = p_s$ for every $s = 1, \ldots, S\ (= M)$. Then, by SUD:

$$P(\mathbf{p}, c_k) = \sum_{s=1}^{S} \pi_s P(\mathbf{p}^s, c_k) = \sum_{s=1}^{S} p_s P(\mathbf{e}^s, c_k) = \sum_{s=1}^{S} p_s \omega_s. \tag{A1}$$

Now, consider any two $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, such that $p_s = 1$ for some $s \leq k$ and $q_{s'} = 1$ for some $s' > s$. Naturally, $p_t = q_t = 0$ for all $t \neq \{s, s'\}$. Clearly, by OMN, we have $P(\mathbf{q}, c_k) < P(\mathbf{p}, c_k)$. Combined with Equation A1, we then obtain:

$$\omega_{s'} < \omega_s. \tag{A2}$$

The relationship in Equation A2 holds for any s, such that $s \leq k$ and $s < s'$. In other words, $\omega_{s-1} > \omega_s > \omega_{s'}$ for all $s = 2, ..., k$ and even for any $s' > k$, whenever $k \geq 2$. When $S = 2$, then $k = 1$ and so $\omega_1 > \omega_2$.

We next use property SCD. Suppose, $k = 1$. Then, property SCD leads to $P(\mathbf{p}, c_k) = p_1$ and Equation A1 yields

$$P(\mathbf{p}, c_1) = \sum_{s=1}^{S} p_s \omega_s = p_1. \tag{A3}$$

Note, by definition, that $0 \leq p_1 \leq 1$ and so $0 \leq P(\mathbf{p}, c_k) \leq 1$. Consider some $\mathbf{p} \in \mathbb{P}$, such that $p_1 = 1$ and $p_s = 0$ for all $s \neq 1$. Clearly, from Equation A3, $\omega_1 = 1$. Moreover, from Equation A2, it follows that $1 > \omega_s > \omega_{s'}$ for all $s = 2, \ldots, k$ and for any $s' > k$ whenever $k \geq 2$.

In order to complete the proof, we need to show that $\omega_s = 0$ for all $s > k$. For this purpose, consider some $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, such that $p_t = q_u = 1$ for some $t > u > k$. Certainly, $p_s = q_s = 0 \ \forall s \leq k$. By property FOC, we then require $P(\mathbf{p}, c_k) = P(\mathbf{q}, c_k)$. Thus, from Equation A1, we obtain $P(\mathbf{p}, c_k) = \omega_t = \omega_u = P(\mathbf{q}, c_k)$ for $t > u > k$. Since, $p_s = q_s = 0 \ \forall s \leq k$, it follows that $p_1 = q_1 = 0$. Consider $k = 1$. Then, by property SCD, we must have $P(\mathbf{p}, c_k) = P(\mathbf{q}, c_k) = 0$. Hence, it must be the case that $\omega_s = 0$ for all $s > k$, which completes our proof. ∎

Appendix A2. Proof of Theorem 3.1

The sufficiency part is straightforward. We prove the necessity part as follows.

Suppose $k \geq 2$ and $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k - 1$. Now, suppose, $\mathbf{p}'$ and $\mathbf{q}'$ are obtained from $\mathbf{p} \in \mathbb{P}$ as follows. Consider some $s' < t \leq k < S$. Now, $\mathbf{p}'$ is obtained from $\mathbf{p}$, such that $p'_{s'} = p_{s'} - \epsilon$ while $p'_u = p_u \ \forall u \neq \{s', s' + 1\}$. Naturally, $p'_{s'+1} = p_{s'+1} + \epsilon$. Similarly, $\mathbf{q}'$ is obtained from $\mathbf{p}$, such that $q'_t = p_t - \epsilon$ while $q'_u = p_u \ \forall u \neq \{t, t'\}$ for some $t' = \min\{t + \alpha, S\}$. Naturally, again, $q'_{t'} = p_{t'} + \epsilon$.

By property PRE-α, we know that

$$P(\mathbf{p}', c_k) < P(\mathbf{q}', c_k). \tag{A4}$$

Combining Equation 1 and Equation A4, we get

$$\omega_{s'+1} - \omega_{s'} - \omega_{t'} + \omega_t < 0.$$

Substituting $t = s' + 1 = s$ for any $s = 2, \ldots, k$, we obtain

$$\omega_{s-1} - \omega_s > \omega_s - \omega_{t'}. \tag{A5}$$

First, suppose $t' = s + \alpha \leq k < S$ or $s \leq k - \alpha$. Then, $\omega_{t'} = \omega_{s+\alpha} > 0$ by Theorem 2.1 and Equation A5 can be expressed as $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+\alpha}$ for all $s = 2, \ldots, k - \alpha$. Second, suppose $t' = \min\{s + \alpha, S\} > k$ or $s > k - \alpha$. We know that $\omega_s = 0$ for all $s > k$ by Theorem 2.1 and so Equation A5 can be expressed as $\omega_{s-1} - \omega_s > \omega_s$ or $\omega_{s-1} > 2\omega_s$ for all $s = k - \alpha + 1, \ldots, k$. This completes the proof. ■

Appendix A3. Proof of Proposition 3.1

Let us first prove the sufficiency part. Suppose $k \geq 2$. We already know from Theorem 2.1 that $\omega_{s-1} > \omega_s > 0 \ \forall s = 2, \ldots, k$ and $\omega_s = 0 \ \forall s > k$. Suppose additionally that $\omega_{s-1} > 2\omega_s \ \forall s = 2, \ldots, k$. Alternatively, $\omega_{s-1} - \omega_s > \omega_s \ \forall s = 2, \ldots, k$.

For any $\mathbf{p}, \mathbf{p}', \mathbf{q}' \in \mathbb{P}$, for any $c_k \in \mathbf{C}_{-S}$ and for some $\epsilon \in (0, 1)$, let $k \geq t \geq v + \alpha$ and suppose $\mathbf{p}'$ is obtained from $\mathbf{p}$, such that $p'_v = p_v - \epsilon$ and $p'_{v+\alpha} = p_{v+\alpha} + \epsilon$, while $p'_u = p_u \ \forall u \neq \{v, v+\alpha\}$; and $\mathbf{q}'$ is obtained from $\mathbf{p}$, such that $q'_t = p_t - \epsilon$ and $q'_{t+\beta} = p_{t+\beta} + \epsilon$ for some $\beta \in \mathbb{N}$, while $q'_u = p_u \ \forall u \neq \{t, t + \beta\}$.

With the help of Equation 1, we get:

$$P(\mathbf{p}', c_k) - P(\mathbf{q}', c_k)] = \epsilon[\omega_{v+\alpha} - \omega_v - \omega_{t+\beta} + \omega_t] = \epsilon[(\omega_t - \omega_{t+\beta}) - (\omega_v - \omega_{v+\alpha})]. \quad \text{(A6)}$$

By assumption of the sufficiency part: $\omega_{s-1} - \omega_s > \omega_s \; \forall s = 2, \ldots, k$. Combining this assumption with the weight restrictions from Theorem 2.1 we can easily deduce that $(\omega_v - \omega_{v+\alpha}) > (\omega_v - \omega_{v+1}) > \omega_{v+1} > \omega_{v+\alpha}$. Hence, $(\omega_v - \omega_{v+\alpha}) > \omega_{v+\alpha}$. Since $v + \alpha \leq t \leq k$ and $\omega_{s-1} > \omega_s > 0 \; \forall s = 2, \ldots, k$, it also follows that $\omega_{v+\alpha} \geq (\omega_t - \omega_{t+\beta})$. Hence, $(\omega_v - \omega_{v+\alpha}) > (\omega_t - \omega_{t+\beta})$ and $P(\mathbf{p}', c_k) < P(\mathbf{q}', c_k)$.

Next we prove the necessity part starting with Equation A6. By property PRE-G, we know that $P(\mathbf{p}', c_k) < P(\mathbf{q}', c_k)$. Thus,

$$\omega_v - \omega_{v+\alpha} > \omega_t - \omega_{t+\beta}. \quad \text{(A7)}$$

Now the inequality in Equation A7 must hold for any situation in which $t \geq v + \alpha$, including the comparison of the minimum possible improvement for the poorer person, given by $\omega_v - \omega_{v+1}$ (i.e. with $\alpha = 1$), against the maximum possible improvement for the less poor person, given by $\omega_t - \omega_{t+\beta}$ with $t = v + 1$ and $t + \beta > k$. Inserting these values into Equation A7, bearing in mind that $\omega_{t+\beta} = 0$ when $t + \beta > k$, yields

$$\omega_v - \omega_{v+1} > \omega_{v+1}.$$

Substituting $v = s - 1$ for any $s = 2, \ldots, k$ yields $\omega_{s-1} - \omega_s > \omega_s$. Hence, $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$. ■

Appendix A4. Proof of Theorem 4.1

We denote the *difference operator* by Δ and use the following handy definitions: For some $\mathbf{p}, \mathbf{p}' \in \mathbb{P}$, define $\Delta P_k \equiv P(\mathbf{p}, c_k) - P(\mathbf{p}', c_k)$, $\Delta p_s \equiv p_s - p'_s$, and $\Delta F_s \equiv \sum_{\ell=1}^{s} \Delta p_\ell$.

We first prove the sufficiency part. From Theorem 2.1, we know that $\omega_s = 0$ for all $s > k$. Thus, Equation 1 may be presented using the difference operator as $\Delta P_k = \sum_{s=1}^{k} \omega_s \Delta p_s$. Using summation by parts, also known as Abel's lemma (Guenther and Lee, 1988) or formula (Fishburn and Lavalle, 1995, p. 518), it follows that

$$\Delta P_k = \sum_{s=1}^{k-1} [\omega_s - \omega_{s+1}] \Delta F_s + \Delta F_k \omega_k. \tag{A8}$$

We already know from Theorem 2.1 that $\omega_k > 0$ and $\omega_s - \omega_{s+1} > 0$ $\forall s = 1, \ldots, k-1$. Therefore, clearly from Equation A8, the condition that $\Delta F_s \leq 0$ $\forall s \leq k$ and $\Delta F_s < 0$ for at least one $s \leq k$ is sufficient to ensure that $\Delta P_k < 0$ $\forall P \in \mathcal{P}$ and for a given $c_k \in \mathbf{C_{-S}}$.

We next prove the necessity part by contradiction. Consider the situation, where $\Delta F_t > 0$ for some $t \leq k$, $\Delta F_s \leq 0$ for all $s \leq k$ but $s \neq t$, and $\Delta F_s < 0$ for some $s \leq k$ but $s \neq t$. For a sufficiently large value of $\omega_t - \omega_{t+1}$ in Equation A8, it may always be possible to find that $\Delta P_k > 0$. Or, consider the situation $\Delta F_s = 0$ for all $s \leq k$. In this case, $\Delta P_k = 0$. Hence, the necessary condition requires both $\Delta F_s \leq 0$ for all $s \leq k$ and $\Delta F_s < 0$ for some $s \leq k$. This completes the proof. ■

Appendix A5. Proof of Theorem 4.2

Consider some $k \geq 2$ and some $\alpha \in [1, k-1] \subseteq \mathbb{N}$. Let Ω_α^k be the set of all S-dimensional ordinal weighting vectors corresponding to subclass $\mathcal{P}_\alpha$ (see Theorems 3.1 and 2.1), so that:

$$\Omega_\alpha^k = \begin{cases} (\omega_1, \ldots, \omega_S) \mid \omega_1 = 1;\ \omega_{s-1} > 2\omega_s - \omega_{s+\alpha}\ \forall s = 2, \ldots, k-\alpha;\ \omega_{s-1} > 2\omega_s > 0 \\ \qquad \forall s = k-\alpha+1, \ldots, k;\ \text{and } \omega_s = 0\ \forall s > k \text{ whenever } \alpha \leq k-2 \\ (\omega_1, \ldots, \omega_S) \mid \omega_1 = 1;\ \omega_{s-1} > 2\omega_s > 0\ \forall s = 2, \ldots, k;\ \text{and } \omega_s = 0\ \forall s > k \\ \qquad \text{whenever } \alpha = k-1 \end{cases}.$$

Now define $\bar{\Omega}_\alpha^k$ as:

$$\bar{\Omega}_\alpha^k = \begin{cases} (\omega_1, \ldots, \omega_S) \mid \omega_1 = 1;\ \omega_{s-1} \geq 2\omega_s - \omega_{s+\alpha}\ \forall s = 2, \ldots, k-\alpha;\ \omega_{s-1} \geq 2\omega_s \geq 0 \\ \qquad \forall s = k-\alpha+1, \ldots, k;\ \text{and } \omega_s = 0\ \forall s > k \text{ for } \alpha \leq k-2 \\ (\omega_1, \ldots, \omega_S) \mid \omega_1 = 1;\ \omega_{s-1} \geq 2\omega_s \geq 0\ \forall s = 2, \ldots, k;\ \text{and } \omega_s = 0\ \forall s > k \\ \qquad \text{for } \alpha = k-1 \end{cases}.$$

For given values of k and α, $\bar{\Omega}_\alpha^k$ is bounded by a set of linear constraints and it is a convex hull of k extreme points or extreme ordering weighting vector, denoted by $\bar{\boldsymbol{\omega}}^1, \ldots, \bar{\boldsymbol{\omega}}^k$. For the r^{th} extreme point $\bar{\boldsymbol{\omega}}^r$, where $r \in \{1, 2, \ldots, k\}$, *maximum* feasible positive weights are assigned to the first r elements and *minimum* feasible weights are assigned to the rest of the elements (where feasibility refers to satisfying the linear constraints defining the set $\bar{\Omega}_\alpha^k$), such that $\omega_s = \bar{\omega}_s^r > 0$ for all $s = 1, \ldots, r$ and $\omega_s = \bar{\omega}_s^r = 0$ for all $s = r+1, \ldots, S$.[17] Any

[17]The proof is similar in spirit to the proof of Proposition 1 in Seth and McGillivray (2018).

$\boldsymbol{\omega} \in \bar{\Omega}_\alpha^k$ is a convex combination of these k extreme ordering weighting vectors, such that $\boldsymbol{\omega} = \sum_{r=1}^k \theta_r \bar{\boldsymbol{\omega}}^r$, where $\sum_{r=1}^k \theta_r = 1$ and $\theta_r \geq 0$ for all $r = 1, \dots, k$.

The restrictions on the aforementioned θ_r's may be verified as follows. Since, $\omega_1 = 1$ and $\bar{\omega}_1^r = 1$ for all r by Theorem 2.1, then plugging these values in $\omega_1 = \sum_{r=1}^k \theta_r \bar{\omega}_1^r$ yields $\sum_{r=1}^k \theta_r = 1$. Next, define:

$$u_s = \begin{cases} \omega_s - 2\omega_{s+1} + \omega_{s+\alpha+1} \ \forall s = 1, \dots, k-\alpha-1 \text{ and } \omega_s - 2\omega_{s+1} \\ \quad \forall s = k-\alpha, \dots, S-1 \text{ whenever } \alpha \leq k-2 \\ \omega_s - 2\omega_{s+1} \ \forall s = 1, \dots, S-1 \text{ whenever } \alpha = k-1 \end{cases}.$$

Since, $\omega_s = \sum_{r=1}^k \theta_r \bar{\omega}_s^r$ for all s and $\bar{\omega}_s^r = 0$ for all $s > r$, we obtain:

$$u_s = \begin{cases} \sum_{r=s}^k \theta_r \bar{\omega}_s^r - 2\sum_{r=s+1}^k \theta_r \bar{\omega}_{s+1}^r + \sum_{r=s+1+\alpha}^k \theta_r \bar{\omega}_{s+1+\alpha}^r \text{ when } s \leq k-\alpha-1 \\ \sum_{r=s}^k \theta_r \bar{\omega}_s^r - 2\sum_{r=s+1}^k \theta_r \bar{\omega}_{s+1}^r \text{ otherwise} \end{cases}.$$

Rearranging the right hand side of u_s yields:

$$u_s = \theta_s \bar{\omega}_s^s + \sum_{r=s+1}^{s+\alpha} \theta_r [\bar{\omega}_s^r - 2\bar{\omega}_{s+1}^r] + \sum_{r=s+1+\alpha}^{k} \theta_r [\bar{\omega}_s^r - 2\bar{\omega}_{s+1}^r + \bar{\omega}_{s+1+\alpha}^r], \tag{A9}$$

where the third component on the right hand side of Equation A9 exists only when $s+\alpha+1 \leq k$. Given that the extreme weighting vectors satisfy the equality restrictions in $\bar{\Omega}_\alpha^k$, the second component and the third component (whenever in existence) on the right-hand side must be equal to zero. Hence, $u_s = \theta_s \bar{\omega}_s^s$ for all s, which is unrelated to the restrictions of Theorem 3.1. Given that $u_s \geq 0$ for all $s = 1, \dots, k$ by the definition of $\bar{\Omega}_\alpha^k$ and also $\bar{\omega}_s^s > 0$ (i.e., $s = r$) for all $s = 1, \dots, k$ by the definition of extreme weighting vectors, then, substituting $r = s$, it must be the case that $\theta_r \geq 0$ for all $r = 1, \dots, k$.

The primary difference between Ω_α^k and $\bar{\Omega}_\alpha^k$ is that the elements of $\bar{\Omega}_\alpha^k$ satisfy the non-strict versions of the inequality restrictions prescribed by Theorem 3.1. Clearly, $\Omega_\alpha^k \subset \bar{\Omega}_\alpha^k$ and therefore any $\boldsymbol{\omega} \in \Omega_\alpha^k$ is also a convex combination of $\bar{\boldsymbol{\omega}}^1, \ldots, \bar{\boldsymbol{\omega}}^k$, i.e., $\boldsymbol{\omega} = \sum_{r=1}^k \theta_r \bar{\boldsymbol{\omega}}^r$, such that $\sum_{r=1}^k \theta_r = 1$. However, the strict inequality constraints in Ω_α^k require that $u_s > 0$ for all $s = 1, \ldots, k$, which implies that $\theta_r > 0$ for all $r = 1, \ldots, k$ for Ω_α^k.

For any $\mathbf{p} \in \mathbb{P}$, let us denote the poverty level evaluated at some $\boldsymbol{\omega}$ as $P(\mathbf{p}, c_k; \boldsymbol{\omega})$ and, given additive decomposability, we must have $P(\mathbf{p}, c_k; \boldsymbol{\omega}) = \sum_{r=1}^k \theta_r P(\mathbf{p}, c_k; \bar{\boldsymbol{\omega}}^r)$, where $\sum_{r=1}^k \theta_r = 1$ and $\theta_r > 0$ for all $r = 1, \ldots, k$. Clearly, for some $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, if $P(\mathbf{p}, c_k; \bar{\boldsymbol{\omega}}^r) \leq P(\mathbf{q}, c_k; \bar{\boldsymbol{\omega}}^r)$ for all $r = 1, \ldots, k$, with one strict inequality, then it follows directly that $P(\mathbf{p}, c_k; \boldsymbol{\omega}) < P(\mathbf{q}, c_k; \boldsymbol{\omega})$ for any $\boldsymbol{\omega} \in \Omega_\alpha^k$ or, equivalently, for all $P \in \mathcal{P}_\alpha$ for a given c_k. This proves sufficiency.

In order to prove necessity, suppose $P(\mathbf{p}, c_k; \bar{\boldsymbol{\omega}}^r) > P(\mathbf{q}, c_k; \bar{\boldsymbol{\omega}}^r)$ for some $r \in \{1, 2, \ldots, k\}$. Then, for a sufficiently large value of θ_r (i.e., for $\theta_r \to 1$), it is always possible to have $P(\mathbf{p}, c_k; \boldsymbol{\omega}) > P(\mathbf{q}, c_k; \boldsymbol{\omega})$. Furthermore, in case $P(\mathbf{p}, c_k; \bar{\boldsymbol{\omega}}^r) = P(\mathbf{q}, c_k; \bar{\boldsymbol{\omega}}^r)$ for all $r = 1, \ldots, k$, then certainly $P(\mathbf{p}, c_k; \boldsymbol{\omega}) = P(\mathbf{q}, c_k; \boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega_\alpha^k$. Hence, the conditions, $P(\mathbf{p}, c_k; \bar{\boldsymbol{\omega}}^r) \leq P(\mathbf{q}, c_k; \bar{\boldsymbol{\omega}}^r)$ for all $r = 1, \ldots, k$, with one strict inequality, are also necessary.

Finally, the solutions for the k extreme points are obtained as follows. By Theorem 2.1, $\bar{\omega}_1^r = 1$ for all $r = 1, \ldots, k$. From above, we already know that $\bar{\omega}_s^r = 0$ for all $s = r+1, \ldots, S$ and for every r. Therefore, for every $r \in \{1, \ldots, k\}$, $\bar{\omega}_s^r = 0$ for all $s > r$. We also know that $\bar{\omega}_s^r > 0$ for all $s \leq r$. Now, for any α and for every $r \in \{2, \ldots, k\}$, we obtain the following system of $(r-1)$ equations: $\bar{\omega}_1^r = 2\bar{\omega}_2^r - \bar{\omega}_{2+\alpha}^r$, $\bar{\omega}_2^r = 2\bar{\omega}_3^r - \bar{\omega}_{3+\alpha}^r$, $\ldots$, $\bar{\omega}_{r-\alpha-1}^r = 2\bar{\omega}_{r-\alpha}^r - \bar{\omega}_r^r$, $\bar{\omega}_{r-\alpha}^r = 2\bar{\omega}_{r-\alpha+1}^r$, $\ldots$, $\bar{\omega}_{r-2}^r = 2\bar{\omega}_{r-1}^r$, $\bar{\omega}_{r-1}^r = 2\bar{\omega}_r^r$. There are $(r-1)$ unknowns: $\bar{\omega}_2^r, \ldots, \bar{\omega}_r^r$, since $\bar{\omega}_1^r = 1$. Solving the system of equations, we obtain $\omega_s^r = \bar{\omega}_s^r$ for all $s = 2, \dots, r$ and for all $r = 1, \dots, k$ in Equation 2. This completes the proof of the theorem. ■

Appendix A6. Proof of Corollary 4.2

We use the difference operators defined in Appendix A4. Summing by parts the first component on the right-hand side of Equation A8 and then rearranging the terms, we obtain:

$$\Delta P_k = \sum_{s=1}^{k-1} \left(\{[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}]\} \sum_{\ell=1}^{s} \Delta F_\ell \right) + \omega_k \sum_{s=1}^{k} \Delta F_s. \tag{A10}$$

We know from Corollary 3.1 that $[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}] > 0 \ \forall s = 1, ..., k-1$. Likewise, by Theorem 2.1, $\omega_k > 0$. Therefore $\sum_{\ell=1}^{s} \Delta F_\ell \leq 0$ for all $s = 1, ..., k$ with at least one strict inequality is sufficient to ensure that, for any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, $P(\mathbf{p}, c_k) < P(\mathbf{q}, c_k) \ \forall P \in \mathcal{P}_1$. On the contrary, suppose $\sum_{\ell=1}^{s} \Delta F_\ell > 0$ for some $s \in \{1, \dots, k\}$. Now, note that there is no further restriction on whether any of the weight functions in Equation A10 is strictly greater than the others. Then, attaching a sufficiently large weight to this component (i.e. $\sum_{\ell=1}^{s} \Delta F_\ell > 0$ for some $s \in \{1, \dots, k\}$) may result in $P(\mathbf{p}, c_k) > P(\mathbf{q}, c_k)$ for some $P \in \mathcal{P}_1$. Hence, the conditions are also jointly necessary, which completes the proof. ■