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Conciliating Absolute and Relative Poverty: Income Poverty Measurement with Two Poverty Lines

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Conciliating absolute and relative poverty: Income poverty measurement with two poverty lines.*

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Abstract

I study income poverty indices in a framework considering two poverty lines: one absolute line capturing subsistence and one relative line capturing social exclusion. I show that poverty indices accounting for these two lines should be *hierarchical* additive. Hierarchical indices grant a form of priority to subsistence: they always implicitly consider that an absolutely poor individual is more poor than an individual who is only relatively poor, regardless of the income standard in their respective societies. Importantly, classical additive indices are not hierarchical. As a result, they yield debatable poverty comparisons of societies having different income standards. I derive a new (hierarchical) index that generalizes the ubiquitous head-count ratio. This *extended head-count ratio* is equal to the fraction of absolutely poor individuals plus the fraction of individuals who are only relatively poor multiplied by an endogenous weight.

JEL: D63, I32.
Keywords: Income Poverty Measurement, Poverty Line, Relative Poverty, Absolute poverty, Extended Head-Count.

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1 Introduction

There are two different approaches to income poverty measurement: the absolute and the relative approach. An individual is deemed absolutely poor if her income is insufficient to cover her subsistence needs, e.g. being sufficiently nourished or wearing clothes. In a first approximation, the real cost of subsistence does not depend on standards of living. Therefore, the absolute poverty threshold does not dependent on the income standard of the considered society. This is for instance the approach underlying the extreme poverty line of the World Bank, whose threshold is $1.9 per person per day (Ferreira et al., 2016). In turn, an individual is deemed relatively poor if her income is so much smaller than the income standard in her society that she is at risk of social exclusion.[1] The real cost of social participation evolves with standards of living. Therefore, the relative poverty threshold depends on the income standard of the considered society. For instance, most OECD countries use a relative threshold that corresponds to a given fraction of mean or median income.

Unsurprisingly, the main critic raised against any of these two approaches is to ignore either subsistence or social exclusion. On the one hand, any absolute poverty threshold becomes less and less relevant for the identification of the socially excluded as the income standard grows. On the other hand, relative poverty measures often ignore the increase in individual resources that results from growth. Importantly, if a country's growth is such that the income of its poorest citizens becomes sufficient to cover their subsistence needs, its poverty has arguably been reduced, even if these individuals are still socially excluded. Relative measures do not acknowledge such poverty reduction.

There is a need for income poverty measures combining both absolute and relative poverty. Many policymakers, such as the World Bank and the European Commission, aim at reducing both absolute and relative poverty (World Bank, 2015; European Commission, 2015). For such policymakers, using two separate poverty measures, one absolute and one relative, is not a solution. The reason is that two measures would often yield opposing poverty evaluations, in which case no conclusion can be drawn. Such opposing evaluations typically happen when the income of poor individuals grow, but not as fast as their society's income standard.

The research efforts aimed at combining absolute and relative poverty mostly focus on the design of new poverty lines (Foster, 1998; Ravallion and Chen, 2011; Jolliffe and Prydz, 2017). This strand of research has proposed poverty lines whose threshold does depend on the income standard, but is less sensitive to income standard than the threshold of a relative line. These new lines have the potential to better identify the poor, but they cannot resolve on their own all the limitations associated with poverty measures of either approaches. Typically, new lines cannot resolve the important limitation that poverty measures should decrease when poor individuals become able to cover their subsistence needs. This limitation is not so much a problem of whom should be identified as poor, but rather a problem of how poor individuals are compared across societies with different income standards. These inter-personal comparisons primarily rely on the index with which a poverty measure is constructed. Indeed, a poverty measure is defined with two components: a poverty line and a poverty index (Sen, 1976). Poverty indices, like the head-count ratio or the poverty-gap ratio, aggregate the contributions to poverty of all poor individuals in a distribution. The properties of poverty indices have been extensively studied under the assumption that the poverty line is *absolute* (Zheng, 1997). Surprisingly, the properties of poverty indices combined with non-absolute poverty lines have never been rigorously studied. Unfortunately, when combined with a poverty line whose threshold depends on the income standard, these indices provide highly counterintuitive poverty comparisons (Decerf, 2017).

I illustrate these counterintuitive comparisons when comparing poverty using the head-count ratio combined with a relative poverty line. According to this measure, Brazil has in 2015 a larger poverty (42%) than Ivory Coast (40%). Even if income is more unequally distributed in Brazil than in Ivory Coast, this judgment is arguably debatable given that only 4% of individuals are extremely poor in Brazil, while 28% of individuals are extremely poor in Ivory Coast. This debatable comparison is due to the

[1] See Ravallion (2008) for a review of the normative foundations of the relative approach to poverty.

head-count ratio. When using this index, an extremely poor individual earning less than \$1.9 a day in Ivory Coast contributes the same to poverty as a Brazilian whose income is just below the relative poverty threshold in Brazil, i.e. \$10.2 a day.

In this paper, I study poverty indices based on two poverty lines: one absolute line and one relative line. The first main contribution is an axiomatic result showing that indices based on two lines should be *hierarchical additive*. Additive indices sum the contributions to poverty of all poor individuals in a distribution. Hierarchical indices are such that the contribution of an absolutely poor individual is always larger than the contribution of an individual who is only relatively poor, regardless of the income standard in their respective societies. As a result, hierarchical additive indices avoid the counterintuitive comparisons described above. Importantly, mainstream poverty indices are not hierarchical. The second main contribution is the derivation of a hierarchical additive index, which is termed the *extended head-count ratio*. This new index is a continuous generalization of the head-count ratio. This index is equal to the fraction of absolutely poor individuals plus the fraction of individuals who are only relatively poor multiplied by an endogenous weight. An empirical illustration shows how this index affects cross-country poverty comparisons and demonstrates that it yields intuitive judgments on unequal growth.

First, I show that indices based on two lines should be hierarchical additive. The difference between poverty indices and other kinds of normative indices is that the former satisfy a focus property, which forces them to disregard the exact income of non-poor individuals. I consider two poverty lines, which define two different poverty status. An individual is absolutely poor if her income is below the absolute threshold and relatively poor if her income is below the relative threshold. In the presence of two poverty lines, two focus properties should be satisfied. Each focus property is specific to the need captured by its associated poverty line. *Absolute Focus* requires that, when all poor individuals are absolutely poor, the income of non-poor individuals is irrelevant. *Relative Focus* requires that, when all poor individuals are relatively poor, the income of non-poor individuals is irrelevant only as long as the income standard does not change. These two focus properties, together with other classical properties, characterize the family of hierarchical additive indices. This result shows that the contribution of a poor individual may depend on both her income and the income standard. However, the contribution of an *absolutely* poor individual only depends on her income. As the contribution function implicitly ranks individual situations across societies with different income standards, this property has a key implication. If the contribution of absolutely poor individuals does not depend on the income standard, then absolutely poor individuals are always considered more poor than individuals who ore only relatively poor.

Second, I study a particular family of hierarchical additive indices whose mathematical expression features two normative parameters. This family is a hierarchical version of the popular Foster-Greer-Thorbecke (FGT) family of indices (Foster et al., 1984). Only one member of this family, the extended head-count ratio, satisfies classical robustness and monotonicity properties. This index stands out by its simple decomposition. This index is equal to the fraction of absolutely poor individuals plus the fraction of individuals who are only relatively poor multiplied by an endogenous weight. This weight evolves linearly between zero and one as a function of the average income among individuals who are only relatively poor. The closer this average income is to the relative (absolute) threshold, the closer the weight is to zero (one).

Finally, I conduct an empirical illustration of the extended head-count ratio using World Bank data. I contrast its poverty comparisons with those obtained from mainstream poverty measures. I focus on the comparison of societies that have different income standards. When using the hierarchical measure, developed countries are the least poor, sub-saharan countries are the most poor and latin-american countries are poorer than conventionally reported. The evolution of the hierarchical measure in countries whose income standard increases over time can go in either direction. The growth process taking place in urban China over the period 1993-2008 is a particularly telling example of poverty reduction. This process almost entirely eradicates the initial absolute poverty, whose fraction decreases from 20.9% to 1.3%. Over the same period, the fraction of individual who are only relatively poor increases from 4.8% to 25%, implying that the total fraction of poor slightly increases from 25.7% to 26.4%. However, the extended head-count ratio attributes a weight of about 0.5 to individuals who are only relatively poor. As a result, the value taken by this measure is halved over the period, decreasing from 23.3% to 11.5%.

The paper is organized as follows. A succinct literature review is provided in Section 2. I present the framework in Section 3. I characterize the family of hierarchical additive indices and expose an impossibility result for these indices in Section 4. I characterize the new index in a hierarchical version of the FGT family in Section 5. The empirical illustration is presented in Section 6. I make some concluding remarks in Section 7.

2 Literature review

The literature on income poverty measurement studies indicators – called poverty measures – that rank income distributions as a function of poverty. Any poverty measure is composed of two elements: a poverty line and an index. In his groundbreaking paper, Sen (1976) proposes a framework allowing to study the properties inherent to these indices. Following Sen, many authors have proposed particular families of indices and characterized their properties. Among other proposals are the indices studied by Foster et al. (1984), Foster and Shorrocks (1991), Kakwani (1980), Chakravarty (1983) or Duclos and Gregoire (2002). The major results derived in this literature are reviewed in Zheng (1997). This paper extends this literature by departing from the assumption that poverty indices are combined with an absolute line.

A small literature launched by Atkinson and Bourguignon (2001) investigates indices that combine the absolute and relative aspects of income poverty. Atkinson and Bourguignon (2001) suggest to use two poverty lines, an absolute line capturing subsistence and a relative line capturing social exclusion. They propose a family of additive indices – which are not hierarchical – but do not study their properties. The same holds for Anderson and Esposito (2013). Finally, Decerf (2017) considers a different framework with one absolute threshold and one "hybrid" line whose threshold is everywhere above the absolute threshold. He starts from the assumption that being absolutely poor is worse than being relatively poor and proposes a particular hierarchical index. There are three key differences between Decerf (2017) and this paper. First, the priority given to absolutely poor individuals is not assumed but derived from fundamental properties. Second, the two lines studied in this paper are different because they cross each other, i.e. they have the same threshold for a given income standard. This definition of the two lines is more in line with the literature and its implementation requires making fewer normative assumptions. Decerf (2017) shows that, when the hybrid line crosses the absolute line, his index entirely disregards the relative aspect of income poverty. Third, the absolute component of the extended head-count ratio corresponds to the fraction of absolutely poor individuals. As a result, this index can easily complement any absolute head-count based measure, such as the extreme poverty measure of the World Bank or the US official poverty measure.

The design of appropriate absolute, relative and hybrid poverty lines is still an active area of research (Foster, 1998; Ravallion and Chen, 2011, 2017; Allen, 2017). This paper does not contribute to such design. Rather, my starting point is to consider two poverty lines that cross each other, a premise in agreement with this strand of literature.

3 The framework

Let an income distribution $y = (y_1, \dots, y_{n(y)})$ be a list of non-negative incomes. The number of individuals in distribution y is denoted by $n(y)$.

Two poverty lines are considered. Each of these two lines defines a different poverty status. First, there is an absolute line whose threshold defines the minimal income necessary to cover an individual's subsistence needs. Its absolute threshold is denoted by $z_a \geq 0$. The set of individuals who qualify as absolutely poor in distribution y is $Q_a(y) = \{i \leq n(y) \mid y_i < z_a \text{ or } y_i = 0\}$.[2]

[2] For the particular case $z_a = 0$, this definition implies that individual i is absolutely poor if $y_i = 0$.

Second, there is a relative line whose threshold defines the minimal income necessary to be able to participate in social life. The relative poverty threshold is a function of the income standard $\overline{y} = f(y)$. In line with applications, the income standard is either mean income or median income. In the former case $f(y) = \frac{1}{n(y)} \sum y_i$ and in the latter case $f(y) = y_m$, where m denotes the index attached to the median earner in distribution y.[3] The relative line is defined by the threshold function

$$z_r(y) = b + s\overline{y},$$

where $s \in (0, 1)$ defines the slope of the relative line and $b \in [0, z_a(1 - s)]$ defines the lower bound to social participation costs. The line z_r is strongly relative when $b = 0$ and weakly relative when $b > 0$ (Ravallion and Chen, 2011). As I impose that $b \leq z_a(1-s)$, the relative threshold is weakly smaller than the absolute threshold when the income standard takes value z_a. Importantly, this restriction implies that the two poverty lines *cross* (i.e. have equal thresholds) at a level of income standard $\overline{y}^c \geq z_a$ defined as

$$\overline{y}^c = \frac{1}{s}(z_a - b).$$

This crossing property is necessary for the impossibility result established in Theorem 2. Let $\mathcal{Z}$ denote the set of acceptable threshold functions. The set of individuals who qualify as relatively poor in y is $Q_r(y) = \{i \leq n(y) \mid y_i < z_r(y)\}$. Figure 1 illustrates several pairs of poverty lines.

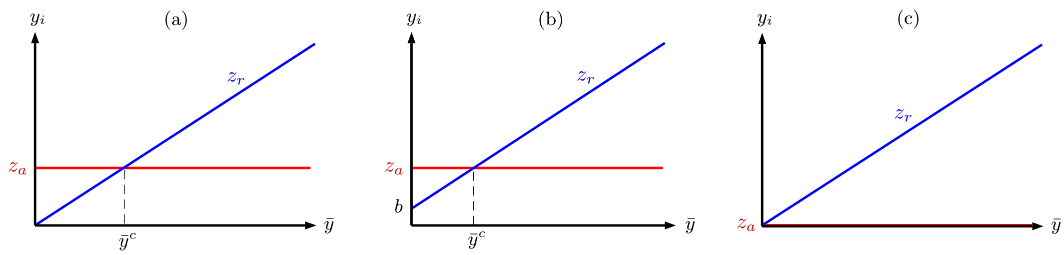


Figure 1: Several pairs of absolute and relative poverty lines.
(a) Positive absolute threshold and strongly relative line. (b) Positive absolute threshold and weakly relative line. (c) Null absolute threshold and strongly relative line.

Together, an individual is poor if her income is below the upper contour of the two lines, which is

$$z(y) = \max\{z_a, z_r(y)\}.$$

The set of individuals who qualify as poor in y is $Q(y) = Q_a(y) \cup Q_r(y)$. The number of poor individuals and the number of absolutely poor individuals are respectively denoted by $q(y)$ and $q_a(y)$. The sets of absolutely poor and relatively poor individuals need not be disjoint. The set of individuals who qualify as only relatively poor in y is $Q(y) \backslash Q_a(y)$. The number of only relatively poor in y is $q(y) - q_a(y)$.

Letting $N = \{n \in \mathbb{N} | n \geq 4\}$, the set of income distributions considered is[4]

$$Y = \left\{ y \in \cup_{n \in N} \mathbb{R}^n_+ \mid \overline{y} \geq z_a \text{ and } \overline{y} > 0 \right\}.$$

This set excludes distributions whose income standard is smaller than the absolute threshold. This restriction is necessary for Theorem 1 to hold when the income standard is median income. Also, this restriction implies that the income of poor individuals is smaller than the income standard, i.e. $y_i < \overline{y}$ for all $i \in Q(y)$.

This convention allows Theorem 1 to cover the particular case $z_a = 0$.

[3] If y is sorted in non-decreasing order $y_1 \leq \cdots \leq y_n$, then $m = \frac{1}{2}(n+1)$ if n is odd and $m = \frac{1}{2}n$ if n is even. This definition of the median when n is even is without loss of generality as the proofs can be easily adapted if m is instead defined as $m = \frac{1}{2}n + 1$.

[4] The requirement $\overline{y} \geq z_a$ does not exclude $\overline{y} = 0$ when $z_a = 0$.

A poverty index is a real-valued function $P : \mathcal{P} \to \mathbb{R}_+$ that ranks income distributions using the two poverty lines as parameters. In general, a poverty index has a domain of definition $\mathcal{P} = Y \times \mathbb{R}_+ \times \mathcal{Z}$. However, almost all the results can be derived when assuming that the two poverty lines are given. As it makes the results more general,[5] I adopt the following much narrower domain of definition

$$\mathcal{P} = Y \times \{(z_a, s, b, f)\}.$$

The poverty in distribution y is simply denoted by $P(y)$. For any two distributions y and y', there is strictly more poverty in y than in y' if $P(y) > P(y')$, and weakly more if $P(y) \geq P(y')$.

4 Hierarchical additive poverty indices

I study which indices should be used to compare poverty in different income distributions. The particularity of the framework is the presence of a second poverty line, whose threshold depends on the income standard. The properties I impose on indices acknowledge this presence in two ways. First, the relevance of each of the two poverty lines is established in a separate focus axiom. Second, several properties are restricted to the comparison of income distributions that have equal income standards.

The particularity of poverty indices is that only the situation of poor individuals matters to poverty comparisons. This property that distinguishes poverty indices from other kinds of normative indices, e.g. inequality or mobility indices, is traditionally encapsulated in a focus axiom. This axiom requires that the exact income of individuals earning more than the poverty threshold is to some extent irrelevant. The extent to which their income is irrelevant depends on the kind of poverty considered. As I consider two kinds of poverty, I impose two separate focus axioms. Each focus axiom is specific to the particular need captured by its associated line.

An individual is absolutely poor if her income is insufficient to meet her subsistence needs. Traditionally, the minimal income necessary to cover subsistence needs is assumed not to depend on the income of non-poor individuals. *Absolute Focus* requires that, when all poor individuals are absolutely poor, the exact income earned by non-poor individuals is irrelevant.

Poverty axiom 1 (*Absolute Focus*).
For all $y, y' \in Y$ with $n(y) = n(y')$ and $Q_a(y) = Q(y) = Q_a(y') = Q(y')$, if $y_i = y'_i$ for all $i \in Q_a(y)$, then $P(y) = P(y')$.

In contrast, the focus axiom associated to the relative line does not completely disregard the income of non-poor individuals. The income necessary for an individual to meet her social participation needs depends on the income standard. In turn, the income standard depends on the income of non-poor individuals. *Relative Focus* requires that, when all poor individuals are relatively poor, the exact income earned by non-poor individuals is irrelevant only as long as the income standard is unchanged.

Poverty axiom 2 (*Relative Focus*).
For all $y, y' \in Y$ with $n(y) = n(y')$, $Q_r(y) = Q(y) = Q_r(y') = Q(y')$ and $\overline{y} = \overline{y}'$, if $y_i = y'_i$ for all $i \in Q_r(y)$, then $P(y) = P(y')$.

The classical monotonicity property requires that poverty is reduced when a poor individual earns an additional amount of income. *Weak Monotonicity* adds the precondition that the other poor individuals are not affected. This precondition is guaranteed by restricting monotonicity comparisons to distributions that have the same income standard.

Poverty axiom 3 (*Weak Monotonicity*).
For all $y, y' \in Y$ with $n(y) = n(y')$, $Q(y') \subseteq Q(y)$ and $\overline{y} = \overline{y}'$, if $y_j < y'_j$ for some $j \in Q(y)$ and $y_i = y'_i$ for all $i \in Q(y') \backslash \{j\}$, then $P(y) > P(y')$.

[5] The results obtained on the narrower domain also constrain indices defined on the larger domain.

Subgroup Consistency is a standard axiom requiring that, if poverty decreases in a subgroup while it remains constant in the rest of the distribution, overall poverty must decline. Sen (1992) questioned the desirability of this axiom by arguing that the incomes in one subgroup may affect poverty in another subgroup. Foster and Sen (1997) recommend not to use this axiom when the index aims at capturing relative aspects of poverty. I subscribe to this point of view. The issue becomes transparent once the channel through which one subgroup affects the other is modeled. In this framework, the incomes in a subgroup impact the income standard, which in turn affects poor individuals in another subgroup. When a poverty line is relative, it is thus not always meaningful to extrapolate the judgments made on a subgroup to the whole population. *Weak Subgroup Consistency* restricts such extrapolations to distributions for which the subgroups have the same income standard. In such cases, the income standard of a subgroup is equal to the income standard of the entire distribution and poverty judgments made on the subgroup are meaningful for the entire distribution.

Poverty axiom 4 (*Weak Subgroup Consistency*)**.**
For all $y^1, y^2, y^3, y^4 \in Y$ with $n(y^1) = n(y^3)$, $n(y^2) = n(y^4)$ and $\overline{y}^1 = \overline{y}^2 = \overline{y}^3 = \overline{y}^4$, if $P(y^1) > P(y^3)$ and $P(y^2) = P(y^4)$, then $P((y^1, y^2)) > P((y^3, y^4))$.

The remaining three auxiliary axioms are standard. *Symmetry* requires that individuals' identities do not matter. Working with sorted distributions is therefore without loss of generality.

Poverty axiom 5 (*Symmetry*)**.**
For all $y, y' \in Y$ with $n(y) = n(y')$, if $y' = y \cdot \pi_{n(y)\times n(y)}$ for some permutation matrix $\pi_{n(y)\times n(y)}$, then $P(y) = P(y')$.

Symmetry implies that individual preferences are irrelevant to the poverty index. This property generates little debate when only the level of own income appears in preferences. If preferences are monotonic, then monotonic indices do not override individual preferences. When both the level of own income and the relative situation matter, the monotonicity of preferences does not entirely define individual preferences and *Symmetry* explicitly requires to completely disregard these preferences. This form of paternalism can be defended on the ground that it prevents poverty indices from giving priority to individuals that are more other-regarding.[6]

Next axiom requires indices to be continuous in incomes. Such continuity requirement is important in empirical applications in order to avoid that measurement errors have an excessive impact on poverty judgments. *Weak Continuity* requires indices to be continuous in all incomes, but only for distributions whose income standard is larger than the income standard at which the two lines cross.[7]

Poverty axiom 6 (*Weak Continuity*)**.**
For all $y \in Y$ with $\overline{y} > \overline{y}^c$, P is continuous in y.

Finally, *Replication Invariance* specifies how to compare poverty across distributions of different population sizes. If a distribution is obtained by replicating another distribution several times, then the two distributions have equal poverty. Formally, for any $k \in \mathbb{N}$, the $k-$replication of a distribution y is the distribution $y^{\times k} = (y, \ldots, y)$ for which $n(y^{\times k}) = kn(y)$.

Poverty axiom 7 (*Replication Invariance*)**.**
For all $y \in Y$ and $k \in \mathbb{N}$, we have $P(y) = P(y^{\times k})$.

[6] For an illustration of the issue, consider two poor individuals living in the same society. Assume that individual 1 has a smaller income than individual 2 but individual 2 has preferences that are more affected by relative income than the preferences of individual 1. If individual preferences matter for the poverty index, it could be that the contribution to poverty of individual 2 is larger than that of individual 1. Potentially, transferring an amount of income from individual 1 to individual 2 decreases poverty. Such conclusion is debatable given that both individuals would agree that individual 2 is better-off than individual 1.

[7] The continuity requirement is restricted to distributions with $\overline{y} > \overline{y}^c$ in order to get around an impossibility result: indices that are continuous on the whole domain Y cannot simultaneously satisfy *Absolute Focus* and *Weak Monotonicity* when $z_a > 0$. The proof of this impossibility is omitted.

Theorem 1 identifies the family of poverty indices characterized by these axioms.

Theorem 1 (Characterization of hierarchical additive poverty indices)**.**
The following two statements are equivalent.

1. *P satisfies Absolute Focus, Relative Focus, Weak Monotonicity, Weak Subgroup Consistency, Symmetry, Weak Continuity and Replication Invariance.*

2. *P is ordinally equivalent to*

$$P'(y) = \frac{1}{n(y)} \sum_{i=1}^{n(y)} p(y_i, \overline{y}), \tag{1}$$

where the poverty contribution function $p : \mathbb{R}_+ \times [z^a, \infty) \to [0, 1]$ is such that (i) $p(0, \overline{y}) = 1$ and $p(y_i, \overline{y}) = 0$ if $i \notin Q(y)$, (ii) p is strictly decreasing in its first argument if $i \in Q(y)$, (iii) p is constant in its second argument if $i \in Q_a(y)$ and (iv) p is continuous in both its arguments if $\overline{y} > \overline{y}^c$.

Proof. It is easy to check that statement 2 implies statement 1. The proof of the converse implication is in Appendix 8.1. ∎

Theorem 1 shows that indices satisfying these axioms sum the poverty contributions of all individuals in a distribution. As usual, non-poor individuals contribute zero. More importantly, the contribution of any poor individual depends on both her income and on the income standard. These two variables summarize the relevant aspects of a poor individual's situation. In this sense, the pair $(y_i, \overline{y})$ defines the "bundle" consumed by the poor individual i. The contribution function ranks all the bundles that poor individuals may consume.[8] Therefore, the contribution function implicitly compares poor individuals across societies with different levels of income standards.

The key message of Theorem 1 is that poor individuals living in different societies must be compared in a specific way. The constraints imposed on these comparisons are revealed by a graphical representation of the *iso-poverty map* defined by the index. An iso-poverty map is a collection of iso-poverty curves. An iso-poverty curve is the set of bundles associated to the same poverty contribution. Implicitly, two individuals whose bundles are on the same iso-poverty curve are deemed equally poor by the contribution function. Figure 2 shows iso-poverty maps satisfying the constraints imposed by restriction (iii). This restriction requires that the contribution of absolutely poor individuals only depends on their own income. As a result, their iso-poverty curves are flat and do not cross the absolute threshold. This implies that the bundle of an absolutely poor individual is always on a lower iso-poverty curve than the bundle of an individual who is only relatively poor. In other words, an individual who is absolutely poor must be deemed more poor than an individual who is only relatively poor, regardless of the income standards in their respective societies. An absolutely poor individual in Ivory Coast must be deemed more poor than an individual who is only relatively poor in Brazil, even if the income standard in Brazil is larger. I call these indices "*hierarchical*" because they grant a particular form of priority to absolute poverty over relative poverty. Below the absolute threshold, the relative aspect of income poverty is irrelevant. It is only when an individual is not absolutely poor that the relative aspect of her poverty becomes relevant.

The normative view according to which being absolutely poor is worse than being only relatively poor is largely shared. It has been expressed in the literature (Atkinson and Bourguignon, 2001; Decerf, 2017) and is largely shared in the population, as appeared from questionnaire studies run all over the world by Corazzini et al. (2011). Theorem 1 provides a normative foundation for this view.

Importantly, mainstream poverty indices are not hierarchical. The classical FGT indices, which are pervasive in empirical applications, are not hierarchical. I emphasize that it is not a mere theoretical issue. When they are combined with relative poverty lines, non-hierarchical indices regularly lead to highly counter-intuitive poverty comparisons (Decerf, 2017). The empirical illustration presented in Section 6 provides additional

[8] Formally, the contribution function defines a complete *ordering* on the space of bundles.

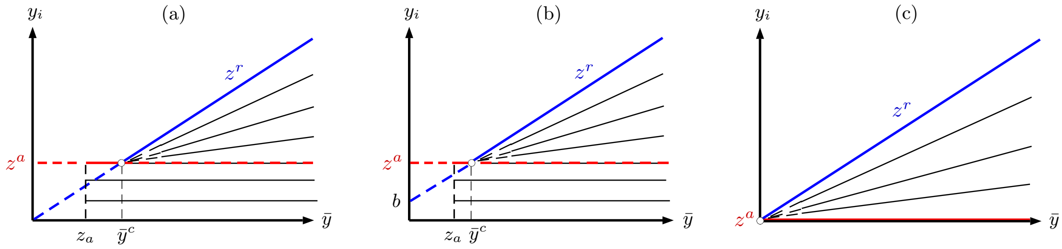


Figure 2: Iso-poverty maps of hierarchical additive indices.
Note: (a) Positive absolute threshold and strongly relative line. (b) Positive absolute threshold and weakly relative line. (c) Null absolute threshold and strongly relative line.

examples when comparing poverty using the head-count ratio combined with a relative poverty line. According to this measure reported in Table 1, Brazil has in 2015 a larger poverty (42%) than Ivory Coast (40%). Even if income is more unequally distributed in Brazil than in Ivory Coast, this judgment is arguably debatable given that only 4% of individuals are extremely poor in Brazil, while 28% of individuals are extremely poor in Ivory Coast. This debatable comparison is due to the head-count ratio, which is non-hierarchical. When using this index, an extremely poor individual earning less than \$1.9 a day in Ivory Coast contributes the same to poverty as a Brazilian whose income is just below the relative poverty threshold in Brazil, i.e. \$10.2 a day. The respective bundles of these two individuals are on the same "thick" iso-poverty curve.

Two remarks are in order. First, Theorem 1 can be extended to relative lines whose threshold function is not linear. This theorem applies as long as the relative line is continuous and monotonically increasing with the income standard and its threshold is always smaller than the income standard. Second, Theorem 1 still holds in the special case $z_a = 0$, i.e. when the index only considers one (relative) poverty line. This theorem thus provides a normative foundation for additive indices used in combination with a relative line. Despites the critics expressed against such (widely used) additive measures (Sen, 1992; Foster and Sen, 1997), indices based on a relative line have never been rigorously studied.

An impossibility result

Theorem 1 places no restriction on the shape that the contribution function takes for a fixed level of income standard. The only requirement is that this function decreases continuously with individual income. Restrictions on its shape emerge from axioms constraining how the index must trade-off the incomes of different poor individuals. I consider two such axioms and show that, unfortunately, hierarchical indices violate at least one of them. This impossibility result requires a strictly positive absolute threshold ($z_a > 0$), which is henceforth assumed.

The first property, *Transfer*, is a classical axiom requiring that a Pigou-Dalton transfer taking place between two poor individuals never unambiguously increases poverty.

Poverty axiom 8 (*Transfer*).
For all $y, y' \in Y$ with $n(y) = n(y')$, $Q(y) = Q(y')$ and $\delta > 0$, if $y_j - \delta = y'_j > y'_k = y_k + \delta$ for some $j, k \in Q(y)$, $y'_i = y_i$ for all $i \neq j, k$ and $\overline{y}' = \overline{y}$, then $P(y) \geq P(y')$.

As is well-known, poverty indices satisfying *Transfer* are based on convex contribution functions.

The second property, *Strong Monotonicity*, considers an increase in the income of some poor individual. When the income standard is mean income, an increase in the income of a poor individual has opposing effects. On the one hand, her poverty contribution decreases. This direct effect reduces poverty. On the other hand, mean income increases.[9] If the relative threshold increases, then the poverty contributions of only

[9] Observe that the larger the number of individuals, the lower is the impact of the increase in income on mean income and, hence, on the poverty contributions of others.

relatively poor individuals increase.[10] Moreover, some individuals who were non-poor might become relatively poor. *Strong Monotonicity* requires that these indirect adverse effects are dominated by the direct effect. Hence, this property imposes that a decrease in the income of some *poor* individual never leads to a reduction of poverty. Indices satisfying this property never deem that a policy whose unique impact is to decrease the income of some poor individuals reduces poverty.

Poverty axiom 9 (*Strong Monotonicity*).
For all $y, y' \in Y$ with $n(y) = n(y')$, if $y_j < y'_j$ for some $j \in Q(y) \cup Q(y')$ and $y_i = y'_i$ for all $i \neq j$, then $P(y) > P(y')$.

Observe that, when the income standard is median income, last axiom is equivalent to *Weak Monotonicity*. The reason is that the income of poor individuals does not impact median income in the domain of distributions considered.

Theorem 2 shows that all hierarchical additive indices violate *Transfer*. Moreover, if the income standard is mean income, then these indices also violate *Weak Monotonicity*.

Theorem 2 (Hierarchical additive indices violate basic properties).
Let P be a hierarchical additive index.

1. *P violates Transfer.*
2. *If the income standard is mean income, then P violates Strong Monotonicity.*

Proof. See Appendix 8.2. ■

Theorem 2 shows that all hierarchical additive indices fail some basic property. In other words, any index based on two poverty lines either fails some basic property or implicitly considers that some absolutely poor individuals are less poor than some individuals who are only relatively poor. There are three possible ways to deal with this normative trade-off.

The first way is to escape the impossibility by changing the definition of the two poverty lines. The impossibility holds because the relative line crosses the absolute line. Decerf (2017) considers a different framework with two lines, one absolute line and one "hybrid" line. His two lines do not cross because his hybrid threshold is always above the absolute threshold. In this alternative framework, the index proposed by Decerf (2017) satisfies both *Transfer* and *Strong Monotonicity*, even when the income standard is mean income. The problem with this stance is that it is at odds with the literature considering two poverty lines. For instance, the societal poverty line recently adopted by the World Bank does cross its extreme poverty threshold (World Bank, 2018).

The second way is to accept that the index implicitly considers that some absolutely poor individuals are less poor than some individuals who are only relatively poor. There exist non-hierarchical indices that satisfy both *Transfer* and *Strong Monotonicity*, even when the income standard is mean income. This is for instance the case of the poverty gap ratio. One problem with this stance is that it leads to the counter-intuitive judgments discussed above. Also, I argue that the minimal form of priority granted to absolute poverty by hierarchical indices is more fundamental than these aggregation axioms. The reason is that the former directly relates to the comparison of poor individuals. Indeed, constraints on the shape of the iso-poverty map define which of any two poor individuals is more poor.[11] In contrast, aggregation axioms relate to the comparisons of gains and losses made by different poor individuals. Therefore, these axioms "merely" relate to the fairness of the index.

[10] The poverty contribution of an absolutely poor individual only depends on her level of income and is therefore not affected by the income of other poor individuals.

[11] The shape of the iso-poverty map may relate to welfare-consistency. In order to get an intuition, assume that the priority granted to absolute poverty corresponds to individual preferences. Under this assumption, individuals prefer to have the possibility to satisfy their subsistence needs, even if it comes at the cost of a worse relative situation. Then, the problem with non-hierarchical indices is that they sometimes conclude that poverty is reduced even when all poor individuals deem that their situation has worsened. For instance, this could happen after an inequality-reducing recession. Such recession may decrease the income of poor individuals below the absolute threshold while improving their relative situation.

The third way out of this impossibility is to use a hierarchical index. However, the poverty measurement literature does not propose any index that satisfies equation (1).[12] The objective of next section is to propose such an index. This index satisfies weakened versions of *Transfer* and *Strong Monotonicity*.

5 The extended head-count ratio

In this section, I consider a parametric family of hierarchical additive indices and show that the value of its parameters can be determined from basic properties. The index characterized turns out to be a continuous extension of the head-count ratio, which makes it particularly interesting to practitioners.

The poverty contribution function considered is inspired from the FGT subfamily of additive indices (Foster et al., 1984). FGT indices have an exponential expression whose properties have been studied by Foster and Shorrocks (1991) as well as Ebert and Moyes (2002). The poverty contribution of individual i in distribution y is given by

$$\hat{p}(y_i, \overline{y}) = (1 - u_\lambda(y_i, \overline{y}))^\alpha , \tag{2}$$

where $\alpha \geq 0$ is the poverty aversion parameter and u_λ is a function that ranks individual bundles. For classical FGT indices, this function is defined by individual i's normalized income, i.e. $u_\lambda(y_i, \overline{y}) = \frac{y_i}{z(y)}$. With this definition, classical FGT indices are not hierarchical. Indeed, the iso-poverty curves associated to bundles below the absolute threshold eventually cross the absolute threshold when the income standard grows sufficiently. In contrast, these indices are hierarchical if function u_λ is defined in two parts as

$$u_\lambda(y_i, \overline{y}) = \begin{cases} \lambda \frac{y_i}{z_a} & \text{if } \; y_i < z_a, \\ \\ \lambda + (1-\lambda) g(y_i, \overline{y}) & \text{if } \; z_a \leq y_i < z_r(y). \end{cases} \tag{3}$$

where $\lambda \in [0, 1]$ is a parameter tuning which fraction of function u_λ's domain of images is attributed to absolute poverty and g is a function ranking the subset of bundles whose income is larger than the absolute threshold. I assume that function g is linear in its first argument. By the properties of the contribution function, the linearity of function g implies that

$$g(y_i, \overline{y}) = \frac{y_i - z_a}{z_r(y) - z_a}. \tag{4}$$

Function g measures the "relative gap" between the two poverty thresholds. The expression for g defines the shape of iso-poverty curves above the absolute threshold. The linear expression for g implies that the iso-poverty curves between the absolute threshold and the relative line are linear and homothetic, as illustrated in Figure 2. This assumption is strong but is also a natural default option.

Together, equations (2) to (4) define a hierarchical version of the FGT family of indices. Any index in this FGT family, which is denoted by $\hat{P}_{\alpha\lambda}$, is defined by a pair of values for the two parameters α and λ.

Observe that index $\hat{P}_{\alpha\lambda}$ violates *Weak Monotonicity* for some values of the parameters. This if for instance the case of the head-count ratio ($\alpha = 0$), the index most used in practice. This is usually not considered as a fundamental issue because this index does satisfy a weaker version of *Weak Monotonicity* for which the inequality sign of the implication is weak. However, this problem becomes more acute when considering two different poverty status, especially when the absolute status is deemed more severe than the relative status. Indeed, poverty indices should acknowledge when an individual changes poverty status, i.e. when her income crosses the absolute poverty line. In the case of hierarchical indices, the poverty contribution of absolutely poor individuals should be strictly larger than that of individuals who are only relatively poor. *Minimal Monotonicity* encapsulates this very weak monotonicity property.

[12] The hierarchical index characterized by (Decerf, 2017) attributes a zero contribution to relatively poor individuals when the two poverty lines cross.

Poverty axiom 10 (*Minimal Monotonicity*).
For all $y, y' \in Y$ with $n(y) = n(y')$, $Q(y') = Q(y)$ and $\overline{y} = \overline{y}'$, if $y_j < z_a < y'_j$ for some $j \in Q(y)$ and $y_i = y'_i$ for all $i \in Q(y') \setminus \{j\}$, then $P(y) > P(y')$.

As *Minimal Monotonicity* is a weakening of *Weak Monotonicity*, all indices satisfying equation (1) satisfy this axiom. In contrast, some $\hat{P}_{\alpha\lambda}$ indices violate *Minimal Monotonicity*. Lemma 1 identifies these unacceptable indices. Its easy proof is omitted.

Lemma 1.
$\hat{P}_{\alpha\lambda}$ satisfies Minimal Monotonicity if and only if $\alpha \neq 0$.

Further requirements allow discriminating among the remaining values for the parameters of index $\hat{P}_{\alpha\lambda}$.

In practice, computing the minimal income level z_a covering subsistence needs is a complex task. Even the procedure to compute z_a is still the topic of ongoing research (Allen, 2017). As a result, the literature on poverty indices has paid particular attention to indices that are robust to common transformations of distributions and the absolute threshold (Foster and Shorrocks, 1991; Zheng, 1994). Transformations of two kinds have been emphasized: scalings and translations. An index satisfies *Scale Invariance* if it is unaffected when all incomes are multiplied by the same factor as the absolute threshold. Then, only the normalized distance to the poverty threshold matters for the contribution of absolutely poor individuals. In turn, an index satisfies *Translation Invariance* if it is unaffected when all incomes are translated by the same amount as the absolute threshold. As a result, only the absolute distance to the poverty threshold matters for the contribution of absolutely poor individuals. An index is called **robust** if it is both scale invariant and translation invariant.[13]

Definition 1 (Robustness to the absolute threshold).
*Index P is **robust** to $z_a > 0$ if for all $y \in Y$ with $Q_a(y) = Q(y)$, all $\gamma > 1$ and all $\delta > 0$ we have*[14]

$$\begin{aligned} P(y, z_a, z_r(y)) &= P(\gamma y, \gamma z_a, z_r(\gamma y)) && \textit{(Scale Invariance)},\\ P(y, z_a, z_r(y)) &= P(y + \delta \cdot \mathbb{1}_n, z_a + \delta, z_r(y + \delta \cdot \mathbb{1}_n)) && \textit{(Translation Invariance)}, \end{aligned}$$

where $\mathbb{1}_n = (1, \ldots, 1)$ is a vector of ones of size $n(y)$.

Foster and Shorrocks (1991) and Zheng (1994) have shown that only head-count related poverty indices are robust. Lemma 2 identifies the members of our family that are head-count related.

Lemma 2.
$\hat{P}_{\alpha\lambda}$ is robust if and only if $\lambda = 0$ or $\alpha = 0$.

Proof. See Appendix 8.3. ■

Taken together, Lemma 1 and 2 characterize an interesting subset of indices. When $\lambda = 0$, the contribution of absolutely poor individuals is one. In turn, the contribution of only relatively poor individuals is a function of their income's "relative gap" between the two poverty thresholds. The closer is their income to the absolute (relative) threshold, the closer is their contribution to one (zero). When $\lambda = 0$, equation (2) becomes

$$\hat{p}(y_i, \overline{y}) = \begin{cases} 1 & \text{if } \; y_i < z_a, \\ \left(\frac{z_r(y) - y_i}{z_r(y) - z_a}\right)^{\alpha} & \text{if } \; z_a \leq y_i < z_r(y), \end{cases} \tag{5}$$

[13] These invariance properties apply to poverty indices whose domain of definition is $\mathcal{P} = Y \times \mathbb{R}_{++} \times \{(s, b, f)\}$.

[14] Recall that the set $\mathcal{Z}$ of acceptable relative lines depends on the absolute threshold z_a through its restriction $b \leq z_a(1 - s)$. Yet, the operations associated to the invariance properties never consider relative lines that violate this restriction. Indeed, the translation (resp. scaling) operation considers a new absolute threshold $z_a + \delta > z_a$ (resp. $\gamma z_a > z_a$) that is larger than the initial threshold. Clearly, if z_r is acceptable given z_a, then z_r is also acceptable given another z'_a larger than z_a.

where $\alpha > 0$. Taken together, equations (1) and (5) show that $\hat{P}_{\alpha 0}$ is the sum of the absolute head-count ratio $HC^A(y) = \frac{q_a(y)}{n(y)}$ and an additional term capturing the contributions of individuals who are only relatively poor. In this sense, index $\hat{P}_{\alpha 0}$ "augments" the absolute head-count ratio. As the official poverty measures of many countries and institutions are based on the absolute head-count, index $\hat{P}_{\alpha 0}$ can easily complement these official measures when relative poverty aspects are deemed relevant. Interestingly, this index satisfies *Weak Continuity*, which makes it less sensitive to measurement errors than the classical head-count ratio.

When taking $\alpha = 1$, the index has a particularly simple decomposition. Indeed, the second term of index $\hat{P}_{10}$ is also based on the head-count ratio. This index can be written as the sum of the absolute head-count ratio HC^A and the (only) relative head-count ratio $HC^R = \frac{q(y)-q_a(y)}{n(y)}$ multiplied by an endogenous weight. For this reason, I call $\hat{P}_{10}$ the *extended head-count ratio*

$$\hat{P}_{10}(y) = \frac{q_a(y)}{n(y)} + \omega(y)\frac{q(y) - q_a(y)}{n(y)}, \tag{6}$$

where

$$\omega(y) = \frac{z_r(y) - \overline{y}^r}{z_r(y) - z_a} \qquad \text{and} \qquad \overline{y}^r = \frac{1}{q(y) - q_a(y)} \sum_{i \in Q(y) \setminus Q_a(y)} y_i.$$

The weight function $\omega : Y \to [0, 1]$ takes as argument the average income among the only relatively poor individuals, denoted by $\overline{y}^r$. More precisely, the weight attributed to relatively poor individuals corresponds to the relative gap of their average income between the two poverty thresholds. The closer their average income is to the absolute (relative) threshold, the closer their weight is to one (zero).

Finally, I show that the extended head-count ratio is characterized by a weakening of *Strong Monotonicity* when the income standard is mean income. With this income standard, Theorem 2 shows that hierarchical indices systematically violate *Strong Monotonicity*. The proof reveals that the impossibility arises when the income of an *absolutely* poor individual is increased. *Strong Monotonicity REL* restricts this axiom to situations for which the income of a relatively poor individual is increased.

Poverty axiom 11 (*Strong Monotonicity REL*).
For all $y, y' \in Y$ with $n(y) = n(y')$, if $z_a \leq y_j < y'_j$ for some $j \in Q(y) \cup Q(y')$ and $y_i = y'_i$ for all $i \neq j$, then $P(y) > P(y')$.

Lemma 3 shows that the extended head-count ratio is the only index $\hat{P}_{\alpha 0}$ that satisfies this axiom.

Lemma 3.
If the income standard is mean income, then $\hat{P}_{\alpha 0}$ satisfies Strong Monotonicity REL if and only if $\alpha = 1$.

Proof. See Appendix 8.4. ∎

The extended head-count ratio also satisfies important weakenings of *Transfer*. Consider for instance the restriction that the transfer takes place between two poor individuals who share the same poverty status, i.e. they are either both absolutely poor or both only relatively poor. It is straightforward to show that $\hat{P}_{10}$ satisfies these transfer properties. Therefore, the extended head-count ratio constitutes a plausible compromise to the impossibility derived in Theorem 2. A direct corollary of Lemma 1, 2 and 3 is that the only index $\hat{P}_{\alpha\lambda}$ that satisfies our properties when the income standard is mean income is the extended head-count ratio. Corollary 1 formalizes this claim.

Corollary 1 (Characterization of index $\hat{P}_{10}$).
$\hat{P}_{10}$ is robust and satisfies Minimal Monotonicity and Strong Monotonicity REL. If the income standard is mean income, then $\hat{P}_{\alpha\lambda}$ is robust and satisfies Minimal Monotonicity and Strong Monotonicity REL if and only if $\lambda = 0$ and $\alpha = 1$.

Corollary 1 characterizes the extended head-count ratio, which is a new hierarchical index. The results leading to this characterization are less general than Theorem 1 and 2, but the relevance of this index is maybe more related to its simple decomposition.

In the next section, I apply this index using World Bank data. The empirical results illustrate that, unlike absolute or relative measures, a poverty measure based on this index leads to nuanced comparisons of societies that differ in their income standards. They also illustrate that the index yields intuitive judgments on unequal growth experiences.

6 Empirical illustration

In this section, I apply the extended head-count ratio using World Bank data. I contrast its poverty comparisons with those obtained from mainstream poverty measures. I focus on the comparison of societies that have different income standards.

The data is taken from PovcalNet, a website built by the World Bank with income and consumption data gathered from surveys of randomly sampled households in most low-, middle- and high-income countries between 1981 and 2010.[15] In order to permit cross-country comparisons, the Bank transforms the survey data using the Purchasing Power Parity (PPP) exchange rates for household consumption from the 2011 International Comparison Program. Chen and Ravallion (2013) provide more information on this database.

Constructing a poverty measure based on index $\hat{P}_{10}$ requires to select two poverty lines: the absolute threshold z_a and the relative line z_r. These choices reflect the practitioner's views on absolute and relative poverty. In this application, I fix the absolute threshold at the level of the World Bank's threshold for extreme poverty: \$1.9 a day (Ferreira et al., 2016). For the sake of simplicity, I use a mean-sensitive relative poverty line. I assume that the relative line is $z_r(y) = 0.4 + 0.5\overline{y}$, which is directly inspired from Ravallion and Chen (2017) who calibrate a weakly relative line on national poverty thresholds in developing countries and obtain $z^{RC}(y) = max\{\$1.9, \$0.396 + 0.485\overline{y}\}$ (see their Figure 5 Panel b). Together, an individual is poor if her income is below the upper-contour of these two lines

$$z(y) = max\{\$1.9, \$0.4 + 0.5\overline{y}\}.$$

The two poverty lines z_a and z_r cross at a value of mean income equal to \$3 a day (see Figure 3). The poverty lines selected are certainly subject to debate, but my objective is simply to pick a reasonable pair of lines for illustration purposes.

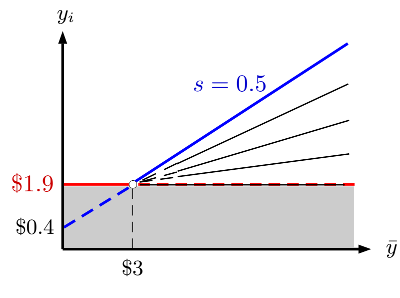


Figure 3: Iso-poverty map associated to $\hat{P}_{10}$.

Note: The absolute threshold is z_a =\$1.9 a day and the relative line is $z_r(y) = 0.4 + 0.5\overline{y}$ where $\overline{y}$ is mean income expressed in \$ a day. All bundles below the absolute threshold belong to a thick iso-poverty curve.

Poverty comparisons based on $\hat{P}_{10}$ are contrasted with those obtained from two other measures. The first measure, denoted by HC^A, is the fraction of individuals whose income is below \$1.9 a day. The second measure, denoted by HC^H, is the fraction of

[15] PovcalNet: the online tool for poverty measurement developed by the Development Research Group of the World Bank. This tool can be found here: www.iresearch.worldbank.org/PovcalNet.

individuals whose income is below the "hybrid" line z, defined as the upper contour of the two lines. From equation (6), we have that

$$\hat{P}_{10}(y) = HC^A(y) + \omega(y)\left(HC^H(y) - HC^A(y)\right),$$

where ω is the endogenous weight given to individuals who are only relatively poor. For countries whose mean income is below \$3 a day, the hybrid threshold is equal to the absolute threshold and the three poverty measures are equal.

Table 1 provides figures for nine countries in recent years. These countries are selected because they differ in their income standard, i.e. in mean income. Also, countries in the middle-income group have been selected for their high income inequality, which implies high relative poverty. Four low-income low- or middle-inequality countries are considered: Madagascar, Ivory Coast, Nepal and Pakistan. The mean incomes in these countries range from \$1.6 to \$4.5 a day, and their Gini coefficients range from 31% to 43%.[16] Two middle-income high-inequality countries are considered: Colombia and Brazil whose mean incomes are equal to \$14.3 and \$17.6 a day respectively and whose Gini coefficients are both equal to 51%. Two developed countries are considered: France and the US whose mean incomes are equal to \$53.3 and \$63.9 a day respectively and whose Gini coefficients are equal to 32% and 41%.

Table 1: Reversals in cross-country poverty comparisons.

Countries	Mean	Gini	HC^A	HC^H	$\hat{P}_{10}$	ω
Madagascar	1.6	43	77.8	77.8	77.8	-
Ivory Coast	3.9	42	27.9	39.9	34.1	0.51
Nepal	4.0	33	15.0	30.7	23.1	0.52
Pakistan	4.5	31	6.1	24.3	14.6	0.47
Colombia	14.3	51	5.5	43.1	24.1	0.50
Brazil	17.6	51	4.3	42.2	22.6	0.48
France	53.3	32	0.0	18.0	4.8	0.27
US	63.9	41	1.0	30.0	11.8	0.37

Note: $\hat{P}_{10}$ seems to balance the size and inequality of income distributions with more nuance than HC^A and HC^H. All poverty measures and the Gini coefficients are expressed in %. Mean incomes are expressed in \$ a day (2011 PPP). Figures are provided for the latest year available for each country: 2012, 2015, 2011, 2014, 2015, 2015, 2014 and 2013 (order of the table). Source: PovcalNet.

In the sample, HC^A is strongly negatively correlated with mean income. Leaving aside Madagascar, for which the hybrid threshold is equal to the absolute threshold, HC^H is strongly positively correlated with inequality as measured by the Gini coefficient. The two middle-income countries have the highest HC^H-poverty because they experience higher inequality. In contrast, the low-income countries have the highest HC^A-poverty. This reversal illustrates that absolute and relative measures typically yield opposing comparisons of distributions with different income standards and levels of inequality. Measure $\hat{P}_{10}$ seems to be more nuanced: the two countries with highest $\hat{P}_{10}$-poverty are low-income countries and $\hat{P}_{10}$ does penalize the middle income countries for their high-inequality. Moreover, the two developed countries have the lowest $\hat{P}_{10}$-poverty in the sample, which is not the case when considering HC^H-poverty.

Pairwise comparisons illustrate the different judgments obtained from $\hat{P}_{10}$, HC^A and HC^H. In the sample, $\hat{P}_{10}$ and HC^A yield opposite rankings between Nepal and Colombia or between Pakistan and Brazil. In both cases, for measure $\hat{P}_{10}$, the lower absolute poverty of the second country is more than compensated by its much higher inequality. In turn, $\hat{P}_{10}$ and HC^H yield opposite rankings between Ivory Coast and Brazil or between Pakistan and the US. In both cases, for measure $\hat{P}_{10}$, the lower absolute poverty of the second country more than compensates its higher inequality. Observe that the US has a smaller $\hat{P}_{10}$-poverty than Pakistan not only because the US

[16] The Gini coefficient is a popular measure of inequality. A larger Gini coefficient implies higher inequality.

has a smaller absolute poverty but also because the weight ω attributed to the relatively poor is smaller in the US than in Pakistan.

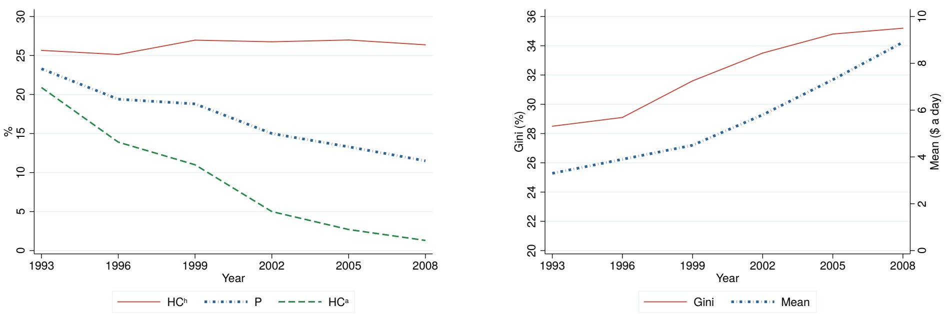
In low- and middle-income countries, this weight is very close to 0.5 because the relative threshold is close to the absolute threshold. As a result, on the narrow income range associated to being only relatively poor, incomes are almost uniformly distributed. This does not hold in many high-income countries where relatively poor individuals earn on average an income that is closer to the relative threshold than to the absolute threshold. This suggests that the simpler index

$$P'(y) = HC^A(y) + 0.5\left(HC^H(y) - HC^A(y)\right),$$

which is based on the exogenously fixed weight $\omega = 0.5$, leads to similar poverty comparisons as those obtained with index $\hat{P}_{10}$ when comparing low- and middle-income countries.

Altogether, Table 1 demonstrates that the $\hat{P}_{10}$-poverty judgments are different from those obtained with classical absolute or relative measures. Moreover, $\hat{P}_{10}$-poverty comparisons seem to be in line with general intuition when both absolute and relative poverty matter.

I now turn to the evaluation of periods of growth. In particular, I consider periods of growth during which inequality increases, because such periods lead to opposing evaluations between absolute measures and relative measures. Urban China experiences a marked unequal growth over the period 1993-2008: its mean income increases from \$3.3 to \$8.9 a day and its Gini coefficient increases from 28% to 35%. The right panel in Figure 4 reveals that urban China undergoes unequal growth over the whole period. The left panel in Figure 4 shows the evolution of $\hat{P}_{10}$, HC^A and HC^H. The evolution of HC^A reveals that growth virtually eradicates absolute poverty in urban China. The evolution of HC^H shows that the fraction of poor individuals barely changes. This means that the individuals who escape absolute poverty are still relatively poor. Yet, the $\hat{P}_{10}$-poverty decreases from 23.3% to 11.5% because, in urban China, an absolutely poor individual contributes on average twice as much to the $\hat{P}_{10}$-poverty as an individual who is only relatively poor. As a result, the unequal growth experienced by urban China is deemed poverty reducing by measure $\hat{P}_{10}$.



Poverty measures | Unequal growth

Figure 4: Evolution of income poverty between 1993 and 2008 in urban China.
Note: The left panel shows the decomposition of poor individuals (HC^H) between absolutely poor (HC^A) and only relatively poor (=the difference $HC^H - HC^A$) and shows the intermediate value taken by $\hat{P}_{10}$. The hybrid threshold is computed based on mean income in urban China. Source: PovcalNet.

Table 2 shows that $\hat{P}_{10}$ can lead to very different conclusions when analyzing periods of unequal growth. This table presents figures for four geographic entities that experiences unequal growth, as signaled by the increases in their mean incomes and in their Gini coefficients. Over the period 2000-2013, urban Indonesia experiences a sizable decline in its $\hat{P}_{10}$-poverty, which dropped from 31.3% to 26.5%. This decline occurs in spite of the increase in income inequality – the Gini coefficient increases from 32% to

43% – that causes the fraction of poor individuals to rise from 31.3% to 41.6%. Overall, the $\hat{P}_{10}$-poverty is reduced in urban Indonesia because of the massive reduction of absolute poverty, which drops from 31.3% to 8.9%. Hence, the reduction in $\hat{P}_{10}$-poverty reflects primarily the drop in absolute poverty, which was the only contributor to the $\hat{P}_{10}$-poverty in 2000.

Table 2: Evaluation with $\hat{P}_{10}$ of several cases of unequal growth.

Geo. entity	Year	Mean	Gini	HC^A	HC^H	$\hat{P}_{10}$	ω
Urban Indonesia	2000	3.0	32	31.3	31.3	31.3	-
	2013	6.3	43	8.9	41.6	26.5	0.54
Honduras	1991	4.6	52	33.4	48.4	41.1	0.51
	2004	7.5	58	26.3	52.0	40.3	0.55
Costa Rica	1986	6.0	34	12.5	27.4	19.6	0.48
	2015	22.7	48	1.6	40.3	18.6	0.44
Bulgaria	2008	16.4	34	1.2	21.1	8.8	0.38
	2014	19.0	37	1.5	24.7	10.8	0.40

Note: The impact of unequal growth on $\hat{P}_{10}$-poverty is decomposed using the three components of $\hat{P}_{10}$, namely HC^A, HC^H and ω. All poverty measures and the Gini coefficients are expressed in %. Mean incomes are expressed in \$ a day (2011 PPP). Source: PovcalNet.

In contrast, the unequal growth experiences reported for Honduras and Costa Rica barely changes their $\hat{P}_{10}$-poverty. In Honduras, mean income increases from \$4.6 to \$7.5 a day, but the income is so unequally distributed that the fraction of absolutely poor individuals only decreases from 33.4% to 26.3%. The increase in inequality leads to an increase in the fraction of poor individuals in Honduras from 48.4% to 52%. Together, the fraction of only relatively poor individuals increases from 15% to 25.7%. Thus, the unequal growth in Honduras barely changes its $\hat{P}_{10}$-poverty because the small reduction in HC^A is almost entirely offset by the increase in HC^H. The analysis of the unequal growth in Costa Rica is very similar, except that the growth and the increase in inequality are stronger. Costa Rica's $\hat{P}_{10}$-poverty barely changes because its large reduction in HC^A is almost entirely offset by its large increase in HC^H.

Finally, the unequal growth experiences reported for Hungary increases its $\hat{P}_{10}$-poverty. There is almost no absolute poverty in Hungary at the beginning of the period and, therefore, growth cannot significantly affect $\hat{P}_{10}$-poverty through HC^A. As a consequence, the increase in inequality results in an increase in $\hat{P}_{10}$-poverty from 8.8% to 10.8%.

Altogether, Table 2 illustrates the different factors affecting whether unequal growth reduces or increases $\hat{P}_{10}$-poverty. Beyond the extent of growth and the size of the increase in inequality, one additional factor is the importance of absolute poverty in the initial distribution.

7 Concluding remarks

Income inequality has recently attracted increasing attention. Abstracting from the impacts that inequality may have on behavior, there exists two main normative reasons why one may care about inequality. The first is fairness. An ethical observer may prefer more equal distributions of resources. The second is that inequality may have intrinsic value for the concerned individual. For instance, their preferences may depend on both their absolute income and their relative income. Alternatively, the social functionings provided by a given amount of resources may depend on the society's standards of living (Sen, 1992). The second reason is the mainstream foundation used to defend relativist poverty measures. Any poverty measure endorsing such foundation must *first* aggregate the absolute and relative aspects of income at the individual level and *second* aggregate individual contributions over the whole population, as proposed by Atkinson and Bourguignon (2001). As these authors suggest, taking onboard the relative aspect of poverty is not only a matter of picking the right poverty line(s) but also a matter of selecting an appropriate index. Following their approach, Decerf (2017) stresses the importance of the iso-poverty maps associated to hybrid poverty measures. In this paper, I show that indices based on two poverty lines should be based on an iso-poverty map that grants some priority to the absolute aspect of income poverty. This result further emphasizes the key role played by iso-poverty maps, which has been little studied in the literature.

Theorem 1 and 2 are general and set the stage for indices based on two poverty lines. The other results are less general but identify the properties of a new index, the extended head-count ratio, which is of particular interest for practitioners. Given its simple decomposition, a measure based on this index could complement official poverty measures in the US, in the EU or yet the extreme poverty measure of the World Bank.

8 Appendix

8.1 Proof of Theorem 1

Consider any poverty index P satisfying the axioms listed in statement 1. Consider first the case for which the income standard is mean income. I explain below how to adapt the proof when the income standard is median income.

The proof is based on the following version of Theorem 2 on page 382 of Gorman (1968).[17]

Theorem 3 (Theorem 2 in Gorman (1968))**.**
Let $\nu = (\nu_1, \dots, \nu_l)$ with $l \in \{n \in \mathbb{N} | n \geq 3\}$ denote a generic element of the product space $\times_{i=1}^{l}[0,1]$. Let $S = \{[0,1]_1, \dots, [0,1]_l\}$ be the set of sectors of this product space. If we have

- ***Assumption 1**: there exists an index P^* that represents a complete and continuous ordering on $\times_{i=1}^{l}[0,1]$,*

  An ordering is continuous on $\times_{i=1}^{l}[0,1]$ if its upper contour set and its lower contour set at any $\nu \in \times_{i=1}^{l}[0,1]$ are closed.

- ***Assumption 2**: index P^* is strictly increasing in each sector $[0,1]_i$,*

  P^* is strictly increasing in each sector if for any i such that $1 \leq i \leq l$, any

$$(\nu_1, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_l) \in \times_{j=1}^{i-1}[0,1] \times_{j=i+1}^{l} [0,1],$$

  and any $\nu_i, \nu_i' \in [0,1]_i$ with $\nu_i > \nu_i'$, we have

$$P^*(\nu_1, \dots, \nu_{i-1}, \nu_i, \nu_{i+1}, \dots, \nu_l) > P^*(\nu_1, \dots, \nu_{i-1}, \nu_i', \nu_{i+1}, \dots, \nu_l).$$

- ***Assumption 3**: any subset of sectors $A \subseteq S$ is separable,*

  Separability means that for all $(u,w), (v,w), (u,t), (v,t) \in \times_{i=1}^{l}[0,1]$, we have

$$P^*(u,w) \geq P^*(v,w) \;\Leftrightarrow\; P^*(u,t) \geq P^*(v,t). \tag{7}$$

then for any $\nu \in \times_{i=1}^{l}[0,1]$ we have

$$P^*(\nu) = \tilde{F}\left(\sum_{i=1}^{l} \tilde{\phi}_i(\nu_i)\right), \tag{8}$$

where $\tilde{F}$ is a strictly increasing function and $\tilde{\phi}_i$ is a continuous and strictly increasing function.

[17] Theorem 2 in Gorman (1968) is more general than the version presented in Theorem 3. For instance, Gorman (1968)'s theorem does not require that each sector be a real interval. It only requires that each sector has a countably dense subset and is arc-connected.

As such, Gorman's theorem cannot be applied because (i) index P represents an ordering on a space Y with a different structure, (ii) this ordering on Y needs not be continuous everywhere, (iii) not all the sectors in Y are strictly essential because distributions in Y always feature non-poor individuals and (iv) for index P, the subsets of sectors in Y need not be separable. In the following few paragraphs, I explain how I connect my framework to Gorman's.

Take any $n \in N$. Let $\overline{y}^*$ be a value of mean income with $\overline{y}^* > \overline{y}^c$. Let Y^* denote the subset of distributions of size n, whose mean income is equal to $\overline{y}^*$ and for which no individual earns more than individual n, i.e.

$$Y^* = \{y \in Y | n(y) = n \text{ and } \overline{y} = \overline{y}^* \text{ and } y_i \leq y_n \text{ for all } i \leq n\}.$$

For notational convenience, let $z^* = z_r(y^*)$ denote the relative threshold associated to mean income $\overline{y}^*$. As $\overline{y}^* > \overline{y}^c$, we have by assumption that $\overline{y}^* > z^a$ and, therefore, $\overline{y}^* > z^*$. This in turn implies that the richest individual is non-poor, i.e. $y_n > z^*$ for all $y \in Y^*$.

Consider the function $D : \mathbb{R}_+ \to [0, 1]$ defined as

$$D(w) = \begin{cases} 1 - \frac{w}{z^*} & \text{if } w \in [0, z^*], \\ 0 & \text{if } w > z^*. \end{cases}$$

Function D enters the construction of the mapping $M : Y^* \to \times_{i=1}^{n-1}[0, 1]$ defined as

$$M(y) = (D(y_1), \ldots, D(y_{n-1})).$$

Mapping M has three important properties. First, mapping M is continuous on Y^* since function D is continuous on its domain. Second, any two $y, y' \in Y^*$ with $M(y) = M(y')$ are such that $P(y) = P(y')$. Indeed, as individual n is non-poor, we have by the construction of M that $M(y) = M(y')$ only if $Q(y) = Q(y')$ and $y_i = y'_i$ for all $i \in Q(y)$. As $z_a < z^*$, *Relative Focus* implies that $P(y) = P(y')$. Third, the domain of images of Y^* through mapping M is the entire product space, i.e. $M(Y^*) = \times_{i=1}^{n-1}[0, 1]$. The definition of mapping M directly implies that $M(Y^*) \subseteq \times_{i=1}^{n-1}[0, 1]$. There remains to show that $\times_{i=1}^{n-1}[0, 1] \subseteq M(Y^*)$, which is proven by the construction of a mapping $M^- : \times_{i=1}^{n-1}[0, 1] \to Y^*$ such that $M(M^-(\nu)) = \nu$ for all $\nu \in \times_{i=1}^{n-1}[0, 1]$. Consider the function $D^- : [0, 1] \to [0, z^*]$ defined as

$$D^-(w) = z^*(1 - w).$$

Function D^- enters the construction of mapping $M^- : \times_{i=1}^{n-1}[0, 1] \to Y^*$ defined as

$$M^-(\nu) = \left(D^-(\nu_1), \ldots, D^-(\nu_{n-1}), n\overline{y}^* - \sum_{k=1}^{n-1} D^-(\nu_k) \right).$$

By definition of its n^{th} component, $M^-(\nu)$ has an income standard equal to $\overline{y}^*$. Therefore, $M^-(\nu) \in Y^*$. As by construction $D(D^-(w)) = w$ for all $w \in [0, 1]$, we have indeed that $M(M^-(\nu)) = \nu$ for all $\nu \in \times_{i=1}^{n-1}[0, 1]$. Observe that mapping M^- is continuous on the product space since function D^- is continuous on its domain and so is the n^{th} component of $M^-(\nu)$.

Mapping M^- and index P are used to define an index P^* on the product space $\times_{i=1}^{n-1}[0, 1]$. For all $\nu \in \times_{i=1}^{n-1}[0, 1]$, index P^* is defined as

$$P^*(\nu) = P(M^-(\nu)).$$

As $M(M^-(\nu)) = \nu$ for all $\nu \in \times_{i=1}^{n-1}[0, 1]$ and as any two $y, y' \in Y^*$ with $M(y) = M(y')$ are such that $P(y) = P(y')$, the definition of P^* implies for all $y \in Y^*$ that

$$P(y) = P^*(M(y)).$$

Therefore, for any $y, y' \in Y^*$ we have

$$P(y) \geq P(y') \Leftrightarrow P^*(M(y)) \geq P^*(M(y')). \tag{9}$$

The proof that statement 1 implies statement 2 is done in three steps. In step 1, I show that index P^* satisfies the three assumptions of Gorman's theorem and use this theorem. In step 2, I show how the expression of P varies with the size $n(y)$ of distributions for which $\overline{y} = \overline{y}^*$. In step 3, I connect the expressions of P for distributions that have different values of income standard.

STEP 1: For all $\nu \in \times_{i=1}^{n-1}[0,1]$, we have

$$P^*(\nu) = \tilde{F}\left(\sum_{i=1}^{n-1} \tilde{\phi}(\nu_i)\right), \tag{10}$$

where $\tilde{F}$ and $\tilde{\phi}$ are two continuous and strictly increasing functions.

First, I show that the ordering on $\times_{i=1}^{n-1}[0,1]$ represented by index P^* meets assumptions 1 to 3.

Assumption 1.
By definition, we have for all ν that $P^*(\nu) = P(M^-(\nu))$. As $M^- : \times_{i=1}^{n-1}[0,1] \to Y^*$ and as P represents a complete ordering on Y^*, we have that index P^* represents a *complete* ordering on the product space $\times_{i=1}^{n-1}[0,1]$.
As we have for all ν that $P^*(\nu) = P(M^-(\nu))$ and as mapping M^- is continuous on $\times_{i=1}^{n-1}[0,1]$, index P^* represents a *continuous* ordering on the product space $\times_{i=1}^{n-1}[0,1]$ if index P represents a continuous ordering on Y^*. Index P represents a continuous ordering on Y^* because P satisfies *Weak Continuity* and all $y \in Y^*$ are such that $\overline{y} > \overline{y}^c$. See below (end of proof of Step 1) for a proof that index P^* is continuous on its domain.

Assumption 2.
Take any i such that $1 \leq i \leq n-1$, any

$$(\nu_1, \ldots, \nu_{i-1}, \nu_{i+1}, \ldots, \nu_{n-1}) \in \times_{j=1}^{i-1}[0,1] \times_{j=i+1}^{n-1} [0,1],$$

and any $\nu_i, \nu_i' \in [0,1]_i$ with $\nu_i > \nu_i'$. We show that

$$P^*(\underbrace{\nu_1, \ldots, \nu_{i-1}, \nu_i, \nu_{i+1}, \ldots, \nu_{n-1}}_{=\nu}) > P^*(\underbrace{\nu_1, \ldots, \nu_{i-1}, \nu_i', \nu_{i+1}, \ldots, \nu_{n-1}}_{=\nu'}).$$

Consider $y = M^-(\nu)$ and $y' = M^-(\nu')$. By the construction of mapping M^-, we have $\overline{y} = \overline{y}'$, $y_i < y_i'$ and $i \in Q(y)$ and $y_j = y_j'$ for all $j \in Q(y') \backslash \{i\}$. By *Weak Monotonicity* we have $P(y) > P(y')$, which by the definition of P^* implies that $P^*(\nu) > P^*(\nu')$.
Observe that P^* is strictly increasing in all sectors of the product space because mapping M does not associate any sector to individual n, who is never poor on Y^*. Yet, her exact income y_n does not affect P by *Relative Focus*.

Assumption 3.
Take any four elements $(u,w), (v,w), (u,t), (v,t) \in \times_{i=1}^{n-1}[0,1]$. We must show that equivalence (7) holds, i.e. $P^*(u,w) \geq P^*(v,w) \;\Leftrightarrow\; P^*(u,t) \geq P^*(v,t)$.
In order to prove this, we construct four particular distributions $y^{1'''}, y^{2'''}, y^{3'''}, y^{4'''} \in Y$ for which $P(y^{1'''}) = P^*(u,w)$, $P(y^{2'''}) = P^*(v,w)$, $P(y^{3'''}) = P^*(u,t)$ and $P(y^{4'''}) = P^*(v,t)$ and we show that

$$P(y^{1'''}) \geq P(y^{2'''}) \;\Leftrightarrow\; P(y^{3'''}) \geq P(y^{4'''}).$$

This is done in two substeps.

Substep A3.1: Construct $y^{1'''}, y^{2'''}, y^{3'''}, y^{4'''} \in Y$ for which $P(y^{1'''}) = P^*(u,w)$, $P(y^{2'''}) = P^*(v,w)$, $P(y^{3'''}) = P^*(u,t)$ and $P(y^{4'''}) = P^*(v,t)$

(1.1) Let $y^1, y^2, y^3, y^4 \in Y^*$ be defined as $y^1 = M^-(u,w)$, $y^2 = M^-(v,w)$, $y^3 = M^-(u,t)$, $y^4 = M^-(v,t)$.

The next operations aim at constructing from y^1 a particular income distribution $y^{1'''}$ with $P(y^{1'''}) = P(y^1)$ but such that the elements of a particular partition of $y^{1'''}$ all have their mean income equal to $\overline{y}^*$, which is a precondition for applying *Weak Subgroup Consistency*.

(1.2) Partition y^1 in three distributions y^u, y^w and y^1_n such that

$$y^1 = (\underbrace{y^1_1, \ldots, y^1_j}_{=y^u}, \underbrace{y^1_{j+1}, \ldots, y^1_{n-1}}_{=y^w}, y^1_n),$$

where $u = (D(y^1_1), \ldots, D(y^1_j))$ and $w = (D(y^1_{j+1}), \ldots, D(y^1_{n-1}))$. By the definition of mapping M^-, all $i \notin Q(y^1) \cup \{n\}$ earn $y^1_i = z^*$ and individual n earns

$$y^1_n = n\overline{y}^* - \sum_{k=1}^{n-1} y^1_k.$$

Typically, we have $\overline{y}^u \neq \overline{y}^w \neq y^1_n \neq \overline{y}^*$ but the next operations aim at obtaining such equality.

(1.3) Construct distribution $y^{1'}$, which is the 3−replication of y^1, i.e.

$$y^{1'} = (y^1, y^1, y^1) = (y^u, y^w, y^1_n, y^u, y^w, y^1_n, y^u, y^w, y^1_n),$$

where I slightly abuse the notation by letting income y^1_n refer to the largest income level in distribution $y^{1'}$. We have $\overline{y}^{1'} = \overline{y}^1$ as the triplication does not affect the mean. By *Replication Invariance*, we have $P(y^{1'}) = P(y^1)$.

(1.4) Construct $y^{1''}$ from $y^{1'}$ by implementing particular transfers among the three individuals earning y^1_n.

Letting $n(u) = j$ and $n(w) = n - (j+1)$ denote the respective sizes of elements u and w, I define three income levels $a_u > z^*$, $a_w > z^*$ and $a_x = \overline{y}^*$ as

$$a_u = (3n(u)+1)\overline{y}^* - 3\sum_{k=1}^{j} y^1_k,$$
$$a_w = (3n(w)+1)\overline{y}^* - 3\sum_{k=j+1}^{n-1} y^1_k,$$
$$a_x = 3y^1_n - a_u - a_w.$$

First, I show that $a_u > z^*$. Recall that all non-poor individuals in y^u earn an income equal to z^*. Therefore, we have that $3\sum_{k=1}^{j} y^1_k \leq 3n(u)z^*$. Given that $\overline{y}^* > z^*$, we have indeed that $a_u > z^*$. The same reasoning shows that $a_w > z^*$. Finally, we get $a_x = \overline{y}^*$ when replacing y^1_n, a_u and a_w by their expressions in the expression of a_x and using the identity $n(u) + n(w) + 1 = n$.

Let the distribution

$$y^{1''} = (y^u, y^w, a_u, y^u, y^w, a_w, y^u, y^w, a_x)$$

be constructed from $y^{1'}$ by implementing balanced transfers among the three non-poor individuals whose income is y^1_n. As balanced transfers do not affect the mean, we have $\overline{y}^{1''} = \overline{y}^{1'} = \overline{y}^*$. As $Q(y^{1''}) = Q(y^{1'})$ and $y^{1''}_i = y^{1'}_i$ for all $i \in Q(y^{1'})$, we have by *Relative Focus* that $P(y^{1''}) = P(y^{1'})$.

(1.5) Let the distribution

$$y^{1'''} = (y^u, y^u, y^u, a_u, y^w, y^w, y^w, a_w, a_x)$$

be constructed from $y^{1''}$ by using an appropriate permutation matrix. By *Symmetry*, we have $P(y^{1'''}) = P(y^{1''})$. Importantly, the distribution $y^{1'''}$ is easily partitioned in three distributions y^{3u} ,y^{3w} and a_x, i.e.

$$y^{1''} = (\underbrace{y^u, y^u, y^u, a_u}_{=y^{3u}}, \underbrace{y^w, y^w, y^w, a_w}_{=y^{3w}}, a_x)$$

where the distributions have the same mean income $\overline{y}^{3u} = \overline{y}^{3w} = a_x = \overline{y}^*$.

Using procedure (1.1) to (1.5), we construct successively $y^{1'''}, y^{2'''}, y^{3'''}, y^{4'''}$ such that:

$$\begin{aligned} y^{1'''} &= (y^{3u}, y^{3w}, a_x) \quad && \text{with } P(y^{1'''}) = P^*(u, w), \\ y^{2'''} &= (y^{3v}, y^{3w}, a_x) \quad && \text{with } P(y^{2'''}) = P^*(v, w), \\ y^{3'''} &= (y^{3u}, y^{3t}, a_x) \quad && \text{with } P(y^{3'''}) = P^*(u, t), \\ y^{4'''} &= (y^{3v}, y^{3t}, a_x) \quad && \text{with } P(y^{4'''}) = P^*(v, t), \end{aligned}$$

where by construction $\overline{y}^{3u} = \overline{y}^{3w} = \overline{y}^{3v} = \overline{y}^{3t} = a_x = \overline{y}^*$.

Substep A3.2: Show that $P(y^{1'''}) \geq P(y^{2'''}) \Leftrightarrow P(y^{3'''}) \geq P(y^{4'''})$.

To keep notation simple, I re-label distribution y^{3u} as y^u, distribution y^{3w} as y^w, and so on. Then, last equivalence can be rewritten as

$$P\left((y^u, y^w, a_x)\right) \geq P\left((y^v, y^w, a_x)\right) \Leftrightarrow P\left((y^u, y^t, a_x)\right) \geq P\left((y^v, y^t, a_x)\right).$$

It is sufficient to prove that

$$P\left((y^u, y^w, a_x)\right) \geq P\left((y^v, y^w, a_x)\right) \Rightarrow P\left((y^u, y^t, a_x)\right) \geq P\left((y^v, y^t, a_x)\right),$$

as the converse implication is obtained by the same argument.

Recall that all distributions have their mean income equal to $\overline{y}^*$. By *Symmetry* we have $P\left((y^w, a_x)\right) = P\left((y^w, a_x)\right)$. As by assumption $P\left((y^u, y^w, a_x)\right) \geq P\left((y^v, y^w, a_x)\right)$, *Weak Subgroup Consistency* is violated unless we have[18]

$$P(y^u) \geq P(y^v).$$

Two cases can arise.

- Case 1: $P(y^u) > P(y^v)$.
  As by *Symmetry* we have $P\left((y^t, a_x)\right) = P\left((y^t, a_x)\right)$, *Weak Subgroup Consistency* implies

  $$P\left((y^u, y^t, a_x)\right) > P\left((y^v, y^t, a_x)\right).$$

- Case 2: $P(y^u) = P(y^v)$.
  I show by contradiction that this case is such that $P\left((y^u, y^t, a_x)\right) = P\left((y^v, y^t, a_x)\right)$. Assume to the contrary that we have

  $$P\left((y^u, y^t, a_x)\right) < P\left((y^v, y^t, a_x)\right)$$

  As $P(y^u) = P(y^v)$, *Weak Subgroup Consistency* implies that

  $$P\left((y^u, y^t, a_x, y^v)\right) < P\left((y^v, y^t, a_x, y^u)\right).$$

  This is a contradiction as the two distributions have equal poverty by *Symmetry*.

The two cases lead to $P\left((y^u, y^t, a_x)\right) \geq P\left((y^v, y^t, a_x)\right)$, which proves separability.

[18]By construction, we have $n(y^u) = n(y^v) \geq 4$.

As all three assumptions hold, we can use Theorem 3 and P^* has the expression given in equation (8), which can be rewritten as

$$P^*(\nu) = \tilde{F}^{'}\left(\sum_{i=1}^{n-1}\tilde{\phi}_i(\nu_i)\right) \qquad \text{for all } \nu \in \times_{i=1}^{n-1}[0,1],$$

where $\tilde{F}^{'}$ is a strictly increasing function and $\tilde{\phi}_i$ is a continuous and strictly increasing function. Functions $\tilde{\phi}_i$ cannot depend on the identity i of the individual by *Symmetry*, but they might still depend on the rank i occupied by the individual. Yet, since the ordering on $\times_{i=1}^{n-1}[0,1]$ is separable, we must have $\tilde{\phi}_i(\nu_i) = \tilde{\phi}(\nu_i)+g(i)$. Defining function $\tilde{F}(x) \coloneqq \tilde{F}^{'}(x+\sum_i g(i))$, a translation of function $\tilde{F}^{'}$, we obtain

$$P^*(\nu) = \tilde{F}\left(\sum_{i=1}^{n-1}\tilde{\phi}(\nu_i)\right) \qquad \text{for all } \nu \in \times_{i=1}^{n-1}[0,1],$$

where $\tilde{F}$ is a strictly increasing function and $\tilde{\phi}$ is a continuous and strictly increasing function.

We show that index P^* is continuous on $\times_{i=1}^{n-1}[0,1]$. That is, for all ν^* and all infinite sequence $(\nu^k)_{k\in\mathbb{N}}$ in the product space with $\lim_{k\to\infty}\nu^k = \nu^*$ we have

$$\lim_{k\to\infty} P^*(\nu^k) = P^*(\nu^*),$$

i.e. for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\nu^k - \nu^*\| < \delta$ implies that $\|P^*(\nu^k) - P^*(\nu^*)\| < \epsilon$, where operator $\|\cdot\|$ denotes euclidean distance. As mapping M^- is continuous, for all $\delta' > 0$ there exists a $\delta > 0$ such that $\|\nu^k - \nu^*\| < \delta$ implies that $\|M^-(\nu^k) - M^-(\nu^*)\| < \delta'$. By *Weak Continuity*, index P is continuous on Y^*. Thus, for all $\epsilon' > 0$ there exists a $\delta' > 0$ such that $\|M^-(\nu^k) - M^-(\nu^*)\| < \delta'$ implies that $\|P(M^-(\nu^k)) - P(M^-(\nu^*))\| < \epsilon'$. Together, take any $\epsilon' < \epsilon$, there exists a $\delta' > 0$ for which there exists a $\delta > 0$ such that $\|\nu^k - \nu^*\| < \delta$ implies that $\|P(M^-(\nu^k)) - P(M^-(\nu^*))\| < \epsilon'$. As by definition $P^*(\nu) = P(M^-(\nu))$ for all ν, we have $\|P^*(\nu^k) - P^*(\nu^*)\| < \epsilon' < \epsilon$, the desired result.

As index P^* is continuous on $\times_{i=1}^{n-1}[0,1]$ and as the argument of function $\tilde{F}$ is also continuous on the same domain, we must have that $\tilde{F}$ is a continuous function. This concludes the proof of Step 1.

The expression (10) for P^* is valid for all $y \in Y$ for which $n(y) = n$ and $\overline{y} = \overline{y}^*$. Using the same argument, a similar expression can be used for all $y' \in Y$ for which $n(y') = n' \geq 4$, $n' \neq n$ and $\overline{y}' = \overline{y}^*$. Observe that this similar expression is based on functions $\tilde{F}_{n'}$ and $\tilde{\phi}_{n'}$ that need not be the same as functions $\tilde{F}$ and $\tilde{\phi}$ obtained for size $n \neq n'$. For this reason, a subscript n is henceforth added to these functions: $\tilde{F}_n$ and $\tilde{\phi}_n$.

STEP 2: For all $y \in Y$ with $\overline{y} = \overline{y}^*$, we have

$$P(y) = G\left(\frac{1}{n(y)}\sum_{i=1}^{n(y)} d(y_i)\right), \tag{11}$$

where G is continuous and strictly increasing and $d : \mathbb{R}_+ \to [0,1]$ is continuous and strictly decreasing on $[0, z^*]$, $d(w) = 0$ when $w \geq z^*$ and $d(0) = 1$.

I modify the reasoning of Foster and Shorrocks (1991) in order to show that function $\tilde{\phi}_n$ is independent of n and function $\tilde{F}_n$ is inversely related to n.

Step 2.1: Define transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$ for normalization purposes.

Let F_n and ϕ_n be the following transformations of $\tilde{F}_n$ and $\tilde{\phi}_n$:

$$\phi_n(\nu_i) = n\left[\tilde{\phi}_n(\nu_i) - \tilde{\phi}_n(0)\right],$$

$$F_n(w) = \tilde{F}_n\left[w + (n-1)\tilde{\phi}_n(0)\right].$$

These transformations allow rewritting the expression for P^* obtained in Step 1 as

$$P^*(\nu) = F_n\left(\frac{1}{n}\sum_{i=1}^{n-1}\phi_n(\nu_i)\right),$$

where $\phi_n(0) = 0$.

Let $Q(\nu)$ denote the set of sectors i for which $\nu_i > 0$. By the definition of mapping M^-, we have $Q(\nu) = Q(M^-(\nu))$. For any sector $i \leq n-1$ such that $i \notin Q(\nu)$, we have $\nu_i = 0$ and, therefore, $\phi_n(\nu_i) = 0$. So, last expression may be rewritten as

$$P^*(\nu) = F_n\left(\frac{1}{n}\sum_{i\in Q(\nu)}\phi_n(\nu_i)\right), \tag{12}$$

where F_n and ϕ_n are continuous, strictly increasing and $\phi_n(0) = 0$.

Step 2.2: Prove that functions F_n and ϕ_n do not depend on n using *Replication Invariance*.

From the previous step, we have $\phi_n : [0,1] \to [0, a_n]$ with $\phi_n(0) = 0$ for all $n \geq 4$. Take any $y \in Y$ with $n(y) = 4$ and $\overline{y} = \overline{y}^*$ such that all individuals except maybe individual 1 are non-poor in y, which is $q(y) \leq 1$. Let $\nu = M(y) = (t, 0, 0)$ be the image of y through mapping M,[19] where t can be any point in $[0,1]$. Consider distribution $y^{\times k} = (y, \ldots, y)$, which is a k-replication of distribution y. Let $\nu' = M(y^{\times k}) = (t, 0, 0, 0, t, 0, 0, 0, \ldots, t, 0, 0)$ be the image of $y^{\times k}$, which features $3k-1$ zeros and k t's. The size of ν is $n(\nu) = 3$, the size of ν' is $n(\nu') = 4k-1$ and we denote the size of $y^{\times k}$ by $r = n(\nu') + 1 = 4k$.

By *Replication Invariance*, we have $P(y) = P\left(y^{\times k}\right)$, which implies by the (extended) definition of P^* that $P^*(\nu) = P^*(\nu')$.[20] Denoting $F = F_4$ and $\phi = \phi_4$, the relationship between ϕ, ϕ_r, F and F_r for all $t \in [0,1]$ is computed from equation (12) as:

$$P^*(\nu) = F\left[\frac{1}{4}\phi(t)\right] = F_r\left[\frac{k}{4k}\phi_r(t)\right] = P^*(\nu'),$$

which implies

$$\phi_r(t) = 4F_r^{-1}\left[F\left(\frac{1}{4}\phi(t)\right)\right].$$

Defining $H_r(w) = F_r^{-1}(F(w))$, which implies $H_4(w) = F^{-1}(F(w)) = w$, with H_r continuous and strictly increasing on $[0, a_4]$, we get

$$\phi_r(t) = 4H_r\left(\frac{1}{4}\phi(t)\right). \tag{13}$$

If $t = 0$, then we have from equation (13) that $H_r(0) = 0$ because $\phi_n(0) = 0$.

Consider now any $y' \in Y$ with $n(y') = 4$ and $\overline{y}' = \overline{y}^*$ such that all individuals except maybe individuals 1 and 2 are non-poor in y, which is $q(y') \leq 2$. Let $\nu'' = M(y') = (t, u, 0)$ be the image of y', where t and u can be any two points in $[0,1]$. Consider $y^{\times k'} = (y', \ldots, y')$ a k-replication of y'. Let $\nu''' = M(y^{\times k'}) = (t, u, 0, 0, t, u, 0, 0, \ldots, t, u, 0)$ be the image of $y^{\times k'}$, which features k t's and k u's.

By *Replication Invariance* again, we have that $F_r^{-1}[P^*(\nu'')] = F_r^{-1}[P^*(\nu''')]$, which by equations (12) and (13) yields:

$$H_r\left(\frac{1}{4}\phi(t) + \frac{1}{4}\phi(u)\right) = H_r\left(\frac{1}{4}\phi(t)\right) + H_r\left(\frac{1}{4}\phi(u)\right),$$

[19] I consider here a generalized version of mapping M defined above. The image of any $y \in Y$ with $\overline{y} = \overline{y}^*$ through the generalized mapping M is an element of $\times_{i=1}^{n(y)-1}[0,1]$.

[20] We consider the extended definitions of P^* and M^- to $\cup_{l\in N'} \times_{i=1}^{l} [0,1]$ with $N' = \{n \in \mathbb{N} | n \geq 3\}$ such that for all $\nu \in \cup_{l\in N'} \times_{i=1}^{l} [0,1]$ we have $P^*(\nu) = P(M^-(\nu))$.

which corresponds to the following Jensen equation

$$H_r(w + w') = H_r(w) + H_r(w'),$$

that admits as general solution $H_r(w) = a_r w + b_r$. As $H_r(0) = 0$ we have $b_r = 0$.

For any $\nu \in \times_{i=1}^{4k-1}[0,1]$, we obtain for $F^{-1}[P^*(\nu)]$ when introducing equation (13) into equation (12):

$$F^{-1}[P^*(\nu)] = F^{-1}\left[F_r\left(\frac{1}{4k}\sum_{i\in Q(\nu)} 4H_r\left(\frac{1}{4}\phi(\nu_i)\right)\right)\right] = H_r^{-1}\left(\frac{1}{4k}\sum_{i\in Q(\nu)} 4H_r\left(\frac{1}{4}\phi(\nu_i)\right)\right).$$

Using the fact that $H_r(w) = a_r w$, we obtain

$$F^{-1}[P^*(\nu)] = \frac{1}{4k}\sum_{i\in Q(\nu)} \phi(\nu_i).$$

Therefore, for any $y \in Y$ with $n(y) = r$ and $\overline{y} = \overline{y}^*$ and its image $\nu = M(y)$:

$$P^*(\nu) = F\left(\frac{1}{r}\sum_{i\in Q(\nu)} \phi(\nu_i)\right). \tag{14}$$

The same expression is valid for all $y \in Y$ with $\overline{y} = \overline{y}^*$ as the same reasoning can be applied between $n(y)$ and the least common multiple between $n(y)$ and 4.

Finally, consider the transformations φ and G of functions ϕ and F respectively. Letting $\varphi(\nu_i) = \frac{\phi(\nu_i)}{\phi(1)}$ and $G(w) = F\big(w\phi(1)\big)$, we have for all $y \in Y$ with $\overline{y} = \overline{y}^*$ and its image $\nu = M(y)$:

$$P^*(\nu) = G\left(\frac{1}{n(y)}\sum_{i\in Q(\nu)} \varphi(\nu_i)\right)$$

where G and φ are continuous and strictly increasing functions and $\varphi : [0,1] \to [0,1]$ is a bijection with $\varphi(0) = 0$ and $\varphi(1) = 1$. By the definition of mapping M, last expression implies

$$P^*(M(y)) = G\left(\frac{1}{n(y)}\sum_{i=1}^{n(y)} d(y_i)\right) = P(y), \tag{15}$$

where $d = \varphi \circ D : \mathbb{R}_+ \to [0,1]$ is continuous and strictly decreasing on $[0, z^*]$, $d(w) = 0$ when $w \geq z^*$ and $d(0) = 1$. This concludes the proof of Step 2.

This expression for P^* is valid for all $y \in Y$ for which $\overline{y} = \overline{y}^*$. Using the same argument, a similar expression can be used for all $y' \in Y$ for which $\overline{y}' > \overline{y}^c$. Observe that this similar expression is based on functions $G_{\overline{y}'}$ and $d_{\overline{y}'}$ that need not be the same as functions G and d, obtained for $\overline{y} = \overline{y}^*$. For this reason, a subscript $\overline{y}$ is henceforth added to these functions: $G_{\overline{y}}$ and $d_{\overline{y}}$.

STEP 3: For all $y \in Y$, P is ordinally equivalent to the index defined in equation (1).

For notational convenience, let $d = d_{\overline{y}^*}$ and let $G = G_{\overline{y}^*}$. Let $Y^c = \{y \in Y | \overline{y} > \overline{y}^c\}$ be the subset of Y on which all distributions have mean income higher than $\overline{y}^c$.

Step 3.1: validity of equation (1) for all $y \in Y \backslash Y^c$.

Take any $y \in Y \backslash Y^c$ with $n \notin Q(y)$. Assuming $n \notin Q(y)$ is without loss of generality as all distributions in Y have at least one non-poor individual. Consider distribution y' with $n(y') = n(y)$ obtained from y such that $y'_i = y_i$ for all $i \leq Q(y)$, $y'_j = z^*$ for all $j \notin Q(y) \cup \{n\}$ and $y'_n = n(y)\overline{y}^* - \sum_{k=1}^{n(y)-1} y'_k$. By construction, we have $\overline{y}' = \overline{y}^*$. By definition of Y^c, $Q(y) = Q_a(y)$ since $\overline{y} \leq \overline{y}^c$ implies $z_a \geq z_r(\overline{y})$. The construction of y' then implies that $Q(y') = Q_a(y') = Q(y) = Q_a(y)$ and $y'_i = y_i$ for all $i \leq Q(y)$. By *Absolute Focus*, we have $P(y') = P(y)$. From equation (11) obtained in Step 2 and as $\overline{y}' = \overline{y}^*$ and $y'_i = y_i$ for all $i \leq q(y)$, we obtain

$$P(y') = G\left(\frac{1}{n(y')}\sum_{i=1}^{n(y')} d(y'_i)\right)$$
$$= G\left(\frac{1}{n(y)}\sum_{i\in Q(y)} d(y_i)\right) = P(y). \tag{16}$$

Letting $p(y_i, \overline{y}) = d(y_i)$ for all $i \in Q(y)$ and $p(y_i, \overline{y}) = 0$ for all $i \notin Q(y)$, equation (16) provides the desired expression for $P(y)$ on $y \in Y \backslash Y^c$. There remains to show that function p has the required properties. We have (i) $p(0, \overline{y}) = 1$ as $d(0) = 1$ and $p(y_i, \overline{y}) = 0$ if $i \notin Q(y)$ by definition. We have (ii) p is strictly decreasing in its first argument if $i \in Q(y)$ since d has this property. We have (iii) p is constant in its second argument if $i \in Q_a(y)$ by the definition of p.

Step 3.2: validity of equation (1) for all $y \in Y^c$.

Take any $y \in Y^c$ with $n \notin Q(y)$. Assuming $n \notin Q(y)$ is again without loss of generality. From step 2, $P(y)$ is given by equation (11), where function d and G are specific to mean income $\overline{y}$, i.e. $d_{\overline{y}}$ and $G_{\overline{y}}$.

First, I show that we have $G_{\overline{y}}(w) = G(w)$ for all $w \in [0, 1)$. Take any *rational* number $r \in [0, 1)$. Rational number r can be expressed as $r = \frac{q}{n}$ with $q, n \in \mathbb{N}$, $q < n$ and $n \geq 4$. I show that $G_{\overline{y}}(r) = G(r)$. If this holds, then the continuity of functions $G_{\overline{y}}$ and G implies that $G_{\overline{y}}(w) = G(w)$ for all $w \in [0, 1)$. There remains to show $G_{\overline{y}}(r) = G(r)$. Construct y' such that $n(y') = n$, $q(y') = q$, $y'_i = 0$ for all $i \in Q(y')$, $y'_j = z_r(\overline{y})$ for all $j \notin Q(y') \cup \{n\}$ and $y'_n = n(y')\overline{y} - \sum_{k=1}^{n(y')-1} y'_k$. By construction, we have $\overline{y}' = \overline{y}$ and thus by equation (11), we have $P(y') = G_{\overline{y}}\left(\frac{q}{n}\right)$ since $d_{\overline{y}}(0) = 1$. Construct y'' such that $n(y'') = n$, $q(y'') = q$, $y''_i = 0$ for all $i \in Q(y'')$, $y''_j = z^*$ for all $j \notin Q(y'') \cup \{n\}$ and $y''_n = n(y'')\overline{y}^* - \sum_{k=1}^{n(y'')-1} y''_k$. By construction, we have $\overline{y}'' = \overline{y}^*$ and thus by equation (11), we have again $P(y'') = G\left(\frac{q}{n}\right)$. By construction, we have $Q_a(y') = Q(y') = Q_a(y'') = Q(y'')$, and $y'_i = y''_i$ for all $i \in Q(y')$. Therefore we have $P(y') = P(y'')$ by *Absolute Focus*. Together, we have $G_{\overline{y}}\left(\frac{q}{n}\right) = G\left(\frac{q}{n}\right)$, the desired result.

Given that functions $G_{\overline{y}}$ and G are identical, we have from step 2 that

$$P(y) = G\left(\frac{1}{n(y)}\sum_{i=1}^{n(y)} d_{\overline{y}}(y_i)\right), \tag{17}$$

where $d_{\overline{y}}$ is continuous and strictly decreasing on $[0, z_r(\overline{y})]$, $d_{\overline{y}}(w) = 0$ when $w \geq z_r(\overline{y})$ and $d_{\overline{y}}(0) = 1$. Letting $p(y_i, \overline{y}) = d_{\overline{y}}(y_i)$, equation (17) provides the desired expression for $P(y)$ on $y \in Y^c$. There remains to show that function p has the required properties. We have (i) $p(0, \overline{y}) = 1$ as $d_{\overline{y}}(0) = 1$ and $p(y_i, \overline{y}) = 0$ if $i \notin Q(y)$ since $d_{\overline{y}}(w) = 0$ when $w \geq z_r(\overline{y})$. We have (ii) p is strictly decreasing in its first argument if $i \in Q(y)$ since $d_{\overline{y}}$ is strictly decreasing on $[0, z_r(\overline{y})]$. We show that (iii) p is constant in its second argument if $i \in Q_a(y)$ by contradiction. Assume that $p(a, \overline{y}) \neq d(a)$ for some $a < z_a$. Then we can construct two distributions $y', y'' \in Y$ such that $n(y') = n(y'')$, $q(y') = q(y'') = 1$, $Q_a(y') = Q(y') = Q_a(y'') = Q(y'')$, $y'_i = y''_i = a$ for the only $i \in Q(y)$ and the remaining incomes of y' and y'' such that $\overline{y}' = \overline{y}$ and $\overline{y}'' = \overline{y}^*$. By *Absolute Focus*, we have $P(y') = P(y'')$, which by equation (17) implies that $p(a, \overline{y}) = d(a)$, a contradiction. As we obtained in step 3.1 that $p(a, \overline{y}) = d(a)$ for all $a < z_a$ when $\overline{y} \leq \overline{y}^c$, property (iii) is established for all $\overline{y} \geq z^a$. Finally, we show that (iv) p is continuous in both its arguments if $\overline{y} > \overline{y}^c$. Function p is continuous in its first argument since $d_{\overline{y}}$ is continuous in y_i. Function p is continuous in its second argument for all $\overline{y} > \overline{y}^c$ because P satisfies

Weak Continuity and mean income is a continuous function of the income of non-poor individuals (all $y \in Y$ feature at least one non-poor individual).

As function G is continuous and strictly increasing, we have shown that P is ordinally equivalent to P' defined by equation (1). This concludes the proof.

Median income

I provide some indications about the way to adapt the proof of Theorem 1 when the income standard is median income y_m instead of mean income.

In steps 1 and 2, we only consider distributions y with size $n(y) \in N^m = \{n \in 2\mathbb{N} | n \geq 8\}$. The set of (even-sized) distributions considered is Y^*_{even}, which is the subset of Y^* for which all individuals $i \leq m$ earn (weakly) less than individual m and all individuals $j \geq m$ earn (weakly) more than individual m, i.e.

$$Y^*_{even} = \{y \in Y | n(y) \in N^m,\ \overline{y} = \overline{y}^*,\ y_i \leq y_m \leq y_j \text{ for all } i \leq m \text{ and all } j \geq m\}.$$

Function D enters the construction of the different mapping $M^m : Y^*_{even} \to \times_{i=1}^{m-1}[0,1]$ defined as

$$M^m(y) = (D(y_1), \ldots, D(y_{m-1})).$$

Distribution y with size $n(y)$ has an image $M^m(y)$ of size $m-1$. As we assume $\overline{y} \geq z_a$, all distributions in Y have at most $m-1$ poor individuals when the income standard is median income. By *Relative Focus*, the exact incomes of the remaining individuals matter only as long as it defines y_m. Then, function D^- enters the construction of mapping $M^{m-} : \times_{i=1}^{m-1}[0,1] \to Y^*_{even}$ defined as

$$M^{m-}(\nu) = \left(D^-(\nu_1), \ldots, D^-(\nu_{m-1}), y_m, \ldots, y_m\right),$$

which features $m+1$ individuals earning y_m.

The reason why mapping M^m (and the reversed mapping M^{m-}) must be limited to even-sized distribution is the following. By the definition of the median m, any $y \in Y^*_{even}$ and the odd distribution (y, y_m), whose size is $n(y)+1$ and (y, y_m) has the same median income as y, would have the *same* image $M^m(y) = M^m(y, y_m)$ (assuming an extended mapping M^m that also applies to odd distributions). Yet, these two distributions do not have the same poverty as (y, y_m) has one additional non-poor individual, the one who earns y_m. By limiting the domain of mapping M^m to even-sized distribution, the problem vanishes.

Equation (11) obtained for distributions in Y^*_{even} in step 2 is extended to distributions $y \in Y \backslash Y^*_{even}$ using *Replication Invariance*: for all $y \in Y \backslash Y^*_{even}$ we have

$$P(y) = P(y^{\times 2}),$$

where $P(y^{\times 2})$ is given by equation (11) as $y^{\times 2} \in Y^*_{even}$.

Observe also that the construction of $y^{1'''}$ from y^1 in step A3.1 is slightly different. In substep (1.2), we must partition y^1 in three distributions y^u, y^w and y^a such that

$$y^1 = (\underbrace{y^1_1, \ldots, y^1_j}_{=y^u}, \underbrace{y^1_{j+1}, \ldots, y^1_{m-1}}_{=y^w}, \underbrace{y^1_m, \ldots, y^1_n}_{=y^a}),$$

where $u = (D(y^1_1), \ldots, D(y^1_j))$ and $w = (D(y^1_{j+1}), \ldots, D(y^1_{m-1}))$ and the $m+1$ incomes in y^a are all equal to y^1_m. We further partition y^a into y^{au} and y^{aw} with $n(y^{au}) = n(y^u)+1$ and $n(y^{aw}) = n(y^w) + 1$. As we define $m = \frac{n(y)}{2}$ when $n(y)$ is even, we have indeed that $n(y) = 2n(y^u) + 2n(y^w) + 2$. In substep (1.3), distribution $y^{1'}$ is a 2-replication of y^1, which implies in substep (1.5) that distributions $y^{2u} = (y^u, y^u, y^{au}, y^{au})$ and $y^{2w} = (y^w, y^w, y^{aw}, y^{aw})$ have at least the minimal size for P and have both their median income equal to y^1_m.

Finally, the construction of any distribution y for which $\overline{y} = \overline{y}^*$ is simpler with median than with mean, as it is sufficient to take $y_j = \overline{y}^*$ for all j with $m \leq j \leq n(y)$.

8.2 Proof of Theorem 2

Take any hierarchical additive index P, i.e take any contribution function p that satisfies the properties stated in equation (1). As $z_a > 0$, we have that the absolute and relative lines cross at an income standard $\overline{y}^c$ such that $\overline{y}^c \geq z_a > 0$.

Claim 1: P violates *Transfer*.

Consider $\overline{y}^* = \overline{y}^c + \frac{z_a}{2s}$, which is such that $z_r(\overline{y}^*) = \frac{3}{2}z_a$. We consider two cases.

Case 1: $p(z_a, \overline{y}^*) \geq \frac{1}{2}$.
This case is such that $p(0, \overline{y}^*) - p(\frac{z_a}{2}, \overline{y}^*) < \frac{1}{2}$. Indeed, we have that $p(0, \overline{y}^*) = 1$ and p is strictly decreasing in its first argument on $[0, z_r(\overline{y}^*)]$. By the continuity of p in its first argument on $[0, z_r(\overline{y}^*)]$, there exists an income level r^* with $z_a < r^* < z_r(\overline{y}^*)$ such that

$$p(r^*, \overline{y}^*) < p(z_a, \overline{y}^*) - \left(p(0, \overline{y}^*) - p\left(\frac{z_a}{2}, \overline{y}^*\right)\right). \tag{18}$$

Consider now two distributions $y, y' \in Y$ with $n(y) = n(y')$, $Q(y) = Q(y') = \{1, 2\}$ and whose income standard is equal to $\overline{y}^*$. Distribution y is such that $y_1 = 0$, and $y_2 = r^*$, while distribution y' is such that $y'_1 = r^* - z_a$, and $y'_2 = z_a$. Distribution y' is obtained from y by a progressive transfer of an amount $r^* - z_a$ from the relatively poor individual 2 to the absolutely poor individual 1. As $r^* < z_r(\overline{y}^*)$ and $z_r(\overline{y}^*) = \frac{3}{2}z_a$, we have $r^* - z_a < \frac{z_a}{2}$, and thus inequality (18) implies that

$$\underbrace{p(0, \overline{y}^*) - p(r^* - z_a, \overline{y}^*)}_{\Delta^a} < \underbrace{p(z_a, \overline{y}^*) - p(r^*, \overline{y}^*)}_{\Delta^r}.$$

Δ^a captures the decrease in poverty contribution of individual 1 consecutive to the progressive transfer. In turn, Δ^r captures the increase in poverty contribution of individual 2. As the latter is larger, we have by equation (1) that $P(y') > P(y)$, a contradiction to *Transfer*.

Case 2: $p(z_a, \overline{y}^*) < \frac{1}{2}$.
Let $a^* > 0$ be the level of income for which $p(a^*, \overline{y}^*) = 1 - p(z_a, \overline{y}^*)$. We have $0 < a^* < z_a$ as p is decreasing in its first argument. Let $\overline{y}^{**} = \overline{y}^c + \frac{a^*}{2s}$, which is such that $z_r(\overline{y}^*) = z_a + \frac{a^*}{2}$. As p is constant in $\overline{y}$ for all $a < z_a$, we have $p(a^*, \overline{y}^{**}) = p(a^*, \overline{y}^*)$. As p is decreasing in its first argument, we have $p(\frac{a^*}{2}, \overline{y}^{**}) > p(a^*, \overline{y}^{**})$. We also have $p(z_a, \overline{y}^{**}) = p(z_a, \overline{y}^*)$ since $\overline{y}^{**} > \overline{y}^c$. This follows from the fact that (iii) $p(a, \overline{y})$ is constant in $\overline{y}$ for all $a < z_a$ and (iv) p is continuous in its first argument when the income standard is larger than $\overline{y}^c$. Together, we have that $p(z_a, \overline{y}^{**}) > p(0, \overline{y}^{**}) - p(\frac{a^*}{2}, \overline{y}^{**})$. By the continuity of p in its first argument on $[0, z_r(\overline{y}^{**})]$, there exists an income level r^{**} with $z_a < r^{**} < z_r(\overline{y}^{**})$ such that

$$p(r^{**}, \overline{y}^{**}) < p(z_a, \overline{y}^{**}) - \left(p(0, \overline{y}^{**}) - p\left(\frac{a^*}{2}, \overline{y}^{**}\right)\right). \tag{19}$$

As in case 1, we can construct two distributions $y, y' \in Y$ with $n(y) = n(y')$, $Q(y) = Q(y') = \{1, 2\}$, whose income standard is equal to $\overline{y}^{**}$, with $y_1 = 0$, and $y_2 = r^{**}$ and $y'_1 = r^{**} - z_a$, and $y'_2 = z_a$. As $r^{**} < z_r(\overline{y}^{**})$ and $z_r(\overline{y}^{**}) = z_a + \frac{a^*}{2}$, we have $r^{**} - z_a < \frac{a^*}{2}$. Therefore, $p(r^{**} - z_a, \overline{y}^{**}) > p\left(\frac{a^*}{2}, \overline{y}^{**}\right)$ and inequality (19) implies that

$$\underbrace{p(0, \overline{y}^{**}) - p(r^{**} - z_a, \overline{y}^{**})}_{\Delta^a} < \underbrace{p(z_a, \overline{y}^{**}) - p(r^{**}, \overline{y}^{**})}_{\Delta^r}.$$

As Δ^r is larger than Δ^a, we have by equation (1) that $P(y') > P(y)$, a contradiction to *Transfer*.

Claim 2: P violates *Strong Monotonicity* when the relative line is mean-sensitive.

Take any distribution $y \in Y$ with $y_1 = 0$, $y_2 = z_a$ and $y_j = \frac{1}{n(y)-2}\left(n(y)\overline{y}^c - y_1 - y_2\right)$ for all $j \geq 3$. As the income standard is mean income, we have $\overline{y} = \overline{y}^c$. By construction, only individual 1 is poor in y as $z_r(y) = b + s\overline{y}^c = z_a$.

Construct distribution y^ϵ from y by increasing the income of individual 1 to $y_1^\epsilon = \epsilon$ where $0 < \epsilon < z_a$. This increment implies that $\overline{y}^\epsilon > \overline{y}^c$. By the definition of the relative line, we have $z_r(y^\epsilon) > z_r(y) = z_a$. Therefore, individual 2 is relatively poor in distribution y^ϵ since $y_2^\epsilon = z_a$. Importantly, individual 2's contribution $p(z_a, \overline{y}^\epsilon) > 0$ is constant in the size of ϵ (this follows from properties (iii) and (iv) of p, as explained in Case 2 of Claim 1).

I show that for sufficiently small ϵ, we have $P'(y^\epsilon) - P'(y) > 0$, which implies that P violates *Strong Monotonicity*. As individual 2 is non-poor in distribution y, we have from equation (1) that

$$P'(y^\epsilon) - P'(y) = \frac{1}{n(y)}\left(p(z_a, \overline{y}^\epsilon) + \underbrace{p\left(\epsilon, \overline{y}^\epsilon\right) - p\left(0, \overline{y}\right)}_{\Delta_\epsilon}\right).$$

By the definition of p, we have $p\left(0, \overline{y}^\epsilon\right) = p\left(0, \overline{y}\right) = 1$. As p is continuous in its first argument, there exists an $\epsilon > 0$ such that $-\Delta_\epsilon < p(z_a, \overline{y}^\epsilon)$, which implies that $P'(y^\epsilon) - P'(y) > 0$, the desired result.

8.3 Proof of Lemma 2

Step 1: $\hat{P}_{\alpha\lambda}$ is robust if $\lambda = 0$ or $\alpha = 0$.

Assume that $\lambda = 0$ or $\alpha = 0$. For any $y \in Y$ with $Q_a(y) = Q(y)$, index $\hat{P}_{\alpha\lambda} = \frac{q(y)}{n(y)}$, i.e. $\hat{P}_{\alpha\lambda}$ is equivalent to the head-count ratio.

Index $\hat{P}_{\alpha\lambda}$ satisfies *Translation Invariance* because a translation of all incomes in distribution y by $\delta > 0$ leaves the poverty status of all individuals unchanged. As $Q_a(y) = Q(y)$, any individual i is either absolutely poor or non-poor. Absolutely poor individuals are still absolutely poor when their income is increased by the same amount $\delta > 0$ as the absolute threshold z_a. Non-poor individuals do not become absolutely poor: if $y_i \geq z_a$, then $y_i + \delta \geq z_a + \delta$. Non-poor individuals do not become relatively poor: if $y_i \geq b + s\overline{y}$, then $y_i + \delta \geq b + s(\overline{y} + \delta)$ because $s < 1$.

The same holds true for *Scale Invariance* for any $\gamma > 1$. Absolutely poor individuals are still absolutely poor when their income is multiplied by the same factor $\gamma > 1$ as the absolute threshold z_a. Non-poor individuals do not become absolutely poor: if $y_i \geq z_a$, then $\gamma y_i \geq \gamma z_a$. Non-poor individuals do not become relatively poor: if $y_i \geq b + s\overline{y}$, then $\gamma y_i \geq b + \gamma s\overline{y}$ as $b \geq 0$.

Step 2: $\hat{P}_{\alpha\lambda}$ is robust only if $\lambda = 0$ or $\alpha = 0$.

The proof adapts the proof of Zheng (1994), which shows that these properties force the index to account only for poverty status.

Take any $z_a > 0$, any $y \in Y$ with $Q_a(y) = Q(y)$ and $\overline{y} > \overline{y}^c$. Construct $y^* \in Y$ from y by equalizing the income of non-poor individuals without changing the income standard. When the income standard is mean income, the construction of y^* is

- $y_i^* = y_i$ for all $i \in Q(y)$,
- $y_j^* = \frac{n(y)\overline{y} - \sum_{k \in Q(y)} y_k}{n(y) - q(y)}$ for all $j \notin Q(y)$.

As $\overline{y} > \overline{y}^c$, we have $\overline{y} > z_r(y) > z_a$, and therefore all $j \notin Q(y)$ are non-poor in y^*. As $\overline{y}^* = \overline{y}$, we have by *Relative Focus* that $\hat{P}_{\alpha\lambda}(y^*, z_a, z_r(y^*)) = \hat{P}_{\alpha\lambda}(y, z_a, z_r(y))$. Let y^{**} be constructed from y^* by sorting the incomes in y^* in non-decreasing order. By *Symmetry*, we have $\hat{P}_{\alpha\lambda}(y^{**}, z_a, z_r(y^{**})) = \hat{P}_{\alpha\lambda}(y, z_a, z_r(y))$.

I construct a sequence of distributions $(y^k)_{k \in \mathbb{N}}$ with $y^0 = y^{**}$ and such that for all $k \in \mathbb{N}$ we have $\overline{y}^k = \overline{y}$, $Q_a(y^k) = Q(y^k) = Q(y)$, $\hat{P}_{\alpha\lambda}(y^k, z_a, z_r(y^k)) = \hat{P}_{\alpha\lambda}(y, z_a, z_r(y))$

and when $k \to \infty$ the sequence tends to:

$$y^{\infty} = \left(z_a, \ldots, z_a, \frac{n(y)\overline{y} - q(y)z_a}{n(y) - q(y)}, \ldots, \frac{n(y)\overline{y} - q(y)z_a}{n(y) - q(y)} \right),$$

where $q(y)$ poor individuals earn z_a and $n(y) - q(y)$ individuals are non-poor. By the continuity of $\hat{P}_{\alpha\lambda}$ in the incomes of poor individuals, we have $\hat{P}_{\alpha\lambda}(y^{\infty}, z_a, z_r(y^{\infty})) = \hat{P}_{\alpha\lambda}(y, z_a, z_r(y))$. Observe that this sequence, when applied to an alternative initial distribution $y' \in Y$ with $n(y') = n(y)$, $Q_a(y') = Q(y') = Q_a(y) = Q(y)$ and $\overline{y}' = \overline{y}$ has the same limit distribution y^{∞}. Therefore, we also have $\hat{P}_{\alpha\lambda}(y', z_a, z_r(y')) = \hat{P}_{\alpha\lambda}(y^{\infty}, z_a, z_r(y^{\infty})) = \hat{P}_{\alpha\lambda}(y, z_a, z_r(y))$. This implies that all distributions with the same number of individuals, the same number of poor individuals, all of which are absolutely poor, have the same poverty. Therefore, $\hat{P}_{\alpha\lambda}$ is equivalent to the head-count ratio when $Q_a(y) = Q(y)$. Hence, we must have $\lambda = 0$ or $\alpha = 0$.

There remains to show that such sequence can be constructed. First, assume that the income standard is *mean income*. Take any $\delta > 0$. For any $k \in \mathbb{N}$, distribution y^{k+1} is obtained from y^k using the following three transformations.

1. Let $y^{k'} = y^k + \delta \cdot \mathbb{1}_n$ where $\mathbb{1}_n$ is a vector of ones of size $n(y)$.
2. Let $y^{k''}$ be constructed from $y^{k'}$ such that $y_i^{k''} = y_i^{k'}$ for all $i \in Q(y^{k'})$ and $y_j^{k''} = y_j^{k'} + \delta \frac{n(y)}{n(y)-q(y)} \frac{\overline{y}-z_a}{z_a}$ for all $j \notin Q(y^{k'})$.
3. $y^{k+1} = \frac{z_a}{z_a+\delta} y^{k''}$.

The intermediate distribution $y^{k''}$ is constructed in order to obtain $\overline{y}^{k''} = \frac{z_a+\delta}{z_a}\overline{y}^k$, which in turn implies that $\overline{y}^{k+1} = \overline{y}^k$.

I show that $Q_a(y^{k+1}) = Q(y^{k+1}) = Q(y^k)$ if $Q_a(y^k) = Q(y^k)$. All absolutely poor individuals in y^k are still absolutely poor in y^{k+1} because $y_j^{k+1} = \frac{z_a}{z_a+\delta}(y_j^k + \delta) < z_a$ for all $j \in Q_a(y^k)$. All non-poor individuals in y^k are still non-poor in y^{k+1} because they all earn the same income and $\overline{y}^{k+1} = \overline{y}^k = \overline{y} > z_r(y) > z_a$.

We have $\hat{P}_{\alpha\lambda}(y^{k+1}, z_a, z_r(y^{k+1})) = \hat{P}_{\alpha\lambda}(y^k, z_a, z_r(y^k))$ since

- $P(y^{k'}, z_a + \delta, z_r(y^{k'})) = P(y^k, z_a, z_r(y^k))$ by *Translation Invariance*, which applies since $y^{k'} = y^k + \delta \cdot \mathbb{1}_n$.
- $P(y^{k''}, z_a + \delta, z_r(y^{k''})) = P(y^{k'}, z_a + \delta, z_r(y^{k'}))$ by *Absolute Focus*, which applies since the translation in the incomes of the non-poor individuals $j \notin Q(y^{k'})$ does not change their poverty status.
- $P(y^{k+1}, z_a, z_r(y^{k+1})) = P(y^{k''}, z_a + \delta, z_r(y^{k''}))$ by *Scale Invariance* since $y^{k''}$ is obtained by scaling y^{k+1} using the factor $\frac{z_a+\delta}{z_a} > 1$.

Finally, I show that the sequence tends to y^{∞}. For each poor person i, we have $y_i^{k+1} = \frac{z_a}{z_a+\delta}(y_i^k + \delta)$. Thus, we have $y_i^0 \leq y_i^1 \leq \cdots \leq y_i^k \leq \cdots \leq z_a$ and the sequence converges to a constant c_i satisfying

$$c_i = (c_i + \delta)\left(\frac{z_a}{z_a + \delta}\right),$$

which implies that

$$\lim_{k \to \infty} y_i^k = z_a.$$

As the incomes of all poor individuals tend to z_a and as the incomes of all non-poor are equal, we have $\overline{y}^{\infty} = \overline{y}$ only if

$$\lim_{k \to \infty} y_j^k = \frac{n(y)\overline{y} - q(y)z_a}{n(y) - q(y)},$$

the desired result.

Second, assume that the income standard is *median income*. The sequence is the same except that, in step 2, we let $y_j^{k''} = y_j^{k'} + \delta \frac{\overline{y}-z_a}{z_a}$ for all $j \notin Q(y^{k'})$.

8.4 Proof of Lemma 3

First, I study the restrictions that *Strong Monotonicity REL* imposes on $\hat{P}_{\alpha 0}$.

Let the partial derivative of any function $F : \mathbb{R}^n \to \mathbb{R}$ in the direction i at point $w \in \mathbb{R}^n$ be denoted by $\partial_i F(w)$. *Strong Monotonicity REL* requires that for all distribution $y \in Y$ and all $i \in Q(y) \backslash Q_a(y)$ we have $\partial_i \hat{P}_{\alpha 0}(y) < 0$. Recall that $\hat{P}_{\alpha 0}(y) = \frac{1}{n(y)} \sum_i \hat{p}(y_i, \overline{y})$ where $\hat{p}$ is defined by equation (5). As the income standard is mean income, we have $\partial_i \overline{y} = \frac{1}{n(y)}$. By chain derivation, the previous inequality becomes:

$$\underbrace{\partial_1 \hat{p}(y_i, \overline{y}) + \frac{1}{n(y)} \sum_{j=1}^{n(y)} \partial_2 \hat{p}(y_j, \overline{y})}_{L_{20}} < 0. \tag{20}$$

Condition (20) relies on the partial derivatives of function $\hat{p}$. For all $i \in Q(y) \backslash Q_a(y)$, the partial derivative with respect to own income is:

$$\partial_1 \hat{p}(y_i, \overline{y}) = -\frac{\alpha}{z_r(y) - z_a} \left(\frac{z_r(y) - y_i}{z_r(y) - z_a} \right)^{\alpha - 1}. \tag{21}$$

For all $i \in Q(y)$, the partial derivative with respect to mean income is:

$$\partial_2 \hat{p}(y_i, \overline{y}) = 0 \qquad \text{if } y_i < z_a, \tag{22}$$

and, recalling that $\frac{\partial z_r(y)}{\partial \overline{y}} = s$,

$$\partial_2 \hat{p}(y_i, \overline{y}) = -s \frac{y_i - z_a}{z_r(y) - z_a} \partial_1 \hat{p}(y_i, \overline{y}) \qquad \text{if } z_a \leq y_i. \tag{23}$$

Observe that if the non-poor individual i has income $y_i = z_r(y)$, then she becomes poor when mean income increases. In that case, the value for $\partial_2 \hat{p}(y_i, \overline{y})$ is defined by equation (23).

Step 1: $\hat{P}_{\alpha 0}$ satisfies *Strong Monotonicity REL* only if $\alpha = 1$.

The proof is by contraposition. Take any $\alpha > 0$ different from 1. Take any mean income $\overline{y}^* > \overline{y}^c$, which implies that $b + s\overline{y}^* > z_a$. There are two cases to consider.

- Case 1: $0 < \alpha < 1$.

  Consider a distribution $y \in Y$ with $Q(y) = \{1, 2\}$, $z_a < y_1 < y_2 \leq z_r(y)$ and $y_j = \frac{1}{n(y)-2} (n(y)\overline{y}^* - y_1 - y_2)$. By construction, we have $\overline{y} = \overline{y}^*$. When the income of individual 1 increases marginally, we have that

  $$L_{20}(y) = \partial_1 \hat{p}(y_1, \overline{y}) + \frac{1}{n(y)} \partial_2 \hat{p}(y_1, \overline{y}) + \frac{1}{n(y)} \partial_2 \hat{p}(y_2, \overline{y}).$$

  As $\alpha < 1$ and $s > 0$, we have from equations (23) and (21) that

  $$\lim_{w \to z_r(y)} \partial_2 \hat{p}(w, \overline{y}) = \infty.$$

  For a fixed y_1, there exists hence a value of y_2 sufficiently close to $z_r(y)$ such that $L_{20}(y) > 0$ and, therefore, condition (20) is violated. Hence, *Strong Monotonicity REL* is violated.

- Case 2: $1 < \alpha$.

  A similar argument shows that *Strong Monotonicity REL* does not hold. Consider the same distribution as in Case 1. When the income of the relatively poor individual 2 increases marginally, we have

  $$L_{20}(y) = \partial_1 \hat{p}(y_2, \overline{y}) + \frac{1}{n(y)} \partial_2 \hat{p}(y_1, \overline{y}) + \frac{1}{n(y)} \partial_2 \hat{p}(y_2, \overline{y}).$$

As $\alpha > 1$, we have from equation (21) that

$$\lim_{w \to z_r(y^*)} \partial_1 \hat{p}(w, \overline{y}) = 0.$$

As $y_1 > z_a$, we have that $\partial_2 \hat{p}(y_1, \overline{y}) > 0$. Again, for y_2 sufficiently close to $z_r(y)$, we have $L_{20}(y) > 0$ and condition (20) is violated. Hence, *Strong Monotonicity REL* does not hold.

Step 2: $\hat{P}_{\alpha 0}$ satisfies *Strong Monotonicity REL* if $\alpha = 1$.

When $\alpha = 1$, the partial derivatives (21) and (23) simplify to

$$\begin{aligned} \partial_1 \hat{p}(y_i, \overline{y}) &= -\frac{\alpha}{z_r(y) - z_a}, \\ \partial_2 \hat{p}(y_i, \overline{y}) &= -\underbrace{s \frac{y_i - z_a}{z_r(y) - z_a}}_{=K} \partial_1 \hat{p}(y_i, \overline{y}) \qquad \text{if } z_a \leq y_i. \end{aligned}$$

Observe that (A) $\partial_1 \hat{p}(y_i, \overline{y})$ does not depend on y_i for relatively poor individuals and (B) factor $K \leq 1$ for any $y_i \in [z_a, z_r(y)]$. Together, these observations imply that for any $i \in Q(y) \backslash Q_a(y)$ and any $j \leq n(y)$ that we have $-\partial_1 \hat{p}(y_i, \overline{y}) > \partial_2 \hat{p}(y_j, \overline{y})$. As a result, we have

$$L_{20}(y) < \partial_1 \hat{p}(y_i, \overline{y}) - \frac{1}{n(y)} \sum_{j=1}^{n(y)} \partial_1 \hat{p}(y_j, \overline{y}) \leq \partial_1 \hat{p}(y_i, \overline{y}) - \partial_1 \hat{p}(y_i, \overline{y}) = 0,$$

which shows that condition (20) and *Strong Monotonicity REL* hold.

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