
# Long-term Social Welfare: Mobility, Social Status, and Inequality

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**Abstract**

Recently there has been a growing interest in the empirical association between income inequality and social mobility. Little is known on the normative nexus between both notions, however. In this paper, we axiomatically characterize a family of multiperiod social evaluation functions that allows to include concerns about income inequality and social mobility in a transparent and explicit way. The two core ideas of our characterization are a requirement of consistency of the social evaluation for the addition of an income source with the same mobility structure, and the idea that the effect of social mobility vanishes when there is no inequality in the society or a subgroup thereof. We obtain a multiperiod rank-dependent social evaluation function that additionally gives a prominent place to the notion of social status in this dynamic context. We discuss various special cases that belong to the characterized family of characterized social evaluation functions, we link them to the existing literature on income inequality or social mobility measurement, and finally present a decomposition in intuitive and meaningful components: average income, income inequality, social status, and mobility.

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# 1 Introduction

Recently, the relation between income inequality and social mobility has been widely discussed in policy debates and academic writings. In his speech in January 2012, for instance, Alan Krueger, chairman of the Council of Economic Advisers, introduced the so-called "Great Gatsby curve". The curve describes the association between income inequality and social mobility for various countries. The association is found to be negative. More unequal countries tend to be less mobile and vice versa.[1] The "Great Gatsby curve" and the relation between income inequality and social mobility received wide media coverage, particularly in the United States, where the opinion is widespread that high income inequality may be more acceptable in a society with a high level of social mobility.[2] Subsequently, concerns about increasing income inequality and low social mobility have been picked up by policy makers and were a central theme in President Obama's State of the Union in 2014, for instance.

In contrast with the recent *descriptive* work on the association between income inequality and social mobility, remarkably little is known about the question how to make *normative* social evaluations that involve multiple periods and include explicitly concerns about income inequality and social mobility (notable exceptions are by Shorrocks (1978) and Kanbur and Stiglitz (1986)). Such an encompassing normative framework would be very useful, however. It allows to compare streams of income distributions while incorporating in a flexible and explicit way different value judgments on the extent to which income inequality indeed may be more acceptable in a society with a high level of social mobility. Most existing normative work, however, focusses on either income inequality in one period (the literature in the wake of Dalton (1920), Atkinson (1970), Kolm (1976) and Sen (1973)) or on social mobility in isolation (with seminal contributions by Atkinson (1981), Dardanoni (1993), D'Agostino and Dardanoni (2009))[3].

In this paper, we consider the problem of making social evaluations of multiperiod income profiles. A multiperiod income profile consists of two building blocks: an instantaneous income distribution for each period, and a mobility (permutation) matrix that captures how the positions of the individuals change over time. Multiperiod income profiles are becoming increasingly available with the recent presence of (long) panels of income data (see Gottschalk and Spolaore (2002), Kopczuk et al. (2010) and Chetty et al. (2014), amongst others).

We obtain a class of rank-dependent social evaluation functions, which are explicitly sensitive to information about the relative position of individuals in each period as well as the reranking across the different periods. An important component of the

[1] The studies by Andrews and Leigh (2009) and Corak (2013) provide a detailed analysis. These findings extend the earlier comparison of mobility between Sweden and the United States by Bjorklund and Jäntti (1997).

[2] See Jencks and Tach (2006), who discuss results from the International Social Justice Project.

[3] Fields and Ok (1999) provide a survey.

social evaluation function will therefore reflect the social status of the individuals in an instantaneous as well as dynamic perspective.

The core results of this paper characterize in a novel and—we believe—appealing way a multivariate Gini social evaluation function and generalize thereby the one-dimensional work of Weymark (1981) and Yaari (1987). Our work is related to earlier work on multivariate extensions of the Gini social evaluation function (see Gajdos and Weymark (2005), Chew and Sagi (2012) and Decancq and Lugo (2012), for instance), and to work on the Gini correlation coefficient (see Schechtman and Yitzhaki (2003), Wodon and Yitzhaki (2003), amongst others).

The paper is structured as follows. Section 2 introduces the notation and the central notion of a multiperiod income profile. Section 3 presents the first main result that requires consistency in social evaluations for addition of an income source to two multiperiod income profiles with the same mobility matrix. In Section 4, we present the second main result building on the idea that the effect of social mobility vanishes when there is no inequality in the society or a subgroup thereof. Section 5 introduces explicitly concerns about mobility and inequality in the framework and studies the restrictions on the obtained family of social evaluation functions. Section 6 discusses various special cases of our general framework and some decompositions of the social evaluation measure in three elementary and interpretable building blocks, namely average income, income inequality and social mobility. Section 7 concludes.

# 2 Notation

The following notation will prove to be useful to make social evaluations of streams of distributions of a continuous economic variable (e.g., income or consumption) for a finite number of individuals (or generations) through discrete time.

Let $\mathcal{N} \subset \mathbb{N}$ denote the set of $n$ individuals. We assume that $n$ is fixed [in the current version of the paper]. We denote the set of periods $\mathcal{T}$. For expositional clarity we assume [in the current version of the paper] that $\mathcal{T} = \{1, 2\}$. Let $x_i^t$ denote the income of individual $i \in \mathcal{N}$ in period $t \in \mathcal{T}$.

Central in our work is the notion of a *multiperiod income profile* $X = (X^1; X^2; P_X)$. Such a profile contains the instantaneous income distribution of each period $X^1$ and $X^2$ and the mobility matrix $P_X$. We discuss both components in detail.

The (instantaneous) *income distribution* for each period $t$ is a strictly ordered vector of incomes $X^t = \left(x_{[1]}^t, x_{[2]}^t, \ldots, x_{[n]}^t\right)$ with $x_{[1]}^t < x_{[2]}^t < \cdots < x_{[n]}^t$. $\mathcal{R}^n$ denotes the set of such ordered income vectors of size $n$. The function $p_X^t : \mathcal{N} \to \mathcal{N}$ maps individual $i \in \mathcal{N}$ on her position in the ordered vector of incomes $X^t$. As the income distribution is strictly ordered, each individual $i$ is assigned a unique position $p_X^t(i)$ and vice versa.

The (exchange) *mobility matrix* $P_X$ associated with multiperiod income profile $X$

is an $n \times n$ permutation matrix such that

$$\begin{cases} P_X(i_1, i_2) = 1 & \text{if } \exists i \in \mathcal{N} \text{ such that } p_X^1(i) = i_1 \text{ and } p_X^2(i) = i_2 \\ P_X(i_1, i_2) = 0 & \text{otherwise.} \end{cases}$$

The identity matrix $I$, for instance, reflects the comonotonic case of perfect statistical dependence where the position of all individuals in the instantaneous income distributions remains the same across all periods (a so-called perfectly immobile society).[4] Let $\mathcal{P}$ be the set of all permutation matrices $P$. $\mathcal{P}$ is a finite set with $n!$ elements.

Example 1 illustrates a simple multiperiod income profile.

**Example 1** *A society with three individuals with income pairs (1,5), (3,4), and (2,6) can be summarized by the multiperiod income profile* $X$

$$X = \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right).$$

The set of all multiperiod income profiles is denoted $\mathcal{X} = \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{P}$. The set of all multiperiod income profiles with the same mobility matrix $P_X$ is denoted $\mathcal{X}(P_X)$.

We assume that there exists a *social evaluation function* (SEF) $W : \mathcal{X} \to \mathbb{R}$ that maps each multiperiod income profile to the real line, so that a higher value can be interpreted as reflecting a socially preferred multiperiod income profile. The social evaluation function $W$ is moreover assumed to be a continuous function of its first two arguments. This property allows to approximate a weakly ordered income distribution arbitrarily close. Blackorby et al. (2001, Theorem 3) provide a representation result on a similar mixed domain of continuous and discrete variables discussing the conditions under which such a social evaluation function $W$ exists.[5] In the following sections we will impose additional requirements upon the social evaluation function $W$ that capture appealing properties.

# 3 Mobility preserving Independence

In order to derive the intertemporal SEF, we start by imposing an axiom that is placing restrictions on the comparison of multiperiod income profiles with the same mobility matrix. The axiom imposes some consistency for addition of an income source to two multiperiod income profiles with the same mobility matrix as long as

[4] A mobility matrix can be seen as a discrete copula density. On using copula's to model mobility structures see Bonhomme and Robin (2009).

[5] To be precise, Blackorby et al. (2001, Theorem 3) show that any reflexive, complete and transitive relation on $\mathcal{X}^*$ (the extension of $\mathcal{X}$ including weakly ordered instantaneous income distributions) that satisfies so-called unconditional continuity can be represented by a function $W : \mathcal{X}^* \to \mathbb{R}$ that is a continuous function of its first two arguments.

the added income source has also the same mobility matrix as the current multiperiod income profiles.

This is a "natural" extension of the univariate "weak independence of income source" property introduced by Weymark (1981) (see also Yaari (1987)). The single period independence property is at the core of all dual linear rank dependent representation models in welfare and risk analysis. Within the multiperiod setting the general specification that we propose is particularly attractive as it does not affect relevant information for our analysis, in fact it allows for intertemporal substitution, and at the same time allows to take both instantaneous and intertemporal information on the positions into account.

**Axiom 1 (Mobility preserving Independence (M-IND))** *For all $X, Y, Z$ in $\mathcal{X}$ with $P_X = P_Y = P_Z$, $W(X) \geq W(Y) \Leftrightarrow W(X+Z) \geq W(Y+Z)$.*

By imposing this property to $W$, we obtain the following result which is a natural multivariate extension of the class of generalized Gini SEFs.

**Theorem 1** *For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, $W(\cdot)$ satisfies M-IND if and only if there exist functions $\omega^1_{P_X}(\cdot,\cdot)$ and $\omega^1_{P_X}(\cdot,\cdot)$ and an increasing and continuous function $V_{P_X}(\cdot)$ such that*

$$W(X) := V_{P_X}\left[\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^1_i + \sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^2_i\right]. \qquad (1)$$

As happens in the single period setting, the axiom imposes quite strong restrictions on the social indifference hyperplanes on $\mathcal{X}(P_X)$. They should be linear and parallel on each $\mathcal{X}(P_X)$. This rules out satiation and marginal decreasing returns of income, for instance. Even if incomes contribute linearly to welfare, as we will see below, there will be still room for concerns about inequality and mobility in the evaluation. These concerns can be captured by the shape of the normalization function $V_{P_X}$ and two sets of weights $\omega^1_{P_X}$ and $\omega^2_{P_X}$.

Intertemporal mobility represented by $P_X$ contributes in different ways to the evaluation since it affects the SEF in three different ways. First, the monotonic transformation function $V_{P_X}$ dependent on $P_X$. Thus the linear evaluation (within brackets) of a distribution can be transformed, in order to make across distribution comparisons, through a function whose shape depends on the mobility exhibited by the whole distribution. If we look instead at the linear evaluation function then $P_X$ plays a double role in affecting the weights $\omega^t$. Mobility has a direct effect by being an argument of the weights, but also covers and indirect effect since as $P_X$ changes the positional history of the individuals in the society are affected and therefore $p^1_X(i)$ and $p^2_X(i)$ may change for many individuals.

Even if the M-IND axiom is remarkably powerful in singling out a class of SEFs, its practical appeal is limited. The social planner needs to define a normalization

function $V_{P_X}$ and two sets of weights $\omega^1_{P_X}$ and $\omega^2_{P_X}$ for each of the $n!$ possible mobility matrices.

Moreover, it would be useful, also for practical and interpretative purposes, to derive a cardinal version of the SEF.

The intertemporal nature of the domain and the concern for mobility calls for the definition of a normalization property that should generalize those traditionally applied for single period evaluations. In the single period setting it is usually required to welfare measures to be cardinalized making use of the notion of equally distributed equivalent income, when all incomes are equally distributed the measure coincides with the average income. Here we consider two periods, moreover the mobility between periods of the individuals also plays a role. The normalization condition we proposes focuses on distributions that converge to equality in each period with the same income in both periods and with each individual that covers the same position in each period, that is the mobility matrix is an identity matrix that depicts the perfect immobility case.

The normalization axiom requires that when evaluating a perfectly immobile multiperiod income profile that converges to perfectly equal distributions in each period, then the welfare level should converge to a value proportional to $\mu$ with scale factor $\lambda$.

We define $\overline{X}^t$ the equal distribution at time $t$ such that all individuals receive the average income.

**Axiom 2 (Normalization (NORM))** *For all $X$ in $\mathcal{X}$ such that $P_X = I$ if $X^1 \to \overline{X}^1$ and $X^2 \to \overline{X}^2$ with $\overline{X}^1 = \overline{X}^2 = \mu \cdot 1^n$, then $W(X) \to \lambda \cdot \mu$ where $\lambda \geq 0$.*

The scale factor is introduced into the axiom because our evaluation considers multiperiod streams of equal incomes ad therefore this evaluation may take into account intertemporal consideration and may not coincide with the single period average income.

Another common axiom is a monotonicity property requiring that a rank-preserving (thereby also mobility preserving) increase in income of an individual at some point in time should increase the social evaluation.

**Axiom 3 (Monotonicity (MON))** *For all $X, Y$, such that $P_X = P_Y$ and $x^t_i = y^t_i$ for all $i, j \in \mathcal{N}$ except for $x^t_h = y^t_h + \varepsilon$ where $\varepsilon > 0$, $t \in \{1, 2\}$ it holds $W(X) > W(Y)$.*

# 4 Mobility and Equality

We add now a second central axiom. It is a natural and relatively weak axiom that requires that if incomes are equally distributed in each period then the mobility matrix is not relevant in the social evaluation. Consistently with our setting the

axiom is defined for distributions that converge to perfect equality and requires that irrespective of the mobility matrix for two intertemporal profiles whose single period distributions converge to the same averages the social evaluations should also converge to the same value.

**Axiom 4 (Irrelevance of Mobility for Equal distributions (IME))** *For all $X, Y$ in $\mathcal{X}$, if $X^t \to \overline{X}^t$ and $Y^t \to \overline{X}^t$ for all $t \in \mathcal{T}$, then $W(X) - W(Y) \to 0$.*

Note that IME does not require that the average incomes of the two periods should be the same. The normative content of the axiom is that mobility plays a role in intertemporal social evaluations if it is attached to some degree of inequality in the periods. Without inequality in the conditions of the individuals in at least one period there is "no label" effect of social status associated to changes in relative positions. As soon as some inequality arises in one period then changes in the position history of the individuals may play a role.

The additional axioms induce a set of restrictions on the characterizations in Theorem 1 leading to the following result.

**Theorem 2** *For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, $W(\cdot)$ satisfies M-IND, NORM, MON and IME if and only if there exist $\gamma_1, \gamma_2 > 0$ such that*

$$W(X) := \gamma_1 \cdot \sum_i w^1_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^1_i + \gamma_2 \cdot \sum_i w^2_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^2_i$$

*where:*

- $\gamma^1 + \gamma^2 = \lambda$,
- $\sum_i w^1_{P_X}(p^1_X(i), p^2_X(i)) = \sum_i w^2_{P_X}(p^1_X(i), p^2_X(i)) = 1$, *and*
- $w^t_{P_X}(p^1_X(i), p^2_X(i)) > 0$ *for all $i$, and all $t$.*

The obtained result is a cardinal two periods extension of the generalized Gini welfare index where every period income is weighted according to the position covered by each individual in each period. The weights can also differ between periods and moreover, the mobility matrix $P_X$ also affects the weights directly not only through the effects of the combinations of positions in each period.

The contribution of each individual condition in the overall evaluation is a weighted average of the incomes of both periods weighted according to the intertemporal factors $\gamma_1$ and $\gamma_2$, and the positional history weights $w^1_{P_X}(p^1_X(i), p^2_X(i))$ and $w^2_{P_X}(p^1_X(i), p^2_X(i))$. The contribution of each individual to the overall welfare does not depend only on his incomes and his positional history but also on the overall mobility experienced in the society. This is and interesting degree of flexibility in the social evaluation, for instance the weight attached to the transition of one individual across different

positions from one period to the other can be amplified or reduced whether these changes take place in a immobile or more mobile society. However, as we will show below, under a very reasonable condition, the direct role of the mobility matrix in the individual weights has to be discarded.

Inspired by the considerations behind IME a stronger version of the axiom can be considered by focussing not on the whole distribution but instead on subgroups of the population.

The axiom requires that if for a group of individuals incomes are equally distributed in both periods then the final evaluation should not depend on the association between their incomes. In other words no matter what is the copula between the incomes of these individuals the evaluation will be the same. If the group coincides with the whole population the axiom boils down to IME.

Let $\mathcal{N}_A$ denote the set of individuals belonging to subgroup A (the set may also coincide with the whole population), and similarly for subgroup B. The distribution of incomes in $X$ for subgroup A at time $t$ is denoted by $X_A^t$, with $\overline{X}_A^t$ representing an equal distribution. The distribution of a population decomposed into two groups A and B with associated intertemporal incomes $X_A$ and $Z_B$ is denoted by $W(X_A, Z_B)$. We will consider two distributions $X, Y \in \mathcal{X}$ partitioned into two subgroups A and B, such that for subgroup A $X_A^t \to \overline{X}_A^t$ and $Y_A^t \to \overline{X}_A^t$ for all periods $t$. That is the individuals in $X$ and $Y$ belonging to subgroup A tend to be identical in both periods even though their positions may differ in the two distributions. The remaining subgroup B is made of the same individuals in both populations, we denote their distribution by $Z_B$. Note that by construction the overall population mobility matrices $P_X$ and $P_Y$ may differ because of switches in the positions of the individuals in subgroup A or eventually these switches may involve also individuals in subgroup B if their incomes coincide with $\overline{X}_A^t$ for some $t$.

**Axiom 5 (Irrelevance of Mobility for Equal Groups (IMEG))** *For all $X, Y$ in $\mathcal{X}$, for all subgroups A and B with $\mathcal{N}_A \cup \mathcal{N}_B = \mathcal{N}$, if $X_A^t \to \overline{X}_A^t$ and $Y_A^t \to \overline{X}_A^t$ for all $t$, then $W(X_A, Z_B) - W(Y_A, Z_B) \to 0$.*

In order to illustrate the axiom we consider some examples with 4 individuals. In the first example the incomes of the individuals that cover the second and third position in the first period and the last two positions in the second period become almost equal in each period, and they converge respectively to $\bar{x}_A^1$ in the first period and to $\bar{x}_A^2$ in the second period. The axiom IMEG requires that the social evaluation is unaffected if the association between the incomes of these two individuals is changed. This is precisely the case if we consider distribution $(\overline{Y}_A, Z_B)$ where the position of the two individuals are permuted w.r.t. $(\overline{X}_A, Z_B)$. We illustrate the distributions by placing at the borders of the mobility matrix the vectors of incomes of the two periods where rows are associated to the first period and columns to the second period. To clarify the notation, for instance $z_{B1}^1$ denotes the lower income (across the overall

population) at time 1 that is also associated to an individual in group B; $\bar{x}^2_A$ instead denotes the average income for subgroup A in period 2.

$$(\overline{X}_A, Z_B) = \begin{array}{c|cccc} & z^2_{B1} & z^2_{B2} & \bar{x}^2_A & \bar{x}^2_A \\ z^1_{B1} & 0 & 1 & 0 & 0 \\ \bar{x}^1_A & 0 & 0 & 0 & \mathbf{1} \\ \bar{x}^1_A & 0 & 0 & \mathbf{1} & 0 \\ z^1_{B4} & 1 & 0 & 0 & 0 \end{array} ; (\overline{Y}_A, Z_B) = \begin{array}{c|cccc} & z^2_{B1} & z^2_{B2} & \bar{x}^2_A & \bar{x}^2_A \\ z^1_{B1} & 0 & 1 & 0 & 0 \\ \bar{x}^1_A & 0 & 0 & \mathbf{1} & 0 \\ \bar{x}^1_A & 0 & 0 & 0 & \mathbf{1} \\ z^1_{B4} & 1 & 0 & 0 & 0 \end{array} .$$

By comparing $(\overline{Y}_A, Z_B)$ to $(\overline{X}_A, Z_B)$ can be noted that the mobility matrix of $(\overline{X}_A, Z_B)$ differs from the one of $(\overline{Y}_A, Z_B)$ by a swap of positions (see the boldface 1s in the two matrices). This operation is going to increase the dependence between the two periods distributions in $(\overline{Y}_A, Z_B)$ in terms of positions, in fact in $(\overline{Y}_A, Z_B)$ the two interested individuals can be ranked unanimously in terms of the income distribution in both periods.

In the former example the axiom has an effect for individuals that cover adjacent positions in both periods. But it is possible do devise distributions where the axiom may impose restrictions also in situations where it does not exist a set of individuals covering adjacent positions. Consider for instance the copula/mobility matrix in distribution $(\overline{X}'_A, Z'_B)$ where

$$(\overline{X}'_A, Z'_B) = \begin{array}{c|cccc} & z^2_{B1} & z^2_{B2} & \bar{x}^2_A & \bar{x}^2_A \\ z^1_{B1} & 0 & \mathbf{1} & 0 & 0 \\ \bar{x}^1_A & 0 & 0 & 0 & \mathbf{1} \\ \bar{x}^1_A & \mathbf{1} & 0 & 0 & 0 \\ z^1_{B4} & 0 & 0 & \mathbf{1} & 0 \end{array} .$$

In this case there are no pairs of individuals covering adjacent positions in the two periods distributions. The axiom may still have some bite also in this case. For instance suppose that $z^2_{B1} \longrightarrow z^2_{B2} \longrightarrow \bar{x}^2_A$ and $z^1_{B4} \longrightarrow \bar{x}^1_A$, that is, all the incomes in the second period converge to equality with value $\bar{x}^2_A$ while the three higher incomes in the first period also converge to equality with value $\bar{x}^1_A$. If this is the case, then any permutation in the positions involving the three first period richest individuals won't affect the evaluation, therefore for instance the evaluation of $(\overline{X}'_A, Z'_B)$ and $(\overline{Y}'_A, Z'_B)$ represented below

$$(\overline{X}'_A, Z'_B) = \begin{array}{c|cccc} & \bar{x}^2_A & \bar{x}^2_A & \bar{x}^2_A & \bar{x}^2_A \\ z^1_{B1} & 0 & 1 & 0 & 0 \\ \bar{x}^1_A & 0 & 0 & 0 & \mathbf{1} \\ \bar{x}^1_A & \mathbf{1} & 0 & 0 & 0 \\ \bar{x}^1_A & 0 & 0 & \mathbf{1} & 0 \end{array} ; (\overline{Y}'_A, Z'_B) = \begin{array}{c|cccc} & \bar{x}^2_A & \bar{x}^2_A & \bar{x}^2_A & \bar{x}^2_A \\ z^1_{B1} & 0 & 1 & 0 & 0 \\ \bar{x}^1_A & 0 & 0 & \mathbf{1} & 0 \\ \bar{x}^1_A & 0 & 0 & 0 & \mathbf{1} \\ \bar{x}^1_A & \mathbf{1} & 0 & 0 & 0 \end{array}$$

should converge to the same value. Note that in this case the differences in the mobility matrices between the two distributions (highlighted by boldface 1s) involve three individuals.

By applying the stronger version IMEG instead of IME we obtain the following result.

**Theorem 3** *For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, $W(\cdot)$ satisfies M-IND, NORM, MON and IMEG if and only if there exist $\gamma_1, \gamma_2 > 0$ such that*

$$\begin{aligned} W(X) \;\; : \;\; &= \gamma_1 \cdot \sum_i \left[\alpha^1(p_X^1(i)) + \beta^1(p_X^2(i))\right] \cdot x_i^1 \\ &+ \gamma_2 \cdot \sum_i \left[\alpha^2(p_X^1(i)) + \beta^2(p_X^2(i))\right] \cdot x_i^2 \end{aligned} \tag{2}$$

*where:*

- $\gamma^1 + \gamma^2 = \lambda$,
- $\sum_i \alpha^t(p_X^1(i)) + \sum_i \beta^t(p_X^2(i)) = 1$ *for all $t$, and*
- $\alpha^t(p_X^1(i)) + \beta^t(p_X^2(i)) > 0$ *for all $i$, and all $t$.*

The role of IMEG is to clarify that the mobility information in $P_X$ should not have a direct impact on the weights. Moreover, in combination with the normalization of the weights to 1 in each period, the IMEG property imposes that each period weight $w^t_{P_X}(p_X^1(i), p_X^2(i))$ should be additively decomposed into two weights $\alpha^t(p_X^1(i))$ and $\beta^t(p_X^2(i))$ that are separately taking into account the positions covered respectively in the first and second period.

## 4.1 Transfer and Correlation axioms

We consider the representation in Theorem 3 our core result. So far we have not introduced in the social evaluation any concern for inequality reduction or mobility preference. We will investigate here what restrictions are imposed on the weighting functions $\alpha^t(\cdot)$ and $\beta^t(\cdot)$ by applying very general notions of inequality aversion and mobility preference.

In order to formalize the concept of inequality aversion we first introduce the notion of a *multiperiod Pigou-Dalton transfer*.

**Definition 1 (Multiperiod Pigou-Dalton (MPD))** *For all $X, Y$ in $\mathcal{X}$ with $P_X = P_Y$, $X$ is obtained from $Y$ by a multiperiod Pigou-Dalton transfer, if there are $k, l \in \mathcal{N}$ such that for all $t \in \mathcal{T}$:*

1. $x_i^t = y_i^t$ *for all* $i \neq k, l$,
2. $x_k^t + x_l^t = y_k^t + y_l^t$,
3. $y_k^t > x_k^t > x_l^t > y_l^t$.

The definition of the multiperiod Pigou-Dalton transfer identifies individual $k$ that is richer than individual $l$ in both periods and implements a rank preserving transfer that does not affect the position of all individuals in both periods from individual $k$ to individual $l$. This transfer may take place in any of the two periods or in each of them.

We require that a multiperiod Pigou-Dalton transfer lead to a (weak) social improvement.

**Axiom 6 (Multiperiod Inequality Aversion (MIA))** *For all $X, Y$ in $\mathcal{X}$ with $P_X = P_Y$, If $X$ can be obtained from $Y$ by a finite sequence of MPD transfers, then $W(X) \geq W(Y)$.*

We impose the above condition only for transfers involving individuals that can be ranked in terms of income in each period. A traditional single period Pigou-Dalton transfer that focus only on the ranking of the individuals in one period in not an ethically attractive transformation. In fact there is no guarantee that the receiver of the transfer that is poorer in a period could be instead very rich in the other period. Without any a-priory on how to measure each individual intertemporal wellbeing the MPD property appears more appropriate.

In line with Atkinson (1981) and Dardanoni, (1993) we introduce in the evaluation the notion of mobility preference by imposing a correlation aversion axiom that keeps the margins $X^1$ and $X^2$ fixed (see also Epstein and Tanny (1980) and Tchen (1980)). We impose that a so-called "correlation increasing switch" leads to a social worsening.

**Definition 2 (Correlation Increasing Switch (CIS))** *For all $X, Y$ in $\mathcal{X}$ with $X^t = Y^t$ for all $t$ in $\mathcal{T}$, $Y$ is obtained from $X$ by a Correlation Increasing Switch, if there are $k, l \in \mathcal{N}$ such that for all $t \in \mathcal{T}$ :*

1. $x_i = y_i$ *for all* $i \neq k, l$,
2. $y_l^t = \min\{x_k^t, x_l^t\}$,
3. $y_k^t = \max\{x_k^t, x_l^t\}$.

After a correlation increasing switch, one individual (individual $k$) is better off than the other individual (individual $l$) in both periods. Such operation increases the degree of persistence across time and therefore reduces mobility. Any SEF that is mobility preferring should therefore reduce after a correlation increasing switch takes place. The corresponding principle is stated as follows:

**Axiom 7 (Mobility preference (MPREF))** *For all $X, Y \in \mathcal{X}$ with $X^t = Y^t$ for all $t \in \mathcal{T}$, if $Y$ can be obtained from $X$ by a finite sequence of CISs, then $W(X) \geq W(Y)$.*

Note that repeated application of CISs leads to a configuration where the maximum welfare given two fixed single period distributions $X^1, X^2$ is obtained if $P_X$ coincides with an antidiagonal permutation matrix with 1s only in the secondary diagonal. This mobility matrix identifies the case of maximum mobility in our setting, compared with perfect immobility obtained when $P_X$ coincides with the identity matrix.

Axioms MIA and MPREF impose restrictions on the weighting functions as highlighted in next theorem.

**Theorem 4** *For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, the SEF $W(\cdot)$ in Theorem 3 satisfies MIA and MPREF if and only if:*

- $\alpha^t(p^1_X(l)) + \beta^t(p^2_X(l)) \geq \alpha^t(p^1_X(k)) + \beta^t(p^2_X(k))$ *for all* $t \in \mathcal{T}$ *if* $p^1_X(l) < p^1_X(k)$ *and* $p^2_X(l) < p^2_X(k)$,
- $\alpha^2(\cdot)$ *is non-increasing in* $p^1_X(i)$ *and* $\beta^1(\cdot)$ *is non-increasing in* $p^2_X(i)$.

It follows that inequality aversion can be formalized by the fact that each period weight $\alpha^t(p^1_X(i)) + \beta^t(p^2_X(i))$ is non-increasing in the position of the individual in both periods. Note that by construction this property should hold by making comparisons with individuals that cover strictly better positions in both periods. The fact that we cannot keep fixed the position in one period implies that we cannot assume that both weights $\alpha^t(\cdot)$ and $\beta^t(\cdot)$ can be non-increasing. In fact in our setting it is not possible to have two individuals in the same position. Further restrictions on the shape of the single weights arise because of MPREF, in this case $\alpha^2(\cdot)$ and $\beta^1(\cdot)$ should not be increasing in the positions. In this case mobility concerns provide restrictions on the weights attached only on the "other period" positions, that is on the weight based on the position in period two used to weight the income in period 1 and on the weight based on the position in period 1 used to weight the income experienced in period 2.

Because of these restrictions it may well be the case that $\alpha^1(\cdot)$ and $\beta^2(\cdot)$ are increasing in the positions covered even if (1) and (2) in Theorem 4 hold.

**Example 2** *Consider a 2 individuals society, that can cover positions 1 or 2 in the two periods. Condition (1) in Theorem 4 requires that $\alpha^t(1) + \beta^t(1) \geq \alpha^t(2) + \beta^t(2)$ should hold for each $t$, while condition (2) requires that $\alpha^2(1) \geq \alpha^2(2)$ and $\beta^1(1) \geq \beta^1(2)$. In addition we have that $\sum_i \alpha^t(p^1_X(i)) + \sum_i \beta^t(p^2_X(i)) = 1$ for all $t$. Take the following set of weights represented by the vectors*

$$\begin{aligned} \alpha^1 &= (0.2, 0.3), \beta^1 = (0.4, 0.1), \\ \alpha^2 &= (0.4, 0.1), \beta^2 = (0.1, 0.4), \end{aligned}$$

*where the first element is the weight attached to the first position. Note that all the above conditions are satisfied, however weights $\alpha^1$ and $\beta^2$ are increasing in the position covered by the individual.*

*This results arises because $\alpha^t(p_X^1(i)) + \beta^t(p_X^2(i))$ may take different values as $t$ changes. If it is required that the sum is independent from $t$ it follows that $\alpha^1 + \theta = \alpha^2$ and $\beta^1 - \theta = \beta^2$ for some admissible real value of $\theta$. If $\alpha^2$ and $\beta^1$ are non-increasing then this is the case also for $\alpha^1$ and $\beta^2$. As a result also Condition (1) holds in this case.*

# 5 Special case and decomposition

By integrating the conditions singled out in Theorem 4 in the structure obtained in Theorem 3, we obtain a family of intertemporal social evaluation functions that are useful for empirical analysis. Furthermore, they make it possible to decompose social welfare in interpretable components: average income, instantaneous inequalities, social mobility across periods and changes in the social status of the individuals.

## 5.1 A special case

As the structure of the SEF remained quite general in Theorem 4, in this section we explore the implications arising from the application of simplifying assumptions. As we will show even under more restrictive conditions the family of SEFs retains desirable properties and sufficient degrees of generality to allow to disentangle interesting information on the evolution of the intertemporal income streams.

Our first verification is the consideration that inequality and mobility concerns in the social evaluation affect the way in which society values the intertemporal trade-off between the incomes of a given individual. In fact one can compute the Social Marginal Rate of Intertemporal Substitution (SMRSIS) for each individual $i$. The SMRIS compares the variations in incomes for the same individual in both periods that do not affect the SEF and can be formalized by:[6]

$$SMRIS_i := \frac{\frac{\partial W(X)}{\partial x_i^2}}{\frac{\partial W(X)}{\partial x_i^1}}.$$

Note that for our model we have

$$SMRIS_i := \frac{\gamma_2 \cdot \left[\alpha^2(p_X^1(i)) + \beta^2(p_X^2(i))\right]}{\gamma_1 \cdot \left[\alpha^1(p_X^1(i)) + \beta^1(p_X^2(i))\right]},$$

therefore SMRIS may depend on the individual position history. We may however separate the intertemporal individual evaluations from the social evaluations that concern mobility. We can for instance assume that the SMRIS is the same for all individuals. This implies that $\left[\alpha^2(p_X^1(i)) + \beta^2(p_X^2(i))\right] = \kappa \cdot \left[\alpha^1(p_X^1(i)) + \beta^1(p_X^2(i))\right]$

[6]Note that according to the formulation in Theorem 3 the SEF is differentiable in the income level of each individual.

for some $\kappa > 0$. By recalling that $\sum_i \alpha^t(p_X^1(i)) + \sum_i \beta^t(p_X^2(i)) = 1$ one obtains that $\kappa = 1$ and therefore $\alpha^t(p_X^1(i)) + \beta^t(p_X^2(i)) = \alpha(p_X^1(i)) + \beta(p_X^2(i))$ for all $t$. In addition as highlighted in the previous example we obtain that both $\alpha(\cdot)$ and $\beta(\cdot)$ should be non-increasing in $p_X^1(i)$ and $p_X^2(i)$ respectively.

The obtained SEF is therefore such that for all $X$ in $\mathcal{X}$:

$$W(X) := \sum_i \left[\alpha(p_X^1(i)) + \beta(p_X^2(i))\right] \cdot \left(\gamma_1 x_i^1 + \gamma_2 x_i^2\right) \tag{3}$$

where $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 = \lambda$, such that$\sum_i \alpha(p_X^1(i)) + \sum_i \beta(p_X^2(i)) = 1$ for all $t$, and $\alpha(p_X^1(i)) + \beta(p_X^2(i)) > 0$ for all $i$, and all $t$, and $\alpha(p_X^1(i))$ and $\beta(p_X^2(i))$ are non-increasing respectively in $p_X^1(i)$ and $p_X^2(i)$.

According to (3) the *individual discounted lifetime income* is evaluated considering two streams of positive and normalized positional weights that are non increasing in the positions covered in each period.

The SEF differs from the generalized Gini SEF applied to discounted lifetime income, because the weights are not necessarily associated with the position covered by each individual in the distribution of the discounted lifetime income. The two values coincide for instance in case of perfect immobility when each period positions are unchanged in time. In this case the SEF coincides with the weighted average (with weights $\gamma$'s) of the generalized Gini SEFs for each period distribution.

In our case the social evaluation coincides with a weighted average of the single period generalized Gini SEFs with weights $\alpha(\cdot)$ for period 1 and $\beta(\cdot)$ for period 2, and of the generalized Gini's where the incomes in one period are weighted according to the weights based on the positions covered in the other periods. It is the difference between these two extra terms and those related to the generalized Gini SEF that is introducing some mobility consideration in the$W(X) := \sum_i \left[\alpha(p_X^1(i)) + \beta(p_X^2(i))\right] \cdot (\gamma_1 x_i^1 + \gamma_2 x_i^2)$ evaluation.

In fact one may rewrite (3) as

$$\begin{aligned} W(X) \;:\; &= \sum_i \left[\alpha(p_X^1(i)) + \beta(p_X^1(i))\right] \cdot \gamma_1 x_i^1 + \sum_i \left[\alpha(p_X^2(i)) + \beta(p_X^2(i))\right] \cdot \gamma_2 x_i^2 \quad (4) \\ &+ \sum_i \left[\beta(p_X^2(i)) - \beta(p_X^1(i))\right] \cdot \gamma_1 x_i^1 + \sum_i \left[\alpha(p_X^1(i)) - \alpha(p_X^2(i))\right] \cdot \gamma_2 x_i^2 \quad (5) \end{aligned}$$

By construction the two summations in (5) are non-negative because the weights are non-increasing with the position of the individuals and therefore when the positions correspond to those in the period whose incomes are considered the weighted average of the incomes is minimized. As a result the terms in (4) identify the value of the SEF in the case of perfect immobility when the two periods positions are the same for each individual, while the terms in (5) measure the positive effect of mobility in the evaluation.

## 5.2 Decomposition

The interplay between single period evaluations and concern for mobility will become clearer if we further assume that the two periods positional weights are identical, that is $\alpha(p) = \beta(p) = 1/2 \cdot v(p)$, with $\sum_i v(p(i)) = 1$.

This additional assumption will simplify the exposition and allow to derive neat decomposition procedures for the overall SEF.

Let denote by $\mu(X^t)$ the average income of period $t$. The function

$$G_v(X^t) := \sum_i v(p_X^t(i)) \cdot \left( \frac{\mu(X^t) - x_i^t}{\mu(X^t)} \right)$$

denotes the relative Generalized *Gini index of inequality* [with weights $v(p)$] for the incomes of period $t$. The index is obtained when the incomes of each period are weighted taking into account the positions covered in that period. The weights $v(p)$ formalize the social evaluation concerns, and by construction are non-increasing in the position of the individuals. A related and less common index is

$$GC_v(X^t) = \sum_i v(p_X^{\mathcal{T}\setminus t}(i)) \cdot \left( \frac{\mu(X^t) - x_i^t}{\mu(X^t)} \right)$$

that represents the relative Generalized *Gini Concentration* index [with weights $v(p)$] for the incomes of period $t$. In this case the incomes of each period $t$ are weighted taking into account the *positions covered in the other period* $\mathcal{T}\setminus t$. As argued before this second term is related to the concept of mobility. The comparison of $G_v(X^t)$ and $GC_v(X^t)$ captures the distributive evaluations due to a "reranking effect" of mobility across periods. This approach is in line with the intuitions in Benabou and Ok (2001), and the analysis in Schechtman and Yitzhaki (2003) and Woodon and Yitzhaki (2003).

**Decomposition 1.** By referring to (3) and assuming $\alpha(p) = \beta(p) = 1/2 \cdot v(p)$, with $\sum_i v(p(i)) = 1$, noting that by construction $\mu(X^t) \cdot [1 - G_v(X^t)] = \sum_i v(p_X^t(i)) \cdot x_i^t$, we can derive the first social evaluation decomposition:

$$\begin{aligned} W(X) \;=\; & \gamma_1 \cdot \mu(X^1) \cdot \{1 - G_v(X^1) + \frac{1}{2} \cdot [G_v(X^1) - GC_v(X^1)]\} \\ & + \gamma_2 \cdot \mu(X^2) \cdot \{1 - G_v(X^2) + \frac{1}{2} \cdot [G_v(X^2) - GC_v(X^2)]\}, \end{aligned}$$

where $[G_v(X^t) - GC_v(X^t)] \geq 0$ quantifies the *Mobility Evaluation*. The term equals 0 for perfectly immobile society where positions are persistent across time.

The situation of each individual is compared symmetrically by looking at the effect of the change in the position from the first to the second period and the effect from the second period to the first. The first effect is weighted according to the incomes

in the first period while the second is taking into account the incomes levels of the second period.

In each period the negative inequality effect $G_v(X^t)$ is mitigated by the non-negative mobility effect $G_v(X^t) - GC_v(X^t)$.

The existence of no inequality in one period [say for instance period 1] eliminates the mobility effect of that period since $G_v(X^1) \to 0$ and $GC_v(X^1) \to 0$ but does not eliminate the overall mobility effect that can be captured by $G_v(X^2) - GC_v(X^2)$ if the second period distribution is unequal.

**Decomposition 2.** Another equivalent formulation can be derived by separating completely the mobility effect from the single period inequality and average income effects. Under the same assumptions as above we can write

$$\begin{aligned} W(X) &= \gamma_1 \cdot \mu(X^1) \cdot [1 - G_v(X^1)] + \gamma_2 \cdot \mu(X^2) \cdot [1 - G_v(X^2)] \\ &\quad + \frac{1}{2} \cdot \sum_i \left[ v(p_X^2(i)) - v(p_X^1(i)) \right] \cdot \{\gamma_2[\mu(X^2) - x_i^2] - \gamma_1[\mu(X^1) - x_i^1]\}. \end{aligned}$$

The component $\frac{1}{2} \cdot \sum_i [v(p_X^2(i)) - v(p_X^1(i))] \cdot \{\gamma_2[\mu(X^2) - x_i^2] - \gamma_1[\mu(X^1) - x_i^1]\}$ denotes the *Mobility social evaluation index* combining: (1) changes in positions [evaluated using weights $v(p)$] and (2) changes in (time adjusted) incomes w.r.t. the average mean of the period. An improvement for an individual in both dimensions or a worsening in both dimensions have a non negative effect on mobility.

The contribution of each individual to the mobility component in the social evaluation is 0 either if he/she does not experience any change in positions across periods, or if he/she has incomes that correspond to the each period means.

**Decomposition 3.** It might be worth to explore the implications of the application of the decompositions in the special case where $v(p)$ is linear in $p$ with

$$v(p) = \frac{1 + 2(n - p)}{n^2}.$$

If this is the case we obtain a formulation based on the *"classical" Gini index* $G(\cdot)$

The mobility welfare component then becomes:

$$\frac{1}{2} \cdot \sum_i \left[ p_X^1(i) - p_X^2(i) \right] \cdot \{\gamma_2[\mu(X^2) - x_i^2] - \gamma_1[\mu(X^1) - x_i^1]\}$$

with maximum value $\gamma_1 \cdot \mu(X^1) \cdot G(X^1) + \gamma_2 \cdot \mu(X^2) \cdot G(X^2)$ obtained when positions are "reversed" across periods, that is when the mobility matrix is antidiagonal.

In the special case of the classical Gini index, it is possible to construct a *relative Gini Mobility index* $M(X)$ obtained dividing the welfare mobility evaluation by its maximum value, that is

$$M(X) = \frac{\frac{1}{2} \cdot \sum_i [p_X^1(i) - p_X^2(i)] \cdot \{\gamma_2[\mu(X^2) - x_i^2] - \gamma_1[\mu(X^1) - x_i^1]\}}{\gamma_1 \cdot \mu(X^1) \cdot G(X^1) + \gamma_2 \cdot \mu(X^2) \cdot G(X^2)} \tag{6}$$

with $0 \leq M(X) \leq 1$. The index is "relative" i.e. it is scale invariant, in fact is not affected by scaling of incomes in each period. It relates to the Gini mobility index of Woodon and Yitzhaki (2003) and Schechtman and Yitzhaki (2003). The use of $M(X)$ allows to derive a neat and useful decomposition:

$$\begin{aligned} W(X) &= \gamma_1 \cdot \mu(X^1) \cdot [1 - G(X^1) \cdot (1 - M(X))] \\ &\quad + \gamma_2 \cdot \mu(X^2) \cdot [1 - G(X^2) \cdot (1 - M(X))]. \end{aligned}$$

Here each period intertemporal evaluation $\gamma_t \cdot \mu(X^t)$ is deflated according to inequality measured by the Gini index $G(X^t)$, however mobility measured precisely according to $M(X)$ in (6) can mitigate the inequality effect and lead to a welfare improvement.

# 6 Conclusions

In this paper we have studied the problem of characterizing a multiperiod social evaluation function. We introduced two core axioms capturing consistency of the social evaluation for the addition of an income source with the same mobility structure and imposing a vanishing effect of social mobility when there is no inequality in the society (or a subgroup thereof). In interplay with two standard axioms, these core axioms lead to an elegant and natural multiperiod extension of the rank-dependent social evaluation function. Imposing some mobility preference and a multiperiod Pigou Dalton requirement leads to further restrictions on the parameters. The obtained family contains various special cases that have been previously studied in the literature.

The resulting social evaluation function gets a natural interpretation in the multiperiod framework of this paper. Yet, alternative interpretations can be considered. The instantaneous income distributions $X^1$ and $X^2$ can be defined as the pre and post-tax income distributions, allowing an interpretation of our core result as a family of measure of tax progressivity which are sensitive to concerns of horizontal and vertical equity in a transparent way. Dropping the axiom monotonicity, on the contrary, offers a framework to analyze income growth while taking account of instantaneous egalitarian concerns and reranking. More generally, the multivariate rank-dependent social evaluation function can be used to study multidimensional well-being distributions with an eye for the inequality in each dimension and correlation across the dimensions.

Moreover, various extensions of the analysis can be considered within the presented multiperiod framework. First, we have assumed a fixed population size, which is obviously an untenable assumption for multiperiod social evaluations. Yet, the model can be extended in order to take variable population sizes into account by imposing a "population replication invariance" property that imposes that the social evaluation is not affected if all individuals in the society are replicated by a finite number of clones. In our framework with strictly ordered income distributions, we

assume that each individual is substituted by a finite number of clones whose incomes converge to that of the initial individual. In that case the SEF of the replicated society is required to converge to the value of the SEF for the original society. By adopting this property, the positional weights will depend on the relative position (normalized by the population size $n$) in the from $p_X^t(i)/n$. All measures considered can be adapted following the procedure used for the single period evaluation in order to incorporate this property. These modifications wont affect the essence of the results presented here. In addition, an extension beyond the two-period case towards a framework with more periods seems is a natural next step.

# References


Andrews, D., and A. Leigh (2009): "More inequality, less social mobility," *Applied Economics Letters*, 16(15), 1489–1492.

Atkinson, A. B. (1970): "On the measurement of inequality," *Journal of Economic Theory*, 2(3), 244–263.

——— (1981): "The measurement of economic mobility," in *Inkomensverdeling en Openbare Financien: Opstellen voor Jan Pen*, ed. by P. Eijgelshoven, and L. van Gemerden, pp. 9–24. Het Spectrum, Utrecht.

Benabou, R., and E. A. Ok (2001): "Social Mobility and the Demand for Redistribution: The Poum Hypothesis," *The Quarterly Journal of Economics*, 116(2), 447–487.

Björklund, A., and M. Jäntti (1997): "Intergenerational Income Mobility in Sweden Compared to the United States," *The American Economic Review*, 87(5), 1009–1018.

Blackorby, C., W. Bossert, and D. Donaldson (2001): "A representation theorem for domains with discrete and continuous variables," Discussion paper, Université de Montreal 2001 - 16.

Bonhomme, S., and J.-M. Robin (2009): "Assessing the Equalizing Force of Mobility Using Short Panels: France, 1990 - 2000," *The Review of Economic Studies*, 76(1), 63–92.

Chetty, R., N. Hendren, P. Kline, and E. Saez (2014): "Where is the Land of Opportunity? The Geography of Intergenerational Mobility in the United States," Discussion paper.

Chew, S. H., and J. S. Sagi (2012): "An inequality measure for stochastic allocations," *Journal of Economic Theory*, 147(4), 1517 – 1544, Inequality and Risk.

Corak, M. (2013): "Income Inequality, Equality of Opportunity, and Intergenerational Mobility," *Journal of Economic Perspectives*, 27(3), 79–102.

D'Agostino, M., and V. Dardanoni (2009): "The measurement of rank mobility," *Journal of Economic Theory*, 144(4), 1783 – 1803.

Dalton, H. (1920): "The measurement of the inequality of incomes," *The Economic Journal*, 30(119), 25–49.

Dardanoni, V. (1993): "Measuring Social Mobility," *Journal of Economic Theory*, 61(2), 372 – 394.

Decancq, K., and M. A. Lugo (2012): "Inequality of Wellbeing: A Multidimensional Approach," *Economica*, 79(316), 721–746.

Epstein, L. G., and S. M. Tanny (1980): "Increasing generalized correlation: a definition and some economic consequences," *The Canadian Journal of Economics / Revue canadienne d'Economique*, 13(1), 16–34.

Fields, G. S., and E. A. Ok (1999): "The measurement of income mobility: an introduction to the literature," in *Handbook of Income Inequality Measurement*, ed. by J. Silber, pp. 557–596. Kluwer Academic Press, Boston.

Gajdos, T., and J. A. Weymark (2005): "Multidimensional generalized Gini indices," *Economic Theory*, 26(3), 471–496.

Gottschalk, P., and E. Spolaore (2002): "On the Evaluation of Economic Mobility," *The Review of Economic Studies*, 69(1), 191–208.

Jencks, C., and L. Tach (2006): "Would equal opportunity mean more mobility?," in *Mobility and Inequality: Frontiers of Research from Sociology and Economics*, ed. by S. Morgan, D. Grusky, and G. Fields. Stanford University Press,, Stanford.

Kanbur, R., and J. Stiglitz (1986): "Intergenerational mobility and dynastic inequality," Discussion paper.

Kolm, S.-C. (1976): "Unequal inequalities. I," *Journal of Economic Theory*, 12(3), 416 – 442.

Kopczuk, W., E. Saez, and J. Song (2010): "Earnings Inequality and Mobility in the United States: Evidence from Social Security Data Since 1937," *The Quarterly Journal of Economics*, 125(1), 91–128.

Schechtman, E., and S. Yitzhaki (2003): "A Family of Correlation Coefficients Based on the Extended Gini Index," *The Journal of Economic Inequality*, 1(2), 129–146.

Sen, A. K. (1973): *On Economic Inequality*. Clarendon Press, Oxford.

Shorrocks, A. (1978): "Income inequality and income mobility," *Journal of Economic Theory*, 19(2), 376 – 393.

Tchen, A. (1980): "Inequalities for distributions with given marginals," *The Annals of Probability*, 8(4), 814–827.

Weymark, J. A. (1981): "Generalized Gini inequality indices," *Mathematical Social Sciences*, 1(4), 409–430.

Wodon, Q., and S. Yitzhaki (2003): "Inequality and the accounting period," *Economics Bulletin*, 4(36), 1–8.

Yaari, M. E. (1987): "The Dual Theory of Choice under Risk," *Econometrica*, 55(1), 95–115.

# A Appendix

## A.1 Proof of Theorem 1

**Theorem 1.** For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, $W : \mathcal{X} \to \mathbb{R}$ is a continuous function of its first two arguments that satisfies M-IND if and only if there exist functions $\omega^1_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ and $\omega^2_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ and an increasing and continuous function $V_{P_X} : \mathbb{R} \to \mathbb{R}$ such that:

$$W(X) = V_{P_X}\left[\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^1_i + \sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^2_i\right]. \qquad (7)$$

**Proof.** The proof hinges on Weymark (1981, Theorem 3).

$\Leftarrow$ One easily demonstrates that once equation (7) is satisfied for some $\omega^1_{P_X}, \omega^2_{P_X}$ and $V_{P_X}$, $W$ is indeed a continuous function of its first two arguments with M-IND satisfied.

$\Rightarrow$ To prove the converse implication, we partition $\mathcal{X}$ in $n!$ subsets $\mathcal{X}(P_X) = \{(Y^1; Y^2; P_Y) \in \mathcal{X} | P_Y = P_X\}$, one for each of the $P_X$ in $\mathcal{P}$. We proceed in three steps.

1. First, we show that the function $W$ is locally non-satiated in its first two arguments or indifferent everywhere, based on a similar argument as Weymark (1981, Lemma 2). Assuming that $W$ is locally satiated, it can be shown that $W(X') = W(X'')$ holds for all $X', X''$ in $\mathcal{X}(P_X)$. This latter case is trivially fulfilled by setting $\omega^1_{P_X} = \omega^2_{P_X} = 0 * 1^n$ in equation (7).

   We therefore consider the other case, i.e., that $W$ is locally non-satiated in its first two arguments.

2. Second, we show that for any $\delta$, the level set of $W$ on $\mathcal{X}(P_X)$ is convex (the level set being defined as $\{X \in \mathcal{X}(P_X) | W(X) = \delta\}$).

   We follow a similar argument as Weymark (1981, lines A.26-A.40), to obtain that

   $$W(X) = W(X') \Leftrightarrow W(rX) = W(rX').$$

   Now call $r = (1 - \alpha)$, so that

   $$W(X) = W(X') \Leftrightarrow W((1-\alpha)X) = W((1-\alpha)X').$$

   This implies that also $W(X) = W(\alpha X + (1-\alpha)X')$ by virtue of M-IND. This establishes that

   $$W(X) = W(\alpha X + (1-\alpha)X') = W(X') \text{ for all } \alpha \in [0, 1],$$

   hence convexity.

3. Third, the fact that $W$ is continuous in its first two arguments, is locally non-satiated in its first two arguments, and has convex level sets, implies that the level sets are linear on $\mathcal{X}(P_X)$ and hence that $W$ must satisfy equation (7).

■

## A.2 Proof of Theorem 2

In order to prove Theorem 2 we will first derive an intermediate result based on the application of the IME and NORM axioms and then we will move to prove Theorem 2 by further imposing the MON property.

**Theorem 5** *For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, $W : \mathcal{X} \to \mathbb{R}$ is a continuous function of its first two arguments that satisfies M-IND, IME and NORM if and only if and one of the following options apply:*

1. *there exist $\omega^1_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ and $\omega^2_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$, and $\gamma^1, \gamma^2 > 0$, such that*

$$W(X) = \gamma^1 \cdot \sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^1_i + \gamma^2 \cdot \sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^2_i, \quad (8)$$

   *where $\gamma^1 + \gamma^2 = \lambda$ and $[\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) = \sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i)) = 1$ or one of the two sets of weights sums to 0 and the other sums to 1],*

2. *there exist $\omega^1_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ and $\omega^2_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$, an increasing and continuous function $V_{P_X} : \mathbb{R} \to \mathbb{R}$ with $V_{P_X}(0) = 0$, such that*

$$W(X) = V_{P_X}\left[\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^1_i + \sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^2_i\right] \quad (9)$$

   *where $\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) = \sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i)) = 0$,*

3. *there exist $\omega^1_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ and $\omega^2_{P_X} : \mathcal{P} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$, and an increasing and continuous function $V_{P_X} : \mathbb{R} \to \mathbb{R}$ with $V_{P_X}(0) = 0$ and $V_{P_X}(0) = V_I\left(x \cdot \frac{\sum_i \omega^1_I(p^1_X(i), p^2_X(i))}{\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i))}\right)$, such that*

$$W(X) = V_{P_X}\left[\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^1_i + \sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i)) \cdot x^2_i\right] \quad (10)$$

   *where $\sum_i \omega^1_{P_X}(p^1_X(i), p^2_X(i)) = -\sum_i \omega^2_{P_X}(p^1_X(i), p^2_X(i))$.*

**Proof.** Consider the result in Theorem 1. Note that the set$\mathcal{P}$ is finite of size $n!$, we then index the elements of the set with $k = 1, 2, ..., n!$, and set $k = 1$ if $P_X = I$.

Moreover, we adopt this indexing to denote the weights $\omega^1_{P_X}(p^1_X(i), p^2_X(i))$ as $\omega^1_k(i_1, i_2)$ where $k$ is the index associated with matrix $P_X$, and $(p^1_X(i), p^2_X(i))$ are denoted in short with $(i_1, i_2)$.

Consider axiom IME, let $\overline{X}^1 = \mu_1 \cdot 1^n$, and $\overline{X}^2 = \mu_2 \cdot 1^n$. Recalling the result in Theorem 1, it follows that IME requires that

$$\begin{aligned} & f_1\left(\sum_i \omega^1_1(i_1, i_2) \cdot \mu_1 + \sum_i \omega^2_1(i_1, i_2) \cdot \mu_2\right) \\ = \quad & f_k\left(\sum_i \omega^1_k(i_1, i_2) \cdot \mu_1 + \sum_i \omega^2_k(i_1, i_2) \cdot \mu_2\right) \end{aligned}$$

for all $k = 1, 2, ...,$ and for all $\mu_1, \mu_2$.

We can rewrite $\sum_i \omega^1_k(i_1, i_2) \cdot \mu_1 = \mu_1 \cdot \Omega^1_k$ where we have denoted $\Omega^1_k := \sum_i \omega^1_k(i_1, i_2)$ and analogously for all the other terms. It follows that IME requires that

$$f_1\left(\mu_1 \cdot \Omega^1_1 + \mu_2 \cdot \Omega^2_1\right) = f_k\left(\mu_1 \cdot \Omega^1_k + \mu_2 \cdot \Omega^2_k\right)$$

for all $k = 1, 2, ...,$ and for all $\mu_1, \mu_2 \in \mathbb{R}$.

Let $\mu_2 = 0$, we obtain

$$f_1\left(\mu_1 \cdot \Omega^1_1\right) = f_k\left(\mu_1 \cdot \Omega^1_k\right)$$

for all $k = 1, 2, ...,$ and for all $\mu_1 \in \mathbb{R}$, and analogously by letting $\mu_1 = 0$ we obtain

$$f_1\left(\mu_2 \cdot \Omega^2_1\right) = f_k\left(\mu_2 \cdot \Omega^2_k\right)$$

for all $k = 1, 2, ...,$ and for all $\mu_2 \in \mathbb{R}$.

Note that these conditions require (if $\mu_i = 0$) that $f_1(0) = f_k(0)$ for all $k = 1, 2, ...$

By setting $\mu_1 \cdot \Omega^1_k = x$, if $\Omega^1_k \neq 0$, one obtains

$$f_1\left(x \cdot \frac{\Omega^1_1}{\Omega^1_k}\right) = f_k(x)$$

for all $k = 1, 2, ...,$ and for all $x \in \mathbb{R}$, and similarly

$$f_1\left(x \cdot \frac{\Omega^2_1}{\Omega^2_k}\right) = f_k(x)$$

for all $k = 1, 2, ...,$ and for all $x \in \mathbb{R}$. It then follows that by construction $\frac{\Omega^2_1}{\Omega^2_k} = \frac{\Omega^1_1}{\Omega^1_k}$, that is $\Omega^2_k = \gamma \cdot \Omega^1_k$ for all $k = 1, 2, ...$ where $\gamma \in \mathbb{R}$ with $\gamma \neq 0$, is independent from $k$.

Note however that if $\Omega^t_1 = 0$ then $f_1(0) = f_k\left(\mu_t \cdot \Omega^t_k\right)$ for all $\mu_t \in \mathbb{R}$ and for all $k = 2, ..., n!$

This implies that either $\Omega_k^t = 0$ or $f_k(x)$ is constant for all $x \in \mathbb{R}$ which cannot be the case because $f_k$ is strictly increasing. It then follows that either $\Omega_k^t = 0$ for all $k$, or $\Omega_k^t \neq 0$ for all $k$.

Alternative solutions are therefore obtained if either $\Omega_k^1 = 0$ for all $k$ or $\Omega_k^2 = 0$ for all $k$, or both.

To summarize we have two cases.

*Case (i):* $\Omega_k^1 = 0$ and $\Omega_k^2 = 0$ for all $k$. In which case IME is satisfied if just $f_1(0) = f_k(0)$ for all $k = 1, 2, ...$

*Case (ii):* all the other cases. If at least $\Omega_k^t \neq 0$ for all $k$ then IME requires that $f_1\left(x \cdot \frac{\Omega_1^t}{\Omega_k^t}\right) = f_k(x)$ for all $k = 1, 2, ...$, and for all $x \in \mathbb{R}$.

Consider now axiom NORM, it requires that

$$f_1\left(\mu \cdot \Omega_1^1 + \mu \cdot \Omega_1^2\right) = \lambda \cdot \mu$$

for all $\mu \in \mathbb{R}$. We can now consider again the 2 cases illustrated above, (i) $\Omega_1^1 = \Omega_1^2 = 0$, and (ii) $\Omega_k^2 = \gamma \cdot \Omega_k^1$ for all $k = 1, 2, ...$, with $\gamma \in \mathbb{R}$ (that incorporates the case where $\Omega_k^2 = 0$). The case where $\Omega_k^1 = 0$ and $\Omega_k^2 \neq 0$ can be treated symmetrically.

Case (i) imposes that $f_1(0) = \lambda \cdot \mu$ for all $\mu \in \mathbb{R}$, therefore we have $\lambda = 0$ that combined to the result obtained above gives $f_1(0) = f_k(0) = 0$ for all $k = 1, 2, ...$, as specified in part II of the theorem.

Case (ii) requires that

$$f_1\left(\mu \cdot \Omega_1^1(1+\gamma)\right) = \lambda \cdot \mu.$$

Within the second case we should consider separately the situation where $\gamma = -1$. In fact if $\Omega_1^1 + \Omega_1^2 = 0$ then we obtain that $f_1(0) = \lambda \cdot \mu$, for all $\mu \in \mathbb{R}$, therefore we have $\lambda = 0$. This result is illustrated in part III of the theorem.

If $\gamma \neq -1$, that is $\Omega_1^1 + \Omega_1^2 \neq 0$, by setting $\mu \cdot \Omega_1^1(1+\gamma) = x \cdot \frac{\Omega_1^1}{\Omega_k^1}$ one obtains $\mu = \frac{x}{(1+\gamma)\cdot\Omega_k^1}$, it then follows that

$$\frac{\lambda}{(1+\gamma)} \cdot \frac{x}{\Omega_k^1} = f_1\left(x \cdot \frac{\Omega_1^1}{\Omega_k^1}\right) = f_k(x)$$

for all $k = 1, 2, ..$ and all $x \in \mathbb{R}$. Going back to the original notation, it then follows that

$$V_{P_X}(W_0(X)) = \lambda \cdot \frac{W_0(X)}{\Omega_k^1 + \Omega_k^2}$$

where $W_0(X) := \sum_i \omega_{P_X}^1(p_X^1(i), p_X^2(i)) \cdot x_i^1 + \sum_i \omega_{P_X}^2(p_X^1(i), p_X^2(i)) \cdot x_i^2$.

We now consider separately the cases where $\Omega_k^1$ and $\Omega_k^2$ are different from 0 and those where one of the two terms equals 0.

If $\Omega_k^1, \Omega_k^2 \neq 0$ then

$$\begin{aligned} V_{P_X}\left(W_0(X)\right) &= \lambda \cdot \left( \frac{\sum_i \omega_{P_X}^1(i_1,i_2) \cdot x_i^1}{(1+\gamma) \cdot \sum_i \omega_{P_X}^1(i_1,i_2)} + \frac{\sum_i \omega_{P_X}^2(i_1,i_2) \cdot x_i^2}{(1+1/\gamma) \cdot \sum_i \omega_{P_X}^2(i_1,i_2)} \right) \\ &= \frac{\lambda}{(1+\gamma)} \cdot \left( \frac{\sum_i \omega_{P_X}^1(i_1,i_2) \cdot x_i^1}{\sum_i \omega_{P_X}^1(i_1,i_2)} + \gamma \cdot \frac{\sum_i \omega_{P_X}^2(i_1,i_2) \cdot x_i^2}{\sum_i \omega_{P_X}^2(i_1,i_2)} \right) \\ &= \gamma_1 \cdot \sum_i w_{P_X}^1(i_1,i_2) \cdot x_i^1 + \gamma_2 \cdot \sum_i w_{P_X}^2(i_1,i_2) \cdot x_i^2 \end{aligned}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1, \gamma_2 \neq 0$ and $\sum_i w_{P_X}^t(i_1, i_2, ) = 1$ for $t = 1, 2$.

If $\Omega_k^2 = 0$ then

$$\begin{aligned} V_{P_X}\left(W_0\right) &= \lambda \cdot \frac{W_0}{\Omega_k^1} \\ &= \lambda \cdot \left( \frac{\sum_i \omega_{P_X}^1(i_1,i_2) \cdot x_i^1}{\sum_i \omega_{P_X}^1(i_1,i_2)} + \frac{\sum_i \omega_{P_X}^2(i_1,i_2) \cdot x_i^2}{\sum_i \omega_{P_X}^1(i_1,i_2)} \right) \\ &= \gamma_1 \cdot \sum_i w_{P_X}^1(i_1,i_2) \cdot x_i^1 + \gamma_2 \cdot \sum_i w_{P_X}^2(i_1,i_2) \cdot x_i^2 \end{aligned}$$

where $\gamma_2 \in \mathbb{R}$, $\gamma_1, \gamma_2 \neq 0$ and $\sum_i w_{P_X}^1(i_1,i_2) = 1$ while $\sum_i w_{P_X}^2(i_1,i_2) = 0$. An analogous result can be derived if $\Omega_k^1 = 0$,

These latter set of results are summarized in part I of the theorem. ■

We now move to prove Theorem 2 by adding the role of the MON axiom.

**Theorem 2 .** For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, $W(\cdot)$ satisfies M-IND, NORM, IME and MON if and only if there exist $\gamma_1, \gamma_2 > 0$ such that

$$W(X) := \gamma_1 \cdot \sum_i w_{P_X}^1(p_X^1(i), p_X^2(i)) \cdot x_i^1 + \gamma_2 \cdot \sum_i w_{P_X}^2(p_X^1(i), p_X^2(i)) \cdot x_i^2$$

where $\gamma^1 + \gamma^2 = \lambda$, $\sum_i w_{P_X}^1(p_X^1(i), p_X^2(i)) = \sum_i w_{P_X}^2(p_X^1(i), p_X^2(i)) = 1$, and $w_{P_X}^t(p_X^1(i), p_X^2(i)) > 0$ for all $i$, and all $t$.

**Proof.** Consider the results in Theorem 5. The results in part I in combination with axiom MON rule out the possibility that $\gamma_1, \gamma_2 < 0$ and also that for a given $t$ one obtains $\sum_i w_{P_X}^t(p_X^1(i), p_X^2(i)) = 0$. Moreover, each person weight should be strictly positive in each period.

In analogy the conditions in part II of Theorem 5 are incompatible with MON, similarly for those in part III that require that there exists a set of weights that sum to a negative value. If this is the case an increase of the same amount in all the incomes of one period reduces the evaluation thereby violating MON.

It can be then concluded that the appropriate representation is the one in part I of Theorem 5 where $\gamma^1 + \gamma^2 = \lambda$, $\sum_i w_{P_X}^1(p_X^1(i), p_X^2(i)) = \sum_i w_{P_X}^2(p_X^1(i), p_X^2(i)) = 1$, and $w_{P_X}^t(p_X^1(i), p_X^2(i)) > 0$ for all $i$, and all $t$. ■

## A.3 Proof of Theorem 3

**Theorem 3.** For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, $W(\cdot)$ satisfies M-IND, NORM, IMEG and MON if and only if there exist $\gamma_1, \gamma_2 > 0$ such that

$$\begin{aligned} W(X) \;\; : \;\; &= \gamma_1 \cdot \sum_i \left[\alpha^1(p_X^1(i)) + \beta^1(p_X^2(i))\right] \cdot x_i^1 \\ &+ \gamma_2 \cdot \sum_i \left[\alpha^2(p_X^1(i)) + \beta^2(p_X^2(i))\right] \cdot x_i^2 \end{aligned}$$

where $\gamma^1 + \gamma^2 = \lambda$, $\sum_i \alpha^t(p_X^1(i)) + \sum_i \beta^t(p_X^2(i)) = 1$ for all $t$, and $\alpha^t(p_X^1(i)) + \beta^t(p_X^2(i)) > 0$ for all $i$, and all $t$.

**Proof.** Consider the result in Theorem 2. We proceed first by proving in part A of the proof that IMEG implies that all the weights $w_P^t(h,j)$ are independent from $P$ for any period $t$ and position pairs $(h,j)$.

We then show, in part B of the proof, how these restrictions affect each period weighting functions considering that in addition it should hold

$$\sum_i w_{P_X}^1(p_X^1(i), p_X^2(i)) = \sum_i w_{P_X}^2(p_X^1(i), p_X^2(i)) = 1.$$

For ease of exposition in the proof we will make explicit the copula/mobility matrix as an argument of the weighting functions, that is for instance we will write $w^t(h,j,P)$ instead of $w_P^t(h,j)$.

*Part A.*

Consider first all distributions such that an individual covers position $(1,j)$ for $j \in \{1,2,...,n\}$. Conditional on the positions of this individual we take into account all the $(n-1)!$ copula/mobility matrices obtained by permuting the positions of all the other individuals. The set of all such matrices is denoted by $\mathcal{P}(1,j)$.

Let all the individual incomes of the second period converge to 0, and let all the incomes of the first period, for all the individuals from position 2 to $n$, converge to $\mu$.

Denote the obtained distribution as $(X_A, Z_B)$ [in short we will denote it with $X$] where $N_A$ includes only the individual in position $(1,j)$. Without loss of generality assume that the index of the individual covering positions $(1,j)$ is set $i = 1$, that is coincides with her position at time 1. The welfare evaluation of this distribution tends to

$$W(X) = \gamma_1 \cdot \left[ w^1(1,j,P_X) \cdot x_1^1 + \mu \cdot \sum_{i \neq 1} w^1(p_X^1(i), p_X^2(i), P_X) \right].$$

Recall that $\sum_i w^1(p_X^1(i), p_X^2(i), P_X) = 1$ for any $P_X$, therefore $\sum_{i \neq 1} w^1(p_X^1(i), p_X^2(i), P_X) = 1 - w^1(1,j,P_X)$ for any $P_X \in \mathcal{P}(1,j)$. It then follows that

$$W(X) = \gamma_1 \cdot \left[ w^1(1,2,P_X) \cdot \left(x_1^1 - \mu\right) + \mu \right]$$

for $P_X \in \mathcal{P}(1,j)$, and $\gamma_1 > 0$.

According to IMEG this is the case also for all distributions $X'$ such that $P_{X'} \in \mathcal{P}(1,j)$ and the income distributions converge in each period to those in distribution $X$. We have that

$$W(X') = \gamma_1 \cdot \left[w^1(1,j,P_{X'}) \cdot \left(x_1^1 - \mu\right) + \mu\right]$$

for $P_X \in \mathcal{P}(1,j)$, and $\gamma_1 > 0$, from which in accordance with IMEG it then follows that $W(X) - W(X') \to 0$ which implies that

$$\begin{aligned} & \gamma_1 \cdot \left[w^1 j1,2,P_X) \cdot \left(x_1^1 - \mu\right) + \mu\right] - \gamma_1 \cdot \left[w^1(1,j,P_{X'}) \cdot \left(x_1^1 - \mu\right) + \mu\right] \\ = \; & \gamma_1 \cdot \left[w^1(1,j,P_X) - w^1(1,j,P_{X'})\right] \cdot \left(x_1^1 - \mu\right) \to 0 \end{aligned}$$

for all $P_X, P_{X'} \in \mathcal{P}(1,j)$. As a result we obtain

$$w^1(1,j,P_X) = w^1(1,j,P_{X'}) \tag{11}$$

for all $P_X$, $P_{X'} \in \mathcal{P}(1,j)$.

We now apply an analogous argument for the individuals in position $(2,j)$. We again assume that all the incomes of the second period converge to 0, and let all the incomes of the first period, for all the individuals from position 3 to $n$, converge to $\mu$. Without loss of generality we assume that the individual that covers position 1 in the first period is covering position $\ell$ in the second period where by construction $\ell \in \{1,2,...,n\}$ with $\ell \neq j$. Again we assume here that the two individuals in the first two positions in the first period are indexed with $i = 1$ and $i = 2$.

By adopting a similar notation as done above we obtain

$$\begin{aligned} W(X) &= \gamma_1 \cdot \left[w^1(1,\ell,P_X) \cdot x_1^1 + w^1(2,j,P_X) \cdot x_2^1 + \mu \cdot \sum_{i \neq 1,2} w^1(p_X^1(i), p_X^2(i), P_X)\right] \\ &= \gamma_1 \cdot \left[\mu + w^1(1,\ell,P_X) \cdot \left(x_1^1 - \mu\right) + w^1(2,j,P_X) \cdot \left(x_2^1 - \mu\right)\right], \end{aligned}$$

where the second formula is derived by considering that $\sum_{i \neq 1,2} w^1(p_X^1(i), p_X^2(i), P_X) = 1 - w^1(1,\ell,P_X) - w^1(2,j,P_X)$.

According to IME the above welfare evaluation should be the same for all distributions $X'$ such that $P_{X'} \in \mathcal{P}(2,j)$ where the income distributions converge in each period to those in distribution $X$. We have therefore that

$$W(X') = \gamma_1 \cdot \left[\mu + w^1(1,\ell,P_{X'}) \cdot \left(x_1^1 - \mu\right) + w^1(2,j,P_{X'}) \cdot \left(x_2^1 - \mu\right)\right]$$

for $P_{X'} \in \mathcal{P}(2,j)$, and $\gamma_1 > 0$, from which in accordance with IMEG it follows that $W(X) - W(X') \to 0$, these considerations imply that

$$[(w^1(1,\ell,P_X) - w^1(1,\ell,P_{X'})] \cdot \left(x_1^1 - \mu\right) + [w^1(2,j,P_X) - w^1(2,j,P_{X'})] \cdot \left(x_2^1 - \mu\right) \to 0$$

for all $P_X, P_{X'} \in \mathcal{P}(2,j)$, for any $\ell \in \{1,2,...,n\}$ with $\ell \neq j$, for all $(x_1^1 - \mu), (x_2^1 - \mu) \in \mathbb{R}$ such that $x_1^1 < x_2^1$.

Recall that according to (11) $w^1(1,\ell,P_X) = w^1(1,\ell,P_{X'})$ for all $P_X, P_{X'} \in \mathcal{P}(1,\ell)$ and therefore this is also the case for the subsets of the set of permutation matrices where a second individual covers position $(2,j)$. It then follows that $[w^1(2,j,P_X) - w^1(2,j,P_{X'})] \cdot (x_2^1 - \mu) \to 0$ for $(x_2^1 - \mu) \in \mathbb{R}$, thereby requiring that

$$w^1(2,j,P_X) = w^1(2,j,P_{X'})$$

for all $P_X, P_{X'} \in \mathcal{P}(2,j)$.

By induction with repeated application of these arguments it can be proved that

$$w^1(h,j,P_X) = w^1(h,j,P_{X'}) = v^1(h,j) \tag{12}$$

for all $P_X, P_{X'} \in \mathcal{P}(h,j)$. Moreover, by adapting the proof technique by substituting period 1 with period 2 one can prove the same independence result also for the weights of period 2.

If all weights are independent from the associated copula matrix then it is immediate that all weights $w^t(p_X^1(i), p_X^2(i), P_X)$ can be written as $v^t(p_X^1(i), p_X^2(i))$

*Part B.*

We now move to prove the explicit restrictions induced by the above condition on the set of weights. We will consider explicitly the first period weights $w^1$, analogous results can be derived for the second period weights also. From (11) we have that all weights $w^1(p_X^1(i), p_X^2(i), P_X)$ can be written as $v^1(h,j)$ where $h = p_X^1(i), j = p_X^2(i)$.

The condition $\sum_i w^1(p_X^1(i), p_X^2(i), P_X) = 1$ for any $P_X \in \mathcal{P}$ can then be rewritten as $\sum_{(h,j)\in P_X} v^1(h,j) = 1$ where $(h,j) \in P_X$ denotes all pairs of positions $(h,j)$ such that $P_X(h,j) = 1$.

Given that $v^1(h,j)$ does not depend on $P_X$ we have that any permutation of the second period positions between two individuals should not affect the overall sum (that in turns should equal to 1) if we keep fixed the positions (in both periods) of all the other $n-2$ individuals. This argument holds for $n \geq 2$. For $n = 2$ it hods by definition because the set of all permutation copula matrices is obtained by permuting the second period positions of the two individuals.

Consider two individuals with positions $(r,s)$ and $(r',s')$ where $r \neq r'$ and $s \neq s'$, if we permute the second period positions of the individuals we obtain $(r,s')$ and $(r',s)$. Denote with $P$ the copula matrix where $P(r,s) = P(r',s') = 1$ and $P'$ the associated copula matrix where $P'(r,s') = P'(r',s) = 1$ and all the remaining individuals cover the same positions as in $P$.

We know that $\sum_{(h,j)\in P} v^1(h,j) = \sum_{(h,j)\in P'} v^1(h,j) = 1$, it follows that if we subtract from all the summations the weights of the individual that cover common positions we obtain

$$v^1(r,s) + v^1(r',s') = v^1(r,s') + v^1(r',s).$$

This condition should hold by construction for any $r \neq r'$ and $s \neq s'$. It can be rewritten as

$$v^1(r,s) - v^1(r,s') = v^1(r',s) - v^1(r',s'),$$

for any $r \neq r'$ and $s \neq s'$. Let $s' = s+1$ for $s \in \{1,2,...,n-1\}$. We then get

$$v^1(r,s) - v^1(r,s+1) = v^1(r',s) - v^1(r',s+1) = g^1(s),$$

for all $r, r' \in \{1,2,...,n\}$ and all $s \in \{1,2,...,n-1\}$. We then derive by construction that

$$v^1(r,s) - v^1(r,n) = g^1(s) + g^1(s+1) + ... + g^1(n-1).$$

Moreover, let $v^1(r,n) := \alpha^1(r) + g^1(n)$ [thereby considering also $s = n$ in addition to all the other values of $s$ taken into account in the previous condition] we then obtain

$$v^1(r,s) = g^1(s) + g^1(s+1) + ... + g^1(n-1) + g^1(n) + \alpha^1(r).$$

By letting $\beta^1(s) := g^1(s) + g^1(s+1) + ... + g^1(n-1) + g^1(n)$ we get:

$$v^1(r,s) := \alpha^1(r) + \beta^1(s), \tag{13}$$

for all $r \in \{1,2,...,n\}$ and all $s \in \{1,2,...,n\}$.

Recall the constraint $\sum_{(h,j)\in P} v^1(h,j) = 1$ for any $P \in \mathcal{P}$, it then follows that $\sum_{(h,j)\in P} \alpha^1(h) + \beta^1(j) = \sum_{h=1}^n \alpha^1(h) + \sum_{j=1}^n \beta^1(j) = 1$. Moreover, the condition $v^1(r,s) > 0$ then requires that $\alpha^1(r) + \beta^1(s) > 0$ for all $r \in \{1,2,...,n\}$ and $s \in \{1,2,...,n\}$.

Note that if (11) holds and $\sum_{h=1}^n \alpha^1(h) + \sum_{j=1}^n \beta^1(j) = 1$ all the required constraints are satisfied.

Similar arguments can be used to derive the restrictions on the second period weights $w^2$. ■

## A.4 Proof of Theorem 4

**Theorem 4.** For all $X = (X^1; X^2; P_X)$ in $\mathcal{X}$, the SEF $W(\cdot)$ in Theorem 3 satisfies MIA and MPREF if and only if

1. $\alpha^t(p_X^1(l)) + \beta^t(p_X^2(l)) \geq \alpha^t(p_X^1(k)) + \beta^t(p_X^2(k))$ for all $t \in \mathcal{T}$ if $p_X^1(l) < p_X^1(k)$ and $p_X^2(l) < p_X^2(k)$.

2. $\alpha^2(\cdot)$ is non-increasing in $p_X^1(i)$ and $\beta^1(\cdot)$ is non-increasing in $p_X^2(i)$.

**Proof.** Consider a MPD of value $\delta > 0$ taking place in period 1 where individual $k$ is the donor and individual $l$ is the receiver. By construction it should be that $p_X^t(k) > p_X^t(l)$ for any $t$. According to (2) and following MIA it should hold that

$$\begin{aligned}\Delta W &= \gamma_1 \cdot \left[\alpha^1(p_X^1(k)) + \beta^1(p_X^2(k))\right] \cdot \left(x_k^1 - \delta\right) + \gamma_1 \cdot \left[\alpha^1(p_X^1(l)) + \beta^1(p_X^2(l))\right] \cdot \left(x_l^1 + \delta\right) \\ &\quad -\gamma_1 \cdot \left[\alpha^1(p_X^1(k)) + \beta^1(p_X^2(k))\right] \cdot x_k^1 - \gamma_1 \cdot \left[\alpha^1(p_X^1(l)) + \beta^1(p_X^2(l))\right] \cdot x_l^1 \\ &= \gamma_1 \cdot \left\{\left[\alpha^1(p_X^1(l)) + \beta^1(p_X^2(l))\right] - \left[\alpha^1(p_X^1(k)) + \beta^1(p_X^2(k))\right]\right\} \cdot \delta \geq 0\end{aligned}$$

That is

$$\alpha^1(p_X^1(l)) + \beta^1(p_X^2(l)) \geq \alpha^1(p_X^1(k)) + \beta^1(p_X^2(k)).$$

If and only if the above condition is satisfied for any $k, l \in \mathcal{N}$ such that for all $t \in \mathcal{T}$ $p_X^t(k) > p_X^t(l)$ the MIA property holds.

For a MPD transfer taking place in period 2 we analogously obtain

$$\alpha^2(p_X^1(l)) + \beta^2(p_X^2(l)) \geq \alpha^2(p_X^1(k)) + \beta^2(p_X^2(k))$$

for any $k, l \in \mathcal{N}$ such that for all $t \in \mathcal{T}$ $p_X^t(k) > p_X^t(l)$.

Consider now MPREF. Take the definition of CIS, to simplify the notation consider $y_l^t < y_k^t$ let $y_l^t = y^t$, $y_k^t = y^t + \delta^t$, where $\delta^t > 0$. Moreover, let $p_Y^t(l) = p_t$, $p_Y^t(k) = p_t'$ with $p_t' > p_t$ by construction. According to (2) and following MPREF it should hold that:

$$\begin{aligned}\Delta W &= \gamma_1 \cdot \left[\alpha^1(p_1) + \beta^1(p_2)\right] \cdot y^1 + \gamma_2 \cdot \left[\alpha^2(p_1) + \beta^2(p_2)\right] \cdot y^2 \\ &\quad +\gamma_1 \cdot \left[\alpha^1(p_1') + \beta^1(p_2')\right] \cdot (y^1 + \delta^1) + \gamma_2 \cdot \left[\alpha^2(p_1') + \beta^2(p_2')\right] \cdot (y^2 + \delta^2) \\ &\quad -\gamma_1 \cdot \left[\alpha^1(p_1) + \beta^1(p_2')\right] \cdot y^1 - \gamma_2 \cdot \left[\alpha^2(p_1) + \beta^2(p_2')\right] \cdot (y^2 + \delta^2) \\ &\quad -\gamma_1 \cdot \left[\alpha^1(p_1') + \beta^1(p_2)\right] \cdot (y^1 + \delta^1) - \gamma_2 \cdot \left[\alpha^2(p_1') + \beta^2(p_2)\right] \cdot y^2 \\ &\leq 0.\end{aligned}$$

After simplifying we get

$$\Delta W = \gamma_1 \cdot \beta^1(p_2') \cdot \delta^1 + \gamma_2 \cdot \alpha^2(p_1') \cdot \delta^2 - \gamma_2 \cdot \alpha^2(p_1) \cdot \delta^2 - \gamma_1 \cdot \beta^1(p_2) \cdot \delta^1 \leq 0,$$

that is the condition can be written as

$$\gamma_1 \cdot \beta^1(p_2') \cdot \delta^1 + \gamma_2 \cdot \alpha^2(p_1') \cdot \delta^2 \leq \gamma_2 \cdot \alpha^2(p_1) \cdot \delta^2 + \gamma_1 \cdot \beta^1(p_2) \cdot \delta^1$$

or in equivalently in more compact terms

$$-\gamma_2 \cdot \left[\alpha^2(p_1) - \alpha^2(p_1')\right] \cdot \delta^2 \leq \gamma_1 \cdot \left[\beta^1(p_2) - \beta^1(p_2')\right] \cdot \delta^1 \tag{14}$$

for all $\gamma_1, \gamma_2, \delta^1, \delta^2 > 0$.

By letting separately $\delta^1 \to 0$ and $\delta^2 \to 0$ we obtain respectively the following necessary conditions:

$$\begin{aligned}\alpha^2(p_1) - \alpha^2(p_1') &\geq 0, \\ \beta^1(p_2) - \beta^1(p_2') &\geq 0\end{aligned}$$

where $p_t' > p_t$. These two conditions are also sufficient in order (14) to hold for all $\gamma_1, \gamma_2, \delta^1, \delta^2 > 0$. ■