



**Inter-temporal Equivalence Scales Based on Stochastic Indifference Criterion: The Case of Poland**

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INTER-TEMPORAL EQUIVALENCE SCALES BASED ON STOCHASTIC  
INDIFFERENCE CRITERION: THE CASE OF POLAND

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Summary

In the paper, the stochastic equivalence scales (*SESs*) are applied to homogenisation of the heterogeneous population of households when analysing the sequence of expenditure distributions in several years. The *SES* is any function that transforms the expenditure distribution of a specific group of households in such a way that the resulting distribution is stochastically indifferent from the expenditure distribution of a reference group of households. The stochastic indifference is a weaker criterion than the *IB*, and it turns useful when *IB* fails. For inter-temporal comparisons of expenditure distributions, we chose one common reference group of households and estimate non-parametric and parametric *SESs* for all years and all groups of households, using the Polish Household Budget Surveys 2005-2010.

JEL codes I300, I320

KEYWORDS: Equivalence scale, stochastic dominance, estimation, expenditure distribution

## I. INTRODUCTION

The aim of the paper is to estimate the inter-temporal equivalence scale for Poland in the years 2005-2010, using micro-data on expenditure distributions. The idea underlying proposed scales consists in choosing one reference group of households (single adults in 2010) for all years and all selected groups of households. Assuming constant prices and ‘benchmark’ reference household group, such equivalence scales allows for homogenisation of expenditure distributions in two dimensions: across time and across household groups.

The stochastic equivalence scales (SES) are applied for aforementioned two-dimensional homogenisation of heterogeneous populations of households. The SESs is a new class of equivalence scales that bases on the concept of stochastic indifference. For the sake of convenience, we will refer other equivalence scales as to ‘conventional

Heterogeneity of household populations raises serious difficulties when addressing inequality, welfare and poverty. A two-step procedure is traditionally applied. In the first step, a reference household group is selected and then the actual expenditures of individual groups of households are adjusted by a chosen equivalence scale (Buhmann et al., 1988, Jones and O’Donnell, 1995). In the second stage, standard measures of inequality, welfare, and poverty are estimated on the basis of the adjusted distribution. These stages are usually perceived as independent.

There are two serious reasons why the two-stage procedure is unsatisfactory. First, conventional equivalence scales are not identifiable<sup>1</sup>, given consumer demand data. Second, two abovementioned stages are not independent.

The problems of equivalence scale identification is well-known in literature (see, in particular, Pollak and Wales, 1979, 1992, Blundell and Lewbel, 1991, Blackorby and Donaldson, 1993, and the surveys of Lewbel, 1997, Slesnick, 1998, Cowell and Mercader-Prats, 1999). It seems that there is no way of selecting an appropriate equivalence scale unless the assumption alternatively called *independence of base (IB)* (Lewbel, 1989) or *equivalence-scale exactness* (Blackorby and Donaldson, 1993) holds. Several papers have tested this assumption, but they ultimately reject it (Blundell and Lewbel, 1991, Blundell et al., 1998, Dickens et al., 1993, Pashardes, 1995, Gozalo, 1997, Pedankur, 1999). Consequently, Coulter, Cowell and Jenkins (1992a) conclude that there is no single correct equivalence scale for adjusting expenditures or incomes. Jäntti and Danziger (2000, p.319) maintain that ‘there is

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<sup>1</sup> In economics, a parameter is said to be ‘identified’ if its numerical value can be determined given enough observable data [Lewbel, Pendakur, 2006].

no optimal method for deriving an equivalence scale.’ For these reasons, conventional equivalence scales, which are used in analyses and in the practice are arbitrary.

There is much evidence that the results of distributional comparisons are sensitive to the choice of equivalence scale (Coulter, Cowell and Jenkins, 1992a,b). This means that the two aforementioned stages are not independent. Thus the arbitrariness of equivalence scales generates the arbitrariness of those comparisons.

The rest of this paper is organised as follows: Section II presents the concept of stochastic indifference and its relation to equivalence scale exactness (or independence of base). Then the concept of *SES* is formally developed. Section III gives a statistical test to verify whether a particular function can be recognised as an *SES* and a method for estimating *SESs*. Section IV contains the empirical results of estimating inter-temporal non-parametric and parametric *SESs* for Poland in the years 2005-2010. Lastly, Section V presents general conclusions.

## II. STOCHASTIC EQUIVALENCE SCALES

### 2.1 Stochastic indifference and equivalence scale exactness.

The concept of the stochastic equivalence scale (SES), applied in this paper, bases on the idea of stochastic indifference (SI), i.e., a symmetric factor of stochastic dominance relation. Suppose that we consider two positive-valued random variables  $W_1$  and  $W_2$  describing two expenditure distributions and characterised respectively by probability distribution functions  $R_1(w)$  and  $R_2(w)$  [ $W_1 \sim R_1(w)$  and  $W_2(x) \sim R_2(w)$ , for short]. Let  $U_1$  denote the class of all von Neuman-Morgenstern type utility functions,  $u$ , such that  $u' \geq 0$  (increasing). Also let  $U_2$  be the class of all utility functions in  $U_1$  for which  $u'' \leq 0$  (strict concavity). Following Linton, Masuomi and Whang (2005), distribution  $W_2$  is said to dominate distribution  $W_1$  stochastically at first order, denoted  $W_2 \geq_{FSD} W_1$ , if and only if either

$$R_1(w) \geq R_2(w) \text{ for all } w \geq 0, \text{ for all } w, \text{ with strict inequality for some } w; \text{ or} \quad (1)$$

$$E[u(W_1)] \leq E[u(W_2)] \text{ for all } u \in U_1, \text{ with strict inequality for some } u, \quad (1a)$$

where  $E[\cdot]$  denotes the mean value operator of a random variable.

Distribution  $W_2$  dominates  $W_1$  at second order, denoted  $W_2 \geq_{SSD} W_1$ , if and only if either

$$\int_0^w R_1(t) dt \geq \int_0^w R_2(t) dt, \text{ for all } w, \text{ with strict inequality for some } w; \text{ or} \quad (2)$$

$$E[u(W_1)] \leq E[u(W_2)] \text{ for all } u \in U_2, \text{ with strict inequality for some } u. \quad (2a)$$

First order dominance implies dominance at all higher orders. However, the implication goes in only one direction.

Distribution  $W_1$  is said to be indifferent to distribution  $W_2$  if and only if  $W_1$  dominates stochastically  $W_2$  and  $W_2$  dominates stochastically  $W_1$ . Stochastic indifference is called the symmetric factor of stochastic dominance relation.

Definition 1.  $W_1$  is first order indifferent to  $W_2$ , i.e.  $W_2 \geq_{FSD} W_1 \wedge W_2 \leq_{FSD} W_1$ , if and only if either

$$R_1(w) = R_2(w) \text{ for all } w \geq 0; \text{ or} \quad (3)$$

$$E[u(W_1)] = E[u(W_2)] \text{ for all } u \in U_1. \quad (3a)$$

Definition 2.  $W_1$  is second order indifferent to  $W_2$ , i.e.  $W_2 \geq_{SSD} W_1 \wedge W_2 \leq_{SSD} W_1$ , if and only if either

$$\int_0^w R_1(t) dt = \int_0^w R_2(t) dt, \text{ for all } w \geq 0; \text{ or} \quad (4)$$

$$E[u(W_1)] = E[u(W_2)], \text{ for all } u \in U_2 \quad (4a)$$

Higher orders of stochastic indifference can be defined in a similar way. It is easy to see that the first order indifference implies the indifference of all higher orders and this implication goes in *both directions*.

There are well known relationships between stochastic dominance and economic inequality and poverty (see, among others, Davidson, Duclos, 2000, Davidson, 2008). All these relationships also valid for stochastic indifference, which can be easily checked when substituting inequality ' $\geq$ ' by strict equality.

The following corollary summarises useful properties of stochastic indifference.

Corollary 1. With the above notations, the following statements are equivalent:

- a)  $R_1(w) = R_2(w)$ , for all  $w \geq 0$ .
- b) Social welfare in  $W_1$  and  $W_2$ , i.e.  $E[u(W_1)]$ , and  $E[u(W_2)]$  is the same for all utility functions  $u \in U_2$ .
- c) Poverty in  $W_1$  and  $W_2$  is the same for all poverty lines and for the Atkinson's (1987) class of poverty indices.
- d) Inequalities in  $W_1$  and  $W_2$  are the same.

Let the general population of all households consists of  $m+1$  disjoint groups (subpopulations),  $m > 1$ , which are selected according to an attribute other than expenditure, e.g. household size, demographic composition, etc. Let  $m+1$  vector  $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_m]$  describe these attributes. Let one of these groups with  $\alpha_0$  attribute be chosen as a reference household group with an expenditure distribution described by positive continuous random variable  $Y$

with probability distribution function  $G(y)$ ,  $Y \sim G(y)$  for short<sup>2</sup>. Without loss of generality, we choose households comprising a single adult as the reference group. Let positive continuous random variables  $X_i \sim F_i(x)$ ,  $i = 1, \dots, m$ , represent the expenditure distributions of the remaining  $m$  household groups.  $X_1, \dots, X_m$  with  $\alpha_1, \dots, \alpha_m$  attributes, respectively. For technical reasons,  $X_1, \dots, X_m$  will be called ‘the evaluated distributions.’

To account for a normative setting, we consider a simple model of utilities and preferences of household members in an economic environment with  $q$  private goods. We assume that any two households with the same attributes have the same preferences and that, if utility levels are comparable across households, individuals in different households with the same attributes and consumption vectors are equally well off (Blackorby, Donaldson, 1993).

Define the cost (expenditure) function,  $c(p, u, \alpha_i)$  corresponding to household utility function  $u(q, \alpha_i)$ ,  $i=0, 1, \dots, m$ , to give the minimum amount of expenditure necessary for a household of  $\alpha_i$  attribute to get utility level  $u$  at prices  $p$ , i.e.,

$$c(u, p, \alpha_i) = \min_q \{p \cdot q \mid u(q, \alpha_i) \geq u\} \quad (5)$$

The indirect utility function  $v(p, x, \alpha_i)$  is given by

$$u = v(p, x, \alpha_i) = \max_q \{u(q, \alpha_i) \mid p \cdot q \leq x\}, i=0, 1, \dots, m, \quad (6)$$

where  $x$  is total expenditure at prices  $p$ . The functions  $c$  and  $v$  are related by the identity

$$x = c(u, p, \alpha_i) \leftrightarrow u = v(p, x, \alpha_i), i=0, 1, \dots, m \quad (7)$$

(Blackorby, Donaldson, 1993).

Let  $d$  be the number of adult equivalents in a household with attributes  $\alpha_i$  and income  $x$  facing prices  $p$ . Blackorby, Donaldson (1993) define  $d_i$  as follows

$$u = v(p, x, \alpha_i) = v\left(p, \frac{x}{d}, \alpha_0\right) \quad (8)$$

where  $\alpha_0$  is the reference household attribute (a single adult). We assume that for all  $\alpha_i$ ,  $i=1, \dots, m$ , there exists  $d$  such that (6) has a unique solution.

We may express (6) in an alternative form, assuming  $y$  as the reference household expenditure

$$u = v(p, y \cdot d, \alpha_i) = v(p, y, \alpha_0). \quad (9)$$

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<sup>2</sup> Henceforth, we reserve capital letters for random variables and lowercase letters for the values of these variables.

Equations (8) or (9) implicitly define a function  $d=eq(p,u,\alpha_i)$ , where  $eq(p,u,\alpha_0)=1$ . Eq. (8) says that if the reference adult has an income  $x/d$  and faces prices  $p$ , then he or she enjoys a utility level exactly equal to the one enjoyed by each member of a household with attributes  $\alpha_i$  and income  $x$ , facing prices  $p$ .

Eq. (9) can be interpreted as follows: the household with attributes  $\alpha_i$ , facing prices  $p$ , should spend amount of  $y \cdot d$  if each of its member to be as well off as the reference adult, spending  $y$  and facing prices  $p$ .

Using (7),  $d$  is given by

$$d = eq(p, u, \alpha_i) = \frac{c(p, u, \alpha_i)}{c(p, u, \alpha_0)}, i=1, \dots, m, \quad (10)$$

(Blackorby, Donaldson, 1993)<sup>3</sup>.

The function  $eq$  depends on the unobservable utility level of members of the household. Blackorby, Donaldson, (1993) suggest to use a single reference level of utility, say  $u_0$ , and to define an index

$$\bar{d} = \bar{eq}(p, \alpha_i) = \frac{c(p, u_0, \alpha_i)}{c(p, u_0, \alpha_0)}, i=1, \dots, m, \quad (11)$$

In general, the index  $d$  is not equal to the index  $\bar{d}$ . Social rankings will be correct if and only if the index is exact, that is if and only if

$$\bar{eq}(p, \alpha_i) = eq(p, u, \alpha_i), \quad (12)$$

for all  $(p, u, \alpha_i)$ ,  $i=1, \dots, m$ .

When (12) holds, Blackorby, Donaldson (1993) say that utilities satisfy *equivalence scale exactness* (ESE). Lewbel (1989) calls this *independence of base (IB)* and shows that if there exists a *base-independent* equivalence scale function  $\Delta(p, \alpha)$  of prices and household attributes, then the cost functions must be related by

$$c(p, u, \alpha_i) = c(p, u, \alpha_0) \Delta(p, \alpha_i), i=1, \dots, m. \quad (13)$$

where  $\Delta(p, \alpha)$  must not depend on  $u$ .

Using (7) and (8), the indirect utility function  $v$  corresponding to  $c$  is

$$v(p, x, \alpha_i) = v\left(p, \frac{x}{\Delta(p, \alpha_i)}, \alpha_0\right), i=1, \dots, m, \quad (14)$$

Defining  $x/\Delta(p, \alpha_i)$  as *equivalent expenditure*, Eq. (14) says that if two households facing the same prices have the same equivalent expenditure, than they are equally well off.

Similarly, using (7) and (9), the indirect utility function  $v$  corresponding to  $c$  is

<sup>3</sup> See also Deaton, Muellbauer, (1980), p.205.

$$v(p, y \cdot \Delta(p, \alpha_i), \alpha_i) = v(p, y, \alpha_o), i=1, \dots, m. \quad (15)$$

Several papers have tested ESE/IB assumption, but they ultimately reject it (Blundell and Lewbel, 1991, Blundell et al., 1998, Dickens et al., 1993, Pashardes, 1995, Gozalo, 1997, Pedankur, 1999). Perhaps, ESE/IB condition is too strong. The following theorem offers a weaker condition.

**Theorem 1.** Let the positive continuous random variables  $X_i \sim F_i(x)$  and  $Y \sim G(y)$  describe expenditure distributions of the households with  $i$ th attributes,  $i=1, \dots, m$ , and the reference households, respectively. Let ESE/IB assumption holds which means that there exists  $\Delta(p, \alpha_i)$  satisfying Eq. (14). Define

$$Z_i = X_i / \Delta(p, \alpha_i) \sim H(z). \quad (16)$$

Then  $Z_i$  is stochastically indifferent to  $Y$ .

Proof. ESE/IB implies (14). This means that expenditure distribution of the households with  $i$ th attributes satisfies the equality:  $X_i = Y \cdot \Delta(p, \alpha_i)$  that implies

$$Y = X_i / \Delta(p, \alpha_i) = Z_i. \quad (17)$$

But the equality of two random variables implies the equality of their distribution functions

$$G(y) = H_i(y), \quad (18)$$

for all  $y > 0$ , which means the validation of the condition (3) of stochastic indifference. QED.

However, the above implication goes in one direction. To see this, it is enough to observe that (18) may not necessary imply (17). It is due to the fact that one probability distribution function may correspond to several random variables (Billingsley, 1995, p.261). This means that indifference criterion is weaker than ESE/IB (see Fig1).

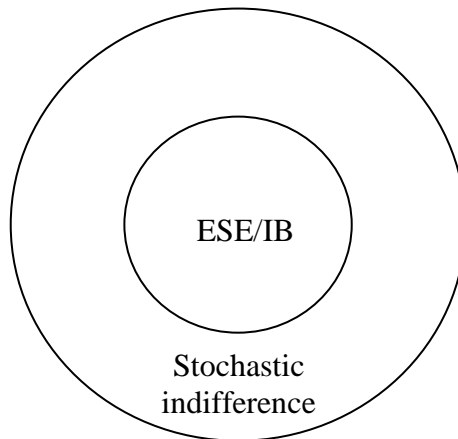


Fig.1. The relation between ESE/IB and stochastic indifference.



Because of that ESE/IB assumption seems to be too strong in applications, we propose to found equivalence scales on the *stochastic indifference* (SI), a weaker criterion than ESE/IB. The SI can be easy to check by Kolmogorov-Smirnov (K-S) test. The K-S statistics may be also used for developing estimators of equivalence scale  $\Delta(p, \alpha_i)$ .

A similar approach has been adopted in time series analysis when defining stationarity. A stochastic process is said to be stationary in the *strict sense* if its joint probability distribution remains unchanged when shifting along the time axis. Unfortunately, stationarity in the strict sense is untestable in most economic analysis where a single time series is usually available. However, stationarity in the strict sense implies some conditions<sup>4</sup>, which can be easily tested on the base of single time series. For this reasons, analysts have adopted these conditions as the definition of stationarity in the weak sense.

## 2.2. The stochastic equivalence scales.

According to the above notations, let  $Y \sim G(y)$  describe the reference expenditure distribution and  $X_i \sim F_i(x)$  be the evaluated expenditure distribution of the  $i$ th household group,  $i=1, \dots, m$ .

Suppose that  $s(\cdot) = [s_1(\cdot), \dots, s_m(\cdot)]$  is a continuous and strictly monotonic real-valued vector function for which the inverse function  $s^{-1}(\cdot) = [s_1^{-1}(\cdot), \dots, s_m^{-1}(\cdot)]$  exists. Let the random variable  $Z_i = s_i(X_i) \sim H_i(z)$  be the transformation of the evaluated expenditure distribution  $X_i$ . Henceforth, the random variable  $Z_i \sim H_i(z)$  will be called the ‘transformed expenditure distribution’. Define the random variable  $Z \sim H(z)$  as a mixture of the  $m$  transformed distributions  $Z_i$

$$H(z) = \sum_{i=1}^m \pi_i H_i(z), \quad (19)$$

where the weights  $\pi_i$  satisfy the condition  $\forall i = 1, \dots, m, \pi_i > 0 \wedge \sum_{i=1}^m \pi_i = 1$ .

Definition 2. With the above notations, the function  $s(\cdot)$  will be called the *stochastic equivalence scale* (SES) if and only if the following equality holds:

$$\forall z \geq 0, H(z) = G(z) \quad (20)$$

When the function  $s(\cdot)$  is an SES, the transformed expenditure distributions  $Z$  will be called ‘the equivalent expenditure distributions’. Naturally, the definition of an SES also applies for  $m = 1$  in Eq. (19), i.e., when only one group of households is compared to the reference group.

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<sup>4</sup> If a stochastic process  $X_t$  is stationary in the strict sense then  $E[X_t] = \mu = \text{const}$ ,  $D^2[X_t] = \sigma^2 = \text{const}$ , and autocovariance function  $\text{cov}(t_1, t_2) = \text{cov}(t_2 - t_1)$ .

The definition of an *SES* is ‘axiomatic’ in the sense that it only defines the *properties of a function* that can be recognised as an *SES*. This definition does not describe how an *SES* should be constructed. In other words, *any* function  $s(\cdot)$  that fulfils the condition (20) has to be recognised as an *SES*.

It might happen that several functions  $s(\cdot)$  satisfy condition (20), i.e., they are *SESs*. Nevertheless, all such functions yield the equivalent expenditure distributions, which in virtue of corollary 1 exhibit the same social welfare, inequality and poverty. In other words, the choice of a particular *SES* from the set of *SESs* does not influence social judgements.

Some classes of *SES* would be of special importance for empirical studies. For example, the class of relative *SES* can be defined in following way. Let  $\mathbf{d} = [d_i], i = 1, \dots, m$ , be the vector of positive numbers called ‘deflators’ that transform the evaluated expenditure distributions  $X_1, \dots, X_m$  thusly:

$$Z_i = X_i/d_i \sim H_i(z), i = 1, \dots, m, \quad (21)$$

Definition 3. Under the above notations, the vector  $\mathbf{d}$  will be called the *relative SES* if and only if the deflators  $d_1, \dots, d_m$  are such that equality (20) holds.

The  $d_i$  deflators are used for calculation the weights  $\pi_i$  appearing in (19), i.e.  $\pi_i = d_i / \sum_{i=1}^m d_i$ . It is due to the fact that the transformation (21) means the pass from the population of individuals to the fictitious population of equivalent adults [Ebert, Moyes, 2003].

Definition 4. The relative *SES* is said to be *nonparametric* if  $d_i$  deflators are simply the numbers of equivalent adults. The relative *SES* is said to be *parametric* if  $d_i$  deflators are functions  $d(\mathbf{h}, \theta)$  of some household attributes,  $\mathbf{h}$ , e.g. the number of household’s members, the number of adults and children, etc<sup>5</sup>.

Certain forms of the parametric deflator  $d(\mathbf{h}, \theta)$  are especially popular in practical applications. The *power deflator* has the following form:

$$d(\mathbf{h}, \theta) = h^\theta, 0 \leq \theta \leq 1, \quad (22)$$

where  $h$  is the household size (Buhmann et al., 1988). The parameter  $\theta$  is set arbitrarily. The per capita (with  $\theta = 1$ ) and square root (with  $\theta = 0.5$ ) equivalence scales appear to be the most popular equivalence scales (OECD, 2008). This and other pragmatic equivalence scales one can treat as potential *SESs*. One will recognise them as *SESs* if condition (20) is fulfilled.

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<sup>5</sup> Jenkins and Cowell (1994) describe the parametric equivalence scale class as ‘...a set of scales sharing a common functional form and for which parametric variations change the scale rate relativities for households of a different type’.

The validation of condition (20) can be tested using nonparametric statistical tests for equality between cumulative distribution functions. In this paper, the Kolmogorov-Smirnov ( $K-S$ ) test is applied. Moreover, the parametric and nonparametric  $SES$  can be estimated when the  $K-S$  statistic is used as a loss function.

### III. STATISTICAL ISSUES CONCERNING $SES$

There are two statistical issues related to  $SES$ s: testing whether a function  $s(\cdot)$  can be recognised as an  $SES$  and estimating parametric and non-parametric  $SES$ s. These two problems require random samples of expenditures per household. Micro-data are the most suitable data for this purpose, though grouped data may also be used. We assume that the general population consists of  $m+1$  disjointed household populations and that each household represents a different type of household. One of these populations, which usually comprises one-person households, is treated as the reference population. The remaining  $m$  populations are called the evaluated populations.

In this section, we will use the notations and symbols defined in Section II. The random samples are defined as follows: the sample of size  $l$  that is from the general reference population will be denoted as  $(y_1, \dots, y_l)$ . Whereas the random sample from the  $i$ -th evaluated general population will be denoted as  $(x_1, \dots, x_{n_i})_i$ , the transformed version of this sample will be denoted as  $(z_1, \dots, z_{n_i})_i$ ,  $i = 1, \dots, m$ . The total sample size of all of the evaluated groups will be denoted as  $n = n_1 + \dots + n_m$ . We calculate the weights  $\pi_i$  in (19) as  $\pi_i = d_i / \sum_{i=1}^m d_i$ , i.e. the shares of *equivalent adults*  $d_i$ , and assign them to the  $i$ th group for each  $i = 1, \dots, m$ . Let  $(z_1, \dots, z_n)$  denote the pooled sample of all of the transformed values with their corresponding weights  $(\pi_1, \dots, \pi_n)$ .

We calculate the empirical distribution function  $\widehat{G}(\cdot)$  for the reference distribution  $Y$  using the sample  $(y_1, \dots, y_l)$ ; similarly, we calculate the empirical distribution function  $\widehat{H}(\cdot)$  using the pooled sample  $(z_1, \dots, z_n)$  and the corresponding weights  $(\pi_1, \dots, \pi_n)$ .

To verify that function  $s(\cdot) = [s_1(\cdot), \dots, s_m(\cdot)]$  is an  $SES$ , i.e., to check whether identity (20) holds, we need to test the null statistical hypothesis

$$H_0: H(z) = G(z) \tag{23}$$

against the alternative hypothesis

$$H_a: H(z) \neq G(z) \tag{24}$$

for all  $z \geq 0$ .

We will verify one of these hypotheses using the Kolmogorov-Smirnov (*K-S*) test:

$$U = \max_z | \hat{H}(z) - \widehat{G}(z) | \sqrt{\frac{l \cdot n}{l+n}}, \text{ for all } z \geq 0 \quad (25)$$

Smirnov (1939).

Under the null hypothesis, the  $U$  statistic (25) has an asymptotic Kolmogorov- $\lambda$  distribution (Kolmogorov, 1933).

The  $p$ -value of the *K-S* test (25), i.e.,  $p = P(U \geq u_{calc})$ , is a convenient tool for testing these hypotheses on the selected significance level  $\alpha$ , where  $u_{calc}$  is the calculated value of the  $U$ -test in the sample. If  $p \leq \alpha$ , we reject the null hypothesis  $H_0$  and accept  $H_a$ , and as a result, the function  $s(\cdot)$  cannot be recognised as an *SES*. If  $p > \alpha$ , we accept the null hypothesis, and therefore, we recognise function  $s(\cdot)$  as an *SES*.

The proposed method of estimating *SESs* uses the  $U$ -test as a loss function. Suppose that the function  $s(\cdot)$  is a potential *SES*. This function may be non-parametric, e.g., it may take the form of a set of deflators  $\mathbf{d} = [d_1, \dots, d_m]$ , or it may be parametric and depend on  $k$  parameters  $\boldsymbol{\theta} = [\theta_1, \dots, \theta_k]$ . We will use the symbols  $s(\cdot/\mathbf{d})$ ,  $s(\cdot/\boldsymbol{\theta})$  or simply  $s$  when the context of the estimation is obvious. Let  $z_1, \dots, z_n$  be the sequence of the evaluated expenditures that are adjusted by the function  $s$ , i.e., let  $z_j = s(x_j/\mathbf{d})$  or  $z_j = s(x_j/\boldsymbol{\theta})$ ,  $j=1, \dots, n$ . Let  $\widehat{G}(z)$  and  $\widehat{H}(z|\mathbf{s})$  denote the empirical distribution functions of the reference expenditures and the adjusted expenditures, respectively. We propose the estimator  $s^*$  of  $s$  that minimises the *K-S* statistic  $U(\mathbf{s})$ , i.e.:

$$U(\mathbf{s}^*) = \min_s \max_z | \widehat{H}(z|\mathbf{s}) - \widehat{G}(z) | \sqrt{\frac{n \cdot l}{n+l}} \text{ for all } z \geq 0. \quad (26)$$

When we compare the reference distribution  $Y$  with a particular evaluated distribution  $Z_i$ , we can substitute for the terms  $\widehat{H}(z|\mathbf{s})$  and  $n$  on the right-hand side of Eq. (26) with  $\widehat{H}_i(z|\mathbf{s})$  and  $n_i$ , respectively, for each  $i = 1, \dots, m$ .

Alternatively, we can estimate the *SES* by using  $p$ -values as a function of  $s$ , i.e., by using  $p(\mathbf{s})$ :

$$p(\mathbf{s}) = P[U(\mathbf{s}) \geq u_{calc}(\mathbf{s})]. \quad (27)$$

This *SES* estimator is  $\mathbf{s}^*$  and it maximises the  $p$ -value (27):

$$p(\mathbf{s}^*) = \max_s P[U(\mathbf{s}) \geq u_{calc}(\mathbf{s})] \wedge p(\mathbf{s}^*) > \alpha \quad (28)$$

The estimator  $\mathbf{s}^*$  can be found using the grid-search method. The following example illustrates this method when estimating nonparametric deflator  $d$  for two-member households.

We assume an initial interval  $[d_1, d_2]$  for the  $d$  deflator and we divide this interval using  $t$  equally spaced values  $d^{(1)}, \dots, d^{(t)}$  (grids). For each  $d^{(j)}$ ,  $j = 1, \dots, t$ , we calculate the  $n$  adjusted expenditures  $z_i^{(j)} = x_i/d^{(j)}, \dots, z_n = x_n/d^{(j)}$  and the  $p$ -value  $p(d^{(j)})$  (27) of the Kolmogorov-Smirnov  $U$  test (26). For a more accurate estimation, we may repeat calculations with the interval for a value  $d$  that is smaller than the initial interval. The deflator  $d^*$  that yields the maximal  $p$ -value, namely,  $p(d^*) = \max[p(d^{(1)}), \dots, p(d^{(t)})]$ , will be the desired estimator of the  $SES$  if  $p(d^*)$  is greater than the significance level  $\alpha$ . Fig. 2 shows the plot of  $p(d^{(j)})$  against  $d^{(j)}$ .

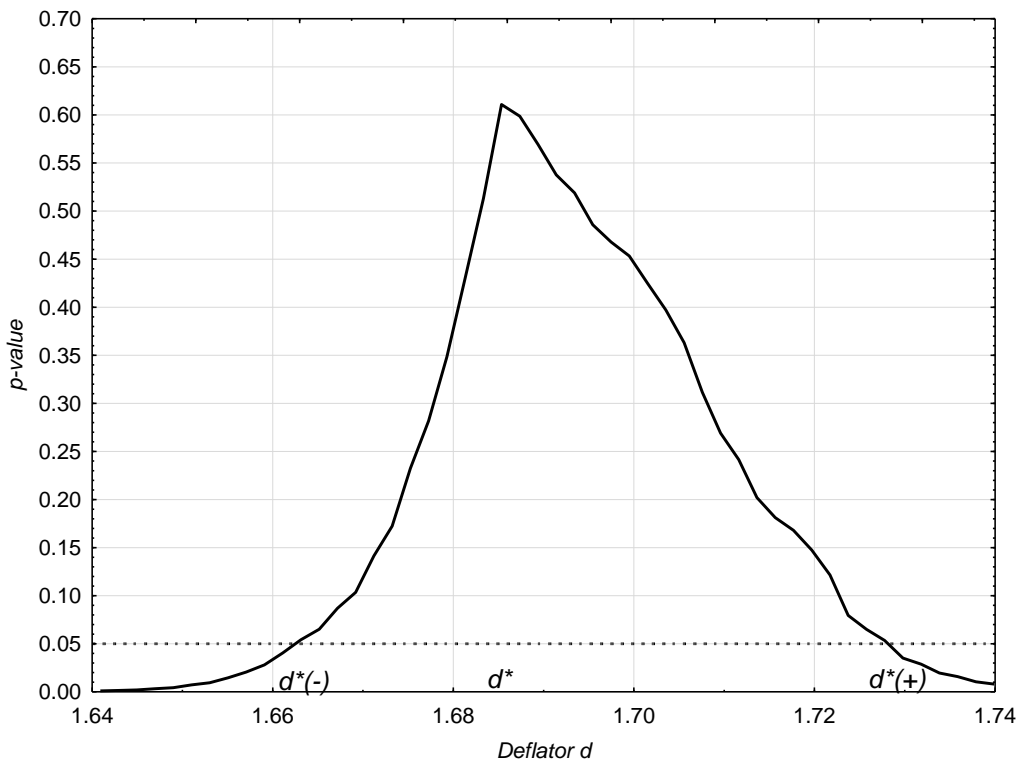


Fig. 2. Estimating the  $d$  deflator of a non-parametric relative  $SES$ .

Source: Author's calculations from Polish Household Budget Survey 2010.

One can see that the plot of  $p^{(j)}$  crosses the critical level  $\alpha = 0.05$  at two points that have the  $d$ -coordinates  $d^{*(-)}$  and  $d^{*(+)}$ . Thus, every relative scale with a deflator  $d$  that belongs to the  $[d^{*(-)}, d^{*(+)}]$  interval can be accepted as an  $SES$ . The interval  $[d^{*(-)}, d^{*(+)}]$  can be interpreted as the  $(1-\alpha)$  confidence interval for the estimated deflator  $d^*$ . In this example,  $d^* = 1.68655$  and the 95% confidence interval is  $[1.663, 1.728]$ .

The above estimation procedure is applied separately to every set of  $m$  deflators  $d_i$ . If every  $d_i$  deflator is an *SES*, overall criterion (20) will be satisfied automatically. However, if some of the estimated deflators are not *SESs*, criterion (20) might be still fulfilled by applying the *K-S* test (26) for pooled samples.

The Kruskal-Wallis (1952) test (which we abbreviate *K-W*) can be applied as a supplementary tool for checking the homogeneity of transformed distributions. The *K-W* test assesses whether different samples were drawn from the same distribution.

It should be noted, however, that  $s^*$  estimators do not always exist. Although  $p(s)$  usually reaches a maximum, the condition  $p(s^*) > \alpha$  might be violated.

#### IV. INTERTEMPORAL EQUIVALENCE SCALES FOR POLAND 2005-2010.

For inter-temporal comparisons of expenditure distributions, we use monthly micro-data that come from the Polish Household Budget Surveys for the years 2005-2010, where household groups are distinguished according to the number of members (household size). The expenditures are expressed in constant 2010 prices. We chose the households of single adults in the year 2010 as the reference group for all household groups and years. We assume a 5% significance level in all of the analysed cases

TABLE 1  
Estimates of non-parametric inter-temporal *SESs*

Household size	2005	2006	2007	2008	2009	2010
1	.88276 (.871;.893) $p=.33497$	.91917 (.911;.927) $p=.19564$	.91381 (.898;.929) $p=.69180$	.94285 (.923;.962) $p=.94631$	.98510 (.969;1.007) $p=.96077$	1
2	1.41601 (1.409;1.437) $p=.13451$	1.47531 (1.463;1.519) $p=.31582$	1.53038 (1.517;1.565) $p=.34443$	1.60348 (1.584;1.646) $p=.61932$	1.65582 (1.639;1.694) $p=.48098$	1.68655 (1.663;1.728) $p=.61811$
3	1.64246 (1.625;1.692) $p=.37444$	1.77607 (1.768;1.805) $p=.14403$	1.85594 (1.838;1.900) $p=.46894$	1.98467 (1.965;2.041) $p=.41149$	2.03721 (2.024;2.074) $p=.19892$	2.06870 (2.062;2.093) $p=.09060$
4	1.78925 (1.780;1.815) $p=.12690$	1.90296 (1.890;1.955) $p=.27833$	2.02864 (2.018;2.065) $p=.15709$	2.16172 (2.153;2.199) $p=.14077$	2.22012 (2.209;2.251) $p=.11610$	2.21725 (2.202;2.276) $p=.22808$
5 or more	1.66465 (1.646;1.697) $p=.21394$	2.00174 (*;*) $p=.00028$	2.12616 (*;*) $p=.00054$	2.26059 (*;*) $p=.00869$	2.32638 (*;*) $p=.00070$	2.37236 (*;*) $p=.00041$
p (K-S)	.06931	.06127	.06895	.13691	.07735	.03988
p(K-W)	.59879	.76587	.48713	.85287	.94596	.90990

*Notes*

reference households: single adults 2010.

$p(K-S)$ : p-value in Kolmogorov-Smirnov test;  $p(K-W)$ : p-value in Kruskal-Wallis test, 95% confidence intervals in parentheses.

*Source*: Author's calculation from Polish Household Budget Surveys, 2005-2010.

Table 1 presents the estimates of nonparametric relative *SESs*. The deflators  $d_i$  are estimated separately for each household group, including single adults, and in each year. The deflator  $d_i=1$  for single adults in the year 2010.

An analysis of the results presented in Table 1 shows that almost all of the estimated deflators can be recognised as *SESs*. The exceptions are five estimates for households with five or more members. The second row from the bottom in Table 1 shows that these exceptions influence the overall *K-S* test for pooled household groups only for the year 2010. However, the p-values of the overall Kruskal-Wallis test, which is presented in the last row of Table 1, are greater than the significance level  $\alpha = 0.05$ . Thus, all of the estimated deflators can be recognised as *SESs*.

Three features of the estimated equivalence scales are remarkable. First, these nonparametric equivalence scales are very flat in comparison with the per capita scale. This means that Polish households exhibited large economies of scale in the years 2005-2010. Second, the 95% confidence intervals are very narrow for the estimated deflators. Consequently, the proposed method to estimate the non-parametric scales is quite accurate. Third, equivalence scales vary over time.

Inter-temporal comparisons of equivalence scales reveal some interesting features. One can see that the number of the units that are equivalent to 2010 single adults, increase over time, within each group of households. This means decreasing economies of scale in the years 2005-2010.

Table 2 contains the estimates of the inter-temporal power scale (22)  $d=h^\theta$ , where  $h$  is the household size. For comparisons, we present the estimates of the ‘current’ power scale, where reference household group contains single adults in current year.

TABLE 2.  
Estimates of parametric inter-temporal and current *SESs*

<i>Year</i>	<i>Inter-temporal SES.</i>		<i>Current SES.</i>	
	<i>Reference households:</i>		<i>Reference households:</i>	
	<i>single adults, 2010</i>		<i>single adults, current year</i>	
	$\theta$	<i>p-value</i>	$\theta$	<i>p-value</i>
2005	0.40127 (0.400; 0.402)	0.06298	0.51872 (0.513; 0.526)	0.38150
2006	0.45899 (0.457; 464)	0.07575	0.53853 (0.537; 0.541)	0.11294
2007	0.50117 (0.499; 0.506)	0.09910	0.58891 (0.582; 0.597)	0.47660
2008	0.55455 (0.550; 0.571)	0.23818	0.61021 (0.606; 0.622)	0.24119
2009	0.58070 (576; 0.595)	0.16811	0.59684 (0.592; 0.609)	0.24987
2010	0.59438 (0.591; 0.603)	0.15262	0.59438 (0.591; 0.603)	0.15262

*Notes:* 95% confidence intervals in parentheses

*Source:* Author's calculations from Polish Household Budget Surveys, 2005-2010.

Analysis of the results in Table 2 shows that all of the power equivalence scales can be recognised as *SESs* because all of the corresponding *p-values* are greater than the significance level of 0.05. The estimates of  $\theta$  are less than one in every year under consideration. This is an indication of the economies of scale that were enjoyed by Polish households in the years 2005-2010. However, the effect of economies of scale seems to diminish because the parameter  $\theta_i$  slowly increases in this period.

One can see in Table 2 that the inter-temporal SES provides lower estimates of  $\theta$  than the current SES. This means that the inter-temporal SES exhibits greater economies of scale than the current SES.

## V. CONCLUSIONS

Thus far, conventional equivalence scales have failed to solve the problem of homogenising a population of households that differ in all respects other than their expenditures. Under a single price regime, the equivalence scales are not identifiable unless the ESE/IB condition holds. Several papers have tested this condition, but they ultimately rejected it. Without *IB*, the standard equivalence scales turn out to be arbitrary, which implies arbitrariness of distributional judgements concerning inequality, poverty and welfare.

Stochastic equivalence scales seems to be a promising alternative to conventional equivalence scales. SES bases on the stochastic indifference criterion, which is a weaker condition than ESE/IB.. The validation of the stochastic indifference can be verified using statistical tests. The stochastic indifference criterion can also be used to develop SES estimators.

The axiomatic formulation of the SES is quite general. It does not specify one definite scale form, but it does define the properties that should be satisfied by a certain function for it to be recognised as an SES. This means that the actual form of an SES function is not



important; only the fact that a function is an SES function matters. Thus, we do not have to search for an optimal SES. The choice of a particular form of SES does not influence distributional judgements, in contrast to arbitrary standard equivalence scales.

The application of SES to inter-temporal comparisons of expenditure distributions has turned out very useful. The choice of bench mark reference group of households for all analysed years and household groups provides more homogeneous platform for normative judgements than that when the current equivalence scales are separately estimated in each year.

The estimated inter-temporal SESs for Poland reveal remarkable features. Polish households exhibited large and diminishing economies of scale in the years 2005-2010. Also, the inter-temporal estimates of economies of scale are lower than the current estimates.

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