Scale vs. Translation Invariant Measures of Inequality of Opportunity when the Outcome is Binary

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Scale vs. translation invariant measures of inequality of opportunity when the outcome is binary[a]

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Abstract

This paper discusses the measurement of ex-ante inequality of opportunity when the outcome is binary. We argue that the use of scale but not translation invariant inequality measures such as the dissimilarity index are problematic, since they rely too much on the average level of access. We propose first a decomposition of these measures in a level and a dispersion effect and second an adapted index satisfying translation invariance. In two short illustrations we show that the conclusions differ substantially between the two methods and that the appropriate between scale and translation invariant measures is crucial.

Keywords: inequality of opportunity, binary indicator, HOI, ordered variables, translation invariance

JEL-Classification: C18, D63, I24, O54

1 Introduction

Inequality of opportunity is defined as being the part of inequality that can be attributed to circumstances beyond the control of the individual, thus the part for which people cannot be held responsible (Roemer, 1998). The remaining inequality in a given outcome might be due to different effort levels and other factors such as luck. Research on the measurement of this decomposition of inequality has flourished over the last years. Many proposals share the common idea of regressing the analyzed outcome exclusively on circumstances beyond individuals' control.

[a]This is a working paper version and we are very happy to receive comments and suggestions. Click here (http://econ.chavezjuarez.com/vcheck.php?i=iopbinary&v=20130815) to see if you have the newest version

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The idea behind this approach is that all variation explained by circumstances can be directly attributed to inequality of opportunity. A metric measure is obtained by applying a common inequality measure to the distribution of outcomes conditional on the set of circumstances. The choice of the used inequality measure depends on the type of the outcome variable and the required properties such as scale or translation invariance[1]. For the case of continuous variables, proposals by Ferreira and Gignoux (2011) and Ferreira and Gignoux (2013) include both *scale* and *translation invariant* inequality measure. In this paper we focus on the case of a binary or an ordered outcome, for which pioneering work was done by Paes de Barros et al. (2008). To the extent of our knowledge, no *translation invariant* measure was used in the context of binary indicator variables. We show that the absence of *translation invariance* in the inequality measure might be problematic, especially when comparing the inequality of opportunity in outcomes with different average levels of access. In an illustration we show that the higher degree of inequality of opportunity in access to higher education as compared to secondary education is mainly due to changes in the average access rates and not to the dispersion of the probability in having access. We propose two ways of analyzing and solving this problem.

First, we show that some inequality measures can be easily decomposed in a level and a dispersion part, allowing us to decompose differences between two outcomes or changes over time in a level- and a dispersion effect. This decomposition does not directly solve the problem, but allows us to understand the importance of it.

Second, we modify the dissimilarity index used in Paes de Barros et al. (2008) in such a way that it becomes translation invariant. The empirical illustration shows that the results are very different compared to what is obtained with the original method and that the issue is therefore potentially of importance. We do not claim that our modified index is better than the original proposal. However, we argue that depending on what we really want to measure, the small changes in the index can have important consequences.

In section 2 we describe first the general approach to assess *ex-ante* inequality of opportunity with a special focus on binary outcome variables. Following this introduction to existent methods, we describe our concerns about the missing translation invariance in the analysis of binary outcomes and finally we introduce the two aforementioned approaches to deal with our concerns. In section 3 we present an illustration of the different methods and finally in section 4 we conclude the paper.

[1] *Scale invariant* and *translation invariant* inequality measures are also known as relative and absolute inequality measures respectively. We use the terms *scale* and *translation invariance* throughout the paper to avoid any confusion with the relative (θ_r) and absolute (θ_a) measures of inequality of opportunity, which we will introduce in section 2.

2 Methodology

2.1 General approach to assess ex-ante inequality of opportunity

Different research proposals over the last years brought forward different methods to assess ex-ante inequality of opportunity. Most of these proposals are of the same family, where inequality is measured on the outcome variable conditioned on circumstances. Let y be the outcome variable of interest and C a matrix of circumstances beyond the control of the individual. We can now compute the conditional expectation of y given C

$$\widehat{y} = E[y|C] \tag{1}$$

There are many possibilities to estimate $\widehat{y}$, including non-parametric methods and parametric methods such as OLS for continuous and probit/logit for dichotomous variables (Ferreira and Gignoux, 2011; Paes de Barros et al., 2007; Checchi and Peragine, 2010). All variation in $\widehat{y}$ is due to the circumstances considered in matrix C. In a situation of perfect equality of opportunity this should be simple the population average for everybody. Thus, all the variability in $\widehat{y}$ can be attributed to inequality of opportunity and therefore we can simply apply a standard inequality measure $I(.)$ on $\widehat{y}$:

$$\theta_a = I(\widehat{y}) \tag{2}$$

The choice of the appropriate inequality measure depends again on the scope of the analysis and the dependent variable. For instance, Paes de Barros et al. (2008) use the dissimilarity index, Ferreira and Gignoux (2011) the mean logarithmic deviation and Ferreira and Gignoux (2013) the variance. By dividing this index by the same inequality measure applied on the actual outcome vector, we obtain a relative measure of inequality of opportunity:

$$\theta_r = \frac{I(\widehat{y})}{I(y)} \tag{3}$$

This last step is only possible when the inequality measure $I(.)$ is equally defined for $\widehat{y}$ and y, which is for example not the case when the actual outcome is binary and $\widehat{y}$ is the estimated probability.

The choice of the appropriate inequality measure $I(.)$ is crucial and depends mainly on the outcome variable. Income is probably the simplest example since most of the inequality measures are designed for this outcome. Ferreira and Gignoux (2011) use the mean logarithmic deviation because it satisfies all the common desired properties of inequality measures and is additively decomposable. This last property is very useful when using the *type*[2] approach. In contrast, estimating inequality of opportunity for an achievement test at school requires very

[2] Types are groups of individuals sharing the same characteristics

different properties, because the outcome variable (test score) has no inherent scale. In this paper we understand a variable with no inherent scale as being a variable that can be multiplied and translated by positive scalars without losing its meaning. Therefore, Ferreira and Gignoux (2013) propose to estimate only the relative measure of inequality of opportunity and to use the variance as inequality measure. The advantage of the variance is its *translation invariance* and once the relative measure is estimated, the ratio of two variances ensures as well *scale invariance* of the inequality measure.

In the context of binary outcome variables the same questions are at stake. Paes de Barros et al. (2008) propose to use the dissimilarity index given by:

$$\theta_a = D(\widehat{y}) = \frac{1}{2N\overline{\widehat{y}}} \sum_{i=1}^{N} |\widehat{y}_i - \overline{\widehat{y}}| \tag{4}$$

where $\widehat{y}_i = E[y|C_i]$ and $\overline{\widehat{y}} = \frac{1}{N}\sum_{i=1}^{N} \widehat{y}_i$. This measure of inequality is *scale* but not *translation invariant*. Paes de Barros et al. (2008) discuss alternative formulations of D along with the properties and argue that the measure has to be invariant to balanced growth. Balanced growth means to distribute any additional outcome (e.g. income) in the same way previous outcome was distributed, which corresponds precisely to the property of *scale invariance*. They argue that this property is very important and suggest therefore to use the dissimilarity index in equation (4) instead of alternative formulations. $D(\widehat{y})$ is used to compute the Human Opportunity Index (HOI), where it serves as an inequality penalty to the average access to the outcome in question (Paes de Barros et al., 2009)

We argue in the next subsection that the lack of *translation invariance* has some potentially unwanted consequences.

2.2 Concerns about missing translation invariance

Compared to continuous outcomes, the assessment of inequality of opportunity for binary or ordered[3] variables is generally more complicated, however, in at least one point these variables have an advantage. The conditional probabilities are defined on the interval [0,1], making the property of scale invariance of the inequality measure dispensable. Since the variable is clearly defined on the same scale for whatever binary outcome we analyze, the measure of inequality does not have to be invariant to changes in the scale.

The possibility of not respecting *scale invariance* enables us to focus on *translation invariance*, since both cannot be satisfied simultaneously by any meaningful inequality measure (Zheng, 1994)[4].

Our concern is that *translation invariance*, as opposed to *scale invariance*, is crucial for the

[3]We generally discuss the case of binary variables where $\widehat{y} = P(y = 1|C)$. However, this method can be easily extended to ordered variables by setting a certain threshold $\tilde{y}$ and computing $\widehat{y} = P(y \geq \tilde{y}|C)$

[4]See Ferreira and Gignoux (2013) for a discussion on this impossibility to satisfy both properties

estimation of inequality of opportunity when the outcome is dichotomous. Assume for instance a country where only one person has access to schooling[5]. Such a situation would be a serious problem of development, since the average level of education would be virtually zero. However, in terms of inequality one cannot say that it's the most extreme case of inequality. However, this is exactly what measures like the Gini index would suggest. On the other extreme, having all but one child in school would be the same in terms of inequality, however, the Gini index, would indicate a completely different result, indicating almost no inequality at all. We argue that in both cases the inequality should not be considered as being high since all but one individual have exactly the same level.

A different way to look at the problem is by comparing inequality in access to education to inequality in exclusion from education. We argue that in this context, exclusion is the contrary to access and therefore we describe the same situation. Hence, the inequality of opportunity measure should be the same in the two definitions.

Proposition 1. *The level of inequality in the access to a certain good equals the level of inequality in the exclusion from the same good. This is*

$$I(p) = I(1-p) \tag{5}$$

where p is a vector of probabilities and therefore $0 \leq p_i \leq 1 \ \forall \ i$.

In the next two sections, we propose two ways to deal with the aforementioned concern and to quantify the level and the dispersion effects.

2.3 Decomposition in dispersion and level effects

Before proposing a translation invariant measure of inequality of opportunity in the next section, we first focus on a decomposition of changes in common inequality measures into a part due to changes in the dispersion of the distribution and to changes in the average level.

Let $d(y)$ be a measure of dispersion and $l(y)$ the average level of the vector y and let us define an inequality measure that takes these two values as argument:

$$I[d(y), l(y)] \tag{6}$$

Now, let us introduce subscripts to indicate two different vectors (e.g. two moments in time, two regions, etc.) and let us simplify the notation by omitting the vector ($d_1 \equiv d(y_1)$). By taking the difference of the inequality measures applied to the two vectors, we have:

$$\Delta I = I(d_1, l_1) - I(d_0, l_0) \tag{7}$$

[5]One person has a probability of 1 and the other children a probability of 0.

where we can add and subtract identical terms and rearrange:

$$I(d_1, l_1) - I(d_0, l_0) = \underbrace{[I(d_1, l_1) - I(d_0, l_1)]}_{\text{dispersion effect}} + \underbrace{[I(d_0, l_1) - I(d_0, l_0)]}_{\text{level effect}} \tag{8}$$

in order to get a decomposition of the total difference in a part due to the dispersion and a part due to the average level.

Coming back to the example of the dissimilarity index introduced before, it is relatively easy to see that the whole index has a component measuring dispersion and a component of the level, which ensures in the end scale invariance. For the sake of readability, we discuss this decomposition of the dissimilarity index on a generic vector y instead of $\widehat{y}$ we used in equation (4)

$$D(y) = \underbrace{\frac{\sum_{i=1}^{N} |y_i - \overline{y}|}{2N}}_{\text{dispersion}} \times \underbrace{\frac{1}{\overline{y}}}_{\text{level}} \tag{9}$$

Using subscript 0 for the first and 1 for the second period, we can decompose the change in the dissimilarity index according to equation 8 as follows:

$$D(y_1) - D(y_0) = \underbrace{\left[\frac{\sum_i^{N_1} |y_{1,i} - \overline{y_1}|}{2N_1\overline{y_1}} - \frac{\sum_i^{N_0} |y_{0,i} - \overline{y_0}|}{2N_0\overline{y_1}}\right]}_{\text{dispersion effect}} + \underbrace{\left[\frac{\sum_i^{N_0} |y_{0,i} - \overline{y_0}|}{2N_0\overline{y_1}} - \frac{\sum_i^{N_0} |y_{0,i} - \overline{y_0}|}{2N_0\overline{y_0}}\right]}_{\text{level effect}} \tag{10}$$

This decomposition of an inequality measure is not limited to the dissimilarity index. Let us take the Gini index given by:

$$G(y) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|}{2\overline{y}n(n-1)} \tag{11}$$

Thus, the decomposition is given by:

$$\begin{aligned} G(y_1) - G(y_0) = & \overbrace{\left[\frac{\sum_{i=1}^{n_1} \sum_{j=1}^{n_1} |y_{1,i} - y_{1,j}|}{2\overline{y_1}n_1(n_1-1)} - \frac{\sum_{i=1}^{n_0} \sum_{j=1}^{n_0} |y_{0,i} - y_{0,j}|}{2\overline{y_1}n_0(n_0-1)}\right]}^{\text{dispersion effect}} \\ & + \underbrace{\left[\frac{\sum_{i=1}^{n_0} \sum_{j=1}^{n_0} |y_{0,i} - y_{0,j}|}{2\overline{y_1}n_0(n_0-1)} - \frac{\sum_{i=1}^{n_0} \sum_{j=1}^{n_0} |y_{0,i} - y_{0,j}|}{2\overline{y_0}n_0(n_0-1)}\right]}_{\text{level effect}} \end{aligned} \tag{12}$$

Such a decomposition does not allow us to eliminate our concerns on the use of the dissimilarity index, but it allows us to quantify and illustrating the sources of the changes. We illustrate this decomposition using data from Mexico and estimating the access to education in section 3. Before that, we first introduce in the next section a translation invariant version of the dissimilarity index.

2.4 Translation invariant version

In order to satisfy proposition 1, we need a translation invariant measure of inequality, in contrast scale invariance is not required, since the (probability of) access to a certain good is defined on the interval [0,1] exclusively. Starting from the dissimilarity index, we can simply leave aside the element ensuring scale invariance (see equation 9), giving us:

$$\tilde{D}(y) = \frac{1}{2N}\sum_{i=1}^{N}|y_i - \overline{y}| \qquad \text{where } 0 \leq \tilde{D}(y) \leq \frac{1}{4} \tag{13}$$

in order to normalize the indicator to the interval of [0,1], we can simply multiply it by 4, which is

$$D^*(y) = \frac{2}{N}\sum_{i=1}^{N}|y_i - \overline{y}| \tag{14}$$

This is simply twice the average absolute distance to the population mean.

Properties of D^*

This slightly modified version of the dissimilarity index shares the basic properties such as anonymity, reflexivity and transitivity with the original dissimilarity index. Properties that are specific to D^* include:

- Normalized scale: $0 \leq D^*(y) \leq 1$ where the value of 0 is attained when everybody has the same (conditional) outcome and 1 whenever half of the population has the value of 1 and the other half zero.
- Translation invariance: $D^*(y) = D^*(y + \lambda)$ where λ is a constant.
- Inversion invariant according to proposition 1: $D^*(y) = D^*(1-y)$ whenever $0 \leq y_i \leq 1 \;\; \forall\, i$
- Proportional to multiplications: $D^*(\varphi y) = \varphi D^*(y)$ where φ is a positive scalar.

The proofs of these properties are presented in appendix A.

3 Illustration: inequality of opportunity in access to education

In section 2 we discussed scale and translation invariance in the measure of inequality of opportunity in an analytical way. Let us now turn to an empirical illustration to show how this discussion actually matters when estimating inequality of opportunity. We propose two related illustrations estimating inequality of opportunity in education using data from Mexico. In the first illustration, we focus on different education levels and the corresponding levels of inequality of opportunity. In the second illustration we focus on the evolution over time of inequality of opportunity in education.

3.1 Different schooling levels, different levels of inequality of opportunity?

Methodology used in this example

We apply the methodology proposed by Paes de Barros et al. (2008). In a first step, we define the conditional probability of having access to a certain level of education as $\widehat{y_i}$:

$$\widehat{y_i} = P(y_i > \tau | C_i) \tag{15}$$

where τ is the threshold education level and C_i is a matrix of circumstances. This probability can be estimated through logit or probit models[6]. This expected probability is then used to compute a measure of inequality. Along with the dissimilarity index proposed by Paes de Barros et al. (2008), we also include the Gini index and our modified dissimilarity index. The differences between our index and the first two helps us highlighting the importance of the different approaches.

Data

The data for this illustration come from the Mexican Family Life Survey[7] (MxFLS). We estimated the conditional probability of achieving different levels of schooling. The thresholds were chosen according to the Mexican Schooling System and represent finished levels of schooling. The first category is to have access to schooling in general, hence the binary indicator is 1 if the individual has at least 1 year of schooling. The subsequent thresholds are 6 for finished primary school, 9 for finished secondary school, 12 for high school and 16 for college. We limit our sample to individuals no longer attending school and not being older than 25 years, giving us a sample of 5'535 people. The matrix of circumstances includes the family log income and father's and mother's years of education, an ability measure and a dummy for indigenous people. Parental years of education are included as a set of dummy variables to achieve the highest possible flexibility of the model.

[6] Alternative estimation methods including non-parametric methods can also be applied in this case.

[7] Available under `http://www.ennvih-mxfls.org` and described in Rubalcava and Teruel (2006)

Results

First, we look at the decomposition of the changes from one threshold to the next in the Gini index by performing the decomposition as proposed in equation (12).

Table 1: Decomposition of changes in the Gini coefficient

	Gini Coeffcient	Total difference	Difference due to dispersion	Difference due to level
At least 1 year of schooling	0.019			
Finished primary or more	0.139	0.121	0.106	0.015
Finished secondary or more	0.229	0.090	-0.043	0.133
Finished high school or more	0.411	0.182	-0.456	0.638
College or more	0.520	0.108	-0.174	0.282

Note: The difference refers to the value of a row minus the value of the preceding row. The decomposition is done using equation 12

Table 1 displays the decomposition of the change in the Gini index from one category to the next. The Gini index constantly increases with the level of education, suggesting that inequality of opportunity becomes higher the higher the evaluated schooling level is. The increase of the Gini index from *"At least 1 year of schooling"* to *"Finished primary or more"* is of 0.121 and almost all of it is due to a larger dispersion, only 0.015 is due to the level effect. For the remaining comparisons, the level effect becomes larger and the dispersion effect smaller and getting even negative. Comparing primary and secondary education shows impressively that the dispersion goes down but the Gini increases a lot due to the lower average. We have therefore contradictory effects for most of the changes and can see clearly that the sharp increase in inequality of opportunity is exclusively due to the level effect.

We can have a look at the same story considering it graphically. Figure 1 displays different measures of inequality of opportunity for different levels of education in Mexico. The clear positive trend from Table 1 can be seen for both the Gini and the Dissimilarity index.

The two remaining lines are corrected measures, where the line entitled *Counterfactual DI* is the counterfactual dissimilarity index when assuming the average access of the population to be equal to the secondary school level for all levels. This is the reason why the line is exactly at the same position for the threshold of 9 years. It can be clearly observed, that the shape is very different, suggesting that the highest dispersion of access probabilities is for primary and secondary school, while it goes down for higher levels of education. The same story is told by the last indicator, the $D^*(y)$ introduced in equation 14. Again, the highest dispersion is observed for the middle education levels.

The differences between the two pattern underline that the discussion on *scale* and *translation invariance* is not anecdotal, since depending on which we put the emphasis, the results are substantially different. According to *scale invariant* measure policy should probably focus on reducing inequality in opportunity at the highest levels of education, while the recommendation

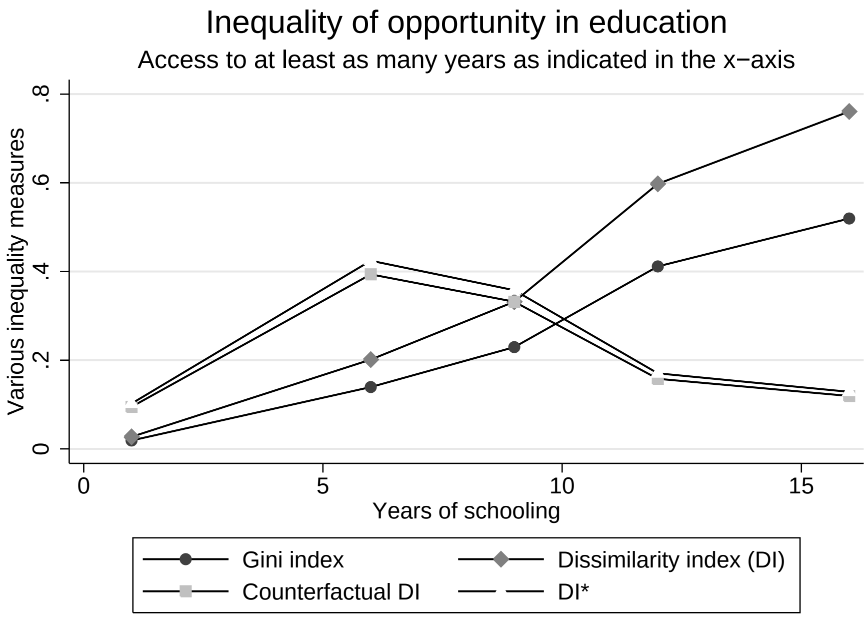

Figure 1: Inequality of opportunity in access to education (different measures)

from the *translation invariant* measure would be to pay special attention to secondary education.

3.2 Decomposing the change over time

In a second illustration, we now focus on the change over time of inequality in the access to secondary education, applying exactly the same methodology as before, but for different samples over time.

Data

We make use of the bi-annual *Encuesta Nacional de Ingresos y Gastos de los Hogares* (ENIGH) from Mexico for the period from 1994 to 2010. The ENIGH is a nationally representative survey with sample sizes of over 50'000 for the most recent years. We construct a dummy variable for people having secondary education or more as a dependent variable. This indicator is then explained by circumstances including gender and parental education and literacy. To ensure comparability across years, we limit the samples to the age range of 15 to 25 years.

Results

Figure 2 displays the level of inequality of opportunity using the Gini index. The line in the middle shows the actual value of the index. The other two lines are counterfactual curves where the average access was set to the levels of 1994 (light gray rhombus) and 2010 (dark gray rectangles) respectively.

Looking at the Gini index, a clear and constant decrease of inequality can be observed. Look-

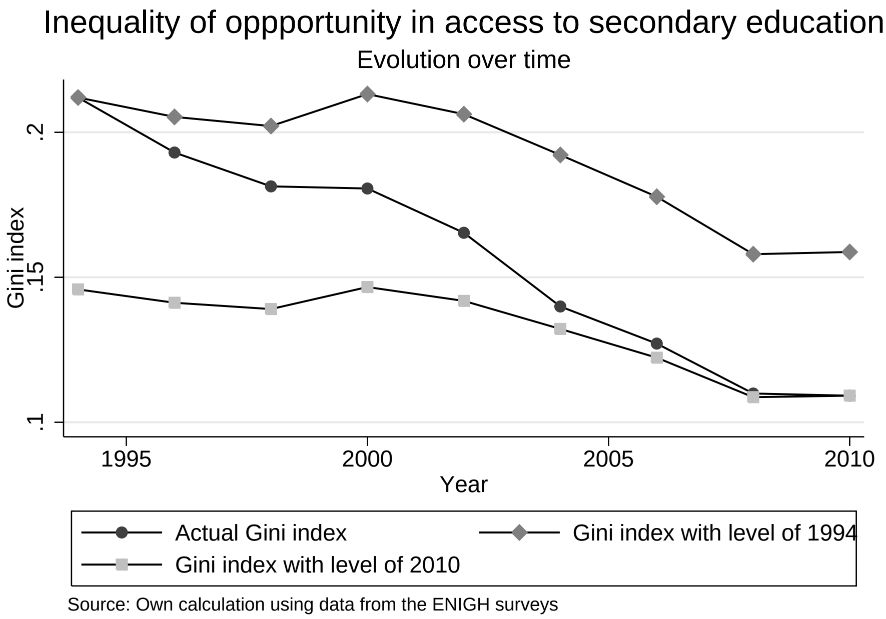

Figure 2: Inequality of opportunity in access to secondary schooling

ing at the two counterfactual curves the story is different. From 1994 to 2000 there was basically no change, while from 2000 on the inequality level went down constantly. The fact of having two counterfactual lines here is due to the general problem of reference in such decomposition exercises. Actually, we could use any of the years as reference. The differences are not very large and more importantly the pattern are essentially the same in the two cases here.

Table 2 provides the numerical analysis of the same exercise by displaying the changes of the Gini index between two time periods and the decomposition in dispersion and level effect. The comparison is done for two consecutive periods and with respect to the first period (1994). Both effects are generally negative, however, the dispersion effect if dominated by the level effect, suggesting that the overall decrease in inequality of opportunity is more due to the change in the average access to secondary schooling than to a reduction in the different probabilities. The relative share of importance is roughly two thirds for the level effect and one third for the dispersion effect.

4 Conclusion

In this paper we rose the concern of unwanted effects in the measurement of inequality of opportunity when the outcome is a binary indicator due to the missing translation invariance of the inequality measures used. First we propose a decomposition of commonly used inequality indexes such as the Gini index into a level and a dispersion effect. This allows us to illustrate that measuring inequality of opportunity with a *scale invariant* measure yields to substantially different results from a measurement based on a *translation invariant* inequality indicator. We

Table 2: Decomposition of changes in the Gini coefficient over time

Year	Gini	Compared to previous period			Compared to 1994		
		Δ_{tot}	Δ_d	Δ_l	Δ_{tot}	Δ_d	Δ_l
1994	0.2120						
1996	0.1930	-0.0190	-0.0063	-0.0127	-0.0190	-0.0063	-0.0127
1998	0.1814	-0.0117	-0.0028	-0.0088	-0.0306	-0.0088	-0.0218
2000	0.1806	-0.0007	0.0093	-0.0101	-0.0314	0.0010	-0.0324
2002	0.1653	-0.0153	-0.0056	-0.0097	-0.0467	-0.0046	-0.0420
2004	0.1399	-0.0254	-0.0102	-0.0152	-0.0721	-0.0144	-0.0576
2006	0.1271	-0.0128	-0.0103	-0.0025	-0.0849	-0.0245	-0.0604
2008	0.1099	-0.0172	-0.0138	-0.0034	-0.1021	-0.0376	-0.0645
2010	0.1092	-0.0007	0.0005	-0.0013	-0.1028	-0.0366	-0.0662

Note: The difference refers to the value of a row minus the value of the preceding row in the first decomposition and minus the value for 1994 in the second. The decomposition is done using equation 12. Δ_{tot} is the total difference, Δ_d the difference due to changes in dispersion and Δ_l the level effect.

propose a modified dissimilarity index which is twice the average absolute distance to the population average. This new indicator is translation invariant, a property we believe to be important in the measurement of inequality of opportunity with binary indicators, since otherwise the results could be driven by the level of access rather the disparities in access.

Based on our illustrations and methodological discussion, we recommend to consider the decomposition of inequality measures into level and dispersion effect whenever comparing two measures of inequality. Understanding where the differences come from is crucial for sound conclusions. In the analysis of inequality of opportunity when the outcome is binary, we recommend the use of the adapted index proposed in this paper, since it satisfies the proposition that inequality of opportunity in the access to a good equals inequality of opportunity in the exclusion from that same good. To ensure a complete understanding of inequality of opportunity, the use of both *translation* and *scale invariant* measures might be needed.

References

Checchi, Daniele and Vito Peragine, "Inequality of Opportunity in Italy," *Journal of Economic Inequality*, 2010, *8*, 429–450.

Ferreira, Francisco H.G. and Jérémie Gignoux, "The Measurement of Inequality of Opportunity: Theory and an Application to Latin America," *The Review of Income and Wealth*, 2011, *57* (4), pp. 622–657.

_ and _ , "The Measurement of Educational Inequality: Achievement and Opportunity," Forthcoming in The World Bank Economic Review. Advance Access published February 20, 2013. doi: 10.1093/wber/lht004 2013.

Paes de Barros, Ricardo, Francisco Ferreira, José Molinas Vega, and Jaime Saavedra Chanduvi, *Measuring Inequality of Opportunity in Latin America and the Caribbean*, The World Bank, Washington DC., 2009.

— , **Jose R. Molinas Vega, and Jaime Saavedra**, "Measuring inequality of Opportunities for Children," in "Measuring Inequality Of Opportunity in Latin America and the Caribbean - Volume 2," The World Bank., March 2008.

— , **Mirela de Carvalho, and Samuel Franco**, "Preliminary Notes on the Measurement of Socially-Determined Inequality of Opportunity when the outcome is Discrete," 2007. Working Paper. Mimeo.

Roemer, John E., *Equality of Opportunity*, Harvard University Press, Cambridge, 1998.

Rubalcava, Luis and Graciela Teruel, "Guía del usuario para la Primera Encuesta Nacional sobre Niveles de Vida de los Hogares," Online available at: www.ennvih-mxfls.org 2006.

Zheng, Buhong, "Can a Poverty Index be Both Relative and Absolute?," *Econometrica*, 1994, *62* (6), pp. 1453–1458.

A Proofs

In this short appendix we present the simple proofs of the properties presented in section 2.4. To simplify the notation, we use the expected value notation instead of the sum:

$$D^*(y) = 2E\left[|y - E[y]|\right] \tag{16}$$

Normalized range between 0 (no inequality) and 1 (highest possible inequality)

To show that the lower bound of the index is 0 is easy, just let $y_i = \bar{y} \; \forall \; i$. In this case we have:

$$D^*(y) = 2E\left[|y - E[y]|\right] = 2E\left[|\bar{y} - \bar{y}|\right] = 0 \quad \square \tag{17}$$

To show, that the maximum value is attained whenever half of the population has the value of zero and the other half the value of 1, we proceed in two steps. First, we show that in this case, the value of $D^*(y)$ is equal to 1. We know that $\bar{y} = \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}$. Thus, the index $D^*(y)$ is given by:

$$D^*(y) = 2\left(\frac{1}{2}|0 - \frac{1}{2}| + \frac{1}{2}|1 - \frac{1}{2}|\right) = 2\left(1| \pm \frac{1}{2}|\right) = 1 \quad \square \tag{18}$$

We can move in two ways away from that situation, first by changing the level of the two groups and second by changing the proportions. It is thus sufficient to show that in both cases we end up with a lower value of $D^*()$ to proof that we actually found the maximum value. We start by changing the values of both groups. Now, let us add a mean preserving transfer from people with the value of 1 to those with the value of 0. In this case we get:

$$D^*(y) = 2\left(\frac{1}{2}|0+\epsilon-\frac{1}{2}|+\frac{1}{2}|1-\epsilon-\frac{1}{2}|\right) = 2|\frac{1}{2}-\epsilon| < 1 \quad \square \tag{19}$$

And finally let us show that whenever the proportions are not equal, the value of the index is also smaller. We change a proportion of ε from the group with 0 outcome to the group of 1 outcome. The average access is then: $\bar{y} = (\frac{1}{2}-\varepsilon)0 + (\frac{1}{2}+\varepsilon)1) = \frac{1}{2}+\varepsilon$. The index is therefore given by:

$$D^*(y) = 2\left((\frac{1}{2}-\varepsilon)|0-\frac{1}{2}-\varepsilon| + (\frac{1}{2}+\varepsilon)|1-\frac{1}{2}-\epsilon|\right) \tag{20}$$

since both terms in absolute values are by definition between 0 and 1, we can drop the absolute values and get:

$$\begin{aligned} D^*(y) &= 2\left((\frac{1}{2}+\varepsilon)(\frac{1}{2}-\varepsilon) + (\frac{1}{2}-\varepsilon)(\frac{1}{2}+\varepsilon)\right) \\ &= 1-4\varepsilon^2 < 1 \quad \square \end{aligned} \tag{21}$$

Translation invariance

To proof that $D^*(y)$ is *translation invariant* we add a vector containing the constant λ to each element of the initial vector and solve to get $D^*(y)$ again.

$$\begin{aligned} D^*(y+\lambda) &= 2E\left[|y+\lambda-E[y+\lambda]|\right] = 2E\left[|y+\lambda-E[y]-\underbrace{E[\lambda]}_{=\lambda}|\right] \\ &= 2E\left[|y+\lambda-E[y]-\lambda|\right] = 2E\left[|y-E[y]|\right] = D^*(y) \quad \square \end{aligned} \tag{22}$$

Inversion invariant

Similarly, we can show easily that $D^*(y)$ satisfied proposition 1 saying that $D(y) = D(1-y)$:

$$\begin{aligned} D^*(1-y) &= 2E\left[|1-y-E[1-y]|\right] = 2E\left[|1-y-E[1]+E[y]\right] \\ &= 2E\left[|1-y-1+E[y]|\right] = 2E\left[|y-E[y]|\right] = D^*(y) \quad \square \end{aligned} \tag{23}$$

Proportional to multiplication

Finally, let us show that $D^*(\varphi y) = \varphi D^*(y)$ where φ is a positive scalar.

$$\begin{aligned} D^*(\varphi y) &= 2E\left[|\varphi y - E[\varphi y]|\right] = 2E\left[|\varphi y - \varphi E[y]|\right] \\ &= 2E\left[|\varphi(y-E[y])|\right] = \varphi 2E\left[|(y-E[y])|\right] = \varphi D^*(y) \end{aligned} \tag{24}$$