When More Does Not Necessarily Mean Better: Health-related Illfare Comparisons with Non-monotone Welfare Relationships

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When more does not necessarily mean better: Health-related illfare comparisons with non-monotone welfare relationships

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Abstract

Most welfare studies are based on the assumption that wellbeing is monotonically related to the variables used for the analysis. While this assumption can be regarded as reasonable for many dimensions of wellbeing like income, education, or empowerment, there are some cases where it is definitively not relevant, in particular with respect to health. For instance, health status is often proxied using the Body Mass Index (BMI). Low BMI values can capture undernutrition or the incidence of severe illness, yet a high BMI is neither desirable as it indicates obesity. Usual illfare indices derived from poverty measurement are then not appropriate. This paper proposes illfare indices that are consistent with some situations of non-monotonic wellbeing relationships and examines the partial orderings of different distributions derived from various classes of illfare indices. An illustration is provided for health poverty as proxied by the BMI and weight-for-age indicators using DHS data for Bangladesh during the period 1997–2007. It is shown inter alia that the gains from the decline of undernutrition for Bangladeshi mothers are undermined by the rapid increase of obesity.

Keywords: Illfare comparisons, poverty measurement, stochastic dominance, monotonicity, Bangladesh, nutrition transition.
JEL Classification: D63, I3.

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§ Please note that this version is an alpha release. It is incomplete and may contain notable mistakes and typos. As a consequence, it is likely to be slightly modified in the future. Please do not quote or cite.

1 Introduction

Target 1.C from the Millennium Development Goals states that the proportion of people who suffer from hunger should be halved between 1990 and 2015. Although this objective is presumed not to be met in 2015, the share of undernourished individuals has declined during the period(de Onis, Blössner, Borghi, Frongillo, and Morris, 2004, Department of Economic and Socia 2012). For instance, the FAO finds that the share of undernourished people in the developing world fell from about 20% to 15% during the period 1990-2010.[1] However, a stylized fact in most developing countries is that progress with respect to undernutrition have often been associated with increase in obesity (Popkin, Adair, and Ng, 2012). This so-called nutrition transition raises the issue of a net gain in social welfare with respect to health. Should we consider that the level of welfare in a society has improved if undernutrition has declined but other forms of malnutrition have become more severe? If we want to perform a global assessment of the social progress with respect to nutrition, then we need to render the situations of underweighted and overweighted individuals socially comparable.

Wellbeing is generally supposed to be monotonically related to the variables used for the analysis in poverty and welfare studies. While this assumption can be regarded as reasonable for many dimensions of wellbeing like income, education, or empowerment, there are some cases where it is definitively not relevant, in particular with respect to health. For instance, health status is often proxied using the Body Mass Index (BMI) in the case of adults,[2] or using weight-for-age or height-for-age in the case of children and adolescents. Low BMI values can capture undernutrition or the incidence of severe illness, yet a high BMI is neither desirable as it indicates obesity. That is why the BMI is usually compared against a left-tail and a right-tail cut-off which work as deprivation lines, e.g. 18.5 kg/m^2 and 25 kg/m^2, respectively. Estimating aggregate illfare using traditional poverty indices, based on a unique (left-tailed) deprivation line, is therefore not appropriate. Likewise several other health indicators are characterized by the use of two deprivation lines for diagnostic purposes because they relate to situations in which either “having too much” or “too little” is detrimental to health. That is the case of several blood tests, including blood pressure, Thyroid function, hemoglobin and total cholesterol.[3]

This paper first proposes illfare indices that are consistent with situations of non-monotonic relationships between wellbeing and its indicators, like the aforementioned examples. These indices are decomposable into two indices that, respectively, measure a concept of “loss” illfare and another one of “excess” illfare. While “loss” illfare is identical to the traditional understanding of poverty as insufficiency, “excess” illfare refers to wellbeing harmed by suboptimal abundance. The family of indices is axiomatically characterized and includes extensions to traditional poverty indices like the Foster-Greer-

[1] Figures are from the 2012 Millenium Development Goals Report (Department of Economic and Social Affairs of the U. N. Secretariat, 2012).

[2] The BMI, also known as the Quetelet index, is defined as the individual’s body mass (in kilograms) divided by the square of his her height (in meters).

[3] As suggested by a seminar participant at the CERDI, our framework can also be applied to free time. Both having too much free time or not having enough of it, could be a source of major stress and result in low wellbeing. On time poverty, see for instance Vickery (1977), Harvey and Mukhopadhyay (2007), Gammage (2010), Bardasi and Wodon (2010).

Thorbecke family and the Watts index. For the purpose of characterization we introduce key alterations to the traditional axioms of focus, monotonicity and transfers.

Indices provide precise and useful informations as well as a complete ordering of observed distributions. However, they are all based on specific underlying welfare functions (Blackorby and Donaldson, 1980) upon which agreement may not be met. Of course, in the health context, risks of death or severe disease may theoretically be precisely estimated for the the different values of the variable under consideration, but it is not so clear how people value such risks in terms of wellbeing. The relationship becomes even more complex once psychological and social aspects of health are taken into account. For these reasons, it is necessary to look for criteria that make it possible to draw robust conclusions about the state of illfare, that is to obtain results that do not depend on the specific functional forms used to assess illfare. The paper also examines the partial orderings of different distributions, according to sub-families of our class of illfare indices, by deriving the required first and second-order stochastic dominance conditions. We also study the conditions for partial orderings when the experience of one form of illfare (e.g. "loss" illfare) is considered to be worse than the other one (e.g. "excess" illfare).

The rest of the paper is organized as follows: The next section introduces the family of non-monotone illfare indices and its associated partial ordering conditions. The third section proposes stochastic dominance conditions when the two forms of illfare are deemed to have differential effects on wellbeing. Section 4 shows how to compute the standard errors for the family of indices and the fifth section provides an empirical illustration using Bangladeshi data from the Demographic Health Survey (DHS) for the period 1997–2007. It is shown that health-related illfare levels have declined during the period for both mothers and under-5 children but that the overall improvement is partly offset by the increase of obesity. The paper concludes with some final remarks.

2 Non-monotone poverty measurement: The general case

2.1 Two classes of poverty indices with revised versions of the focus, monotonicity and transfer axioms

Let x describe an individual attribute defined on the domain $\Omega := [\omega^-, \omega^+] \subset \Re$. Illfare may then be assessed using unidimensional additive poverty indices $P(z)$ that are of the type:

$$P(z) := \int_{\omega^-}^{z} \pi(x,z)\, dF(x), \tag{1}$$

where F is the cumulative distribution function (cdf), $z \in \Omega$ is the poverty line, and $\pi : \Omega \times \Omega \to \Re_+$ is an individual poverty index such that:

$$\pi(x,z) \begin{cases} \geqslant 0 & \text{if } x \leqslant z, \\ = 0 & \text{otherwise.} \end{cases} \tag{2}$$

Indices of the family (1) satisfy the traditional properties of continuity, anonymity, population replication, focus and decomposability. Moreover, they also comply with weak monotonicity if $\frac{\partial \pi}{\partial x} \leqslant 0$. In general the monotonicity axiom enjoys broad consensus and is consistent with poverty assessments based on income.

With indices (1), illfare is associated with insufficient level of the variable x with regard to a norm corresponding to z. However, the relevant space for conceptualizing wellbeing is rarely the one where attribute x is defined. Indeed, the "failure to achieve certain minimum capabilities" (Sen, 1985) does not systematically mean an insufficient value for x. So, in the space of capabilities, illfare can be defined as a lack of resources but potentially not in the space of x. Considering nutrition, a person is health-deprived if she does not have the ability to get an adequate and balanced diet, regarding her physiological, psychological and social needs. Causes of this inability are diverse, including for instance low income, limited access to diversified sources of nutrients, insufficient information on the importance of a balanced diet, severe diseases or handicaps, and mental disorders. Whatever the precise roots of health-related illfare, we consider them to be the expression of low capabilities.[4]

Here we consider illfare indices that do not exhibit the same behaviour as indices (1) because the underlying relationship between variable x and welfare is not supposed to be monotonic. More specifically, we introduce a set of deprivation lines $\{z^L, z^U\} \subset \Omega$, with $z^L < z^U$, such that:[5]

$$\pi(x; z^L, z^U) \begin{cases} \geqslant 0 & \text{if } x \leqslant z^L, \\ = 0 & \text{if } x \in]z^L, z^U[, \text{ and} \\ \geqslant 0 & \text{if } x \geqslant z^U. \end{cases} \tag{3}$$

Hence here illfare relates to situations in which either "having too much" or "having too little" is detrimental for individual wellbeing. We note at the outset that such non-monotone relationship with respect to health has already been investigated regarding health-inequalities (e.g. Dutta, 2007), but, to the best of our knowledge, no tool has yet

[4] Low capabilities, or capability deprivation (Sen, 2001) can be a valuable definition for poverty, but in the present study we prefer using the term "illfare". Indeed, in the case of some health outcomes, even though obesity is often associated with low income in developed countries (see for instance Drewnowski and Darmon, 2005), being overweight in many low-income countries is traditionally regarded as a sign of high socioeconomic status (see, for instance, references in Poterico, Stanojevic, Ruiz-Gross, Bernabe-Ortiz, and Miranda, 2012). Such situations can be seen as an illustration of the discrepancy between the private and social evaluations of life. Though people may value more the exhibition of affluence than the potential risks involved in being overweight, it is reasonable to ask the social evaluator to prioritize the health aspects. However, recent evidence shows that, even in developing countries, obesity tends to become more related to monetary poverty (Popkin, Adair, and Ng, 2012) or educational deprivation (Poterico, Stanojevic, Ruiz-Gross, Bernabe-Ortiz, and Miranda, 2012). Moreover, there is evidence that undernutrition of the mother during pregnancy as well as undernutrition during childhood may result in a higher probability of obesity for the children (e.g. Roseboom and de Rooij, 2006).

[5] Here we suppose that the same deprivation lines z^L and z^U can be applied for each individual within the observed populations, and that they are exogenous with respect to the observed values of x within these populations. The first assumption means that the same thresholds can be applied for each person whatever her sex, age, or any other relevant characteristic. Both for poverty measurement and dominance tests, that assumption can be relaxed, notably by rescaling observed values of x so that all group-specific poverty lines coincide. The second assumption implies that we are measuring absolute poverty. While this focus is reasonable for physiological dimensions of health, it is admittedly contentious when dealing with psychological and social aspects. For instance, we could posit that obesity becomes a more acute concern when its prevalence is rare than when it is widespread among the population. These considerations are however left aside for future work.

been proposed for the social assessment of health poverty.

At the social aggregation level, we consider illfare indices P of the type:

$$P(z^L, z^U) := \int_{\omega^-}^{z^L} \pi(x; z^L, z^U)\, dF(x) + \int_{z^U}^{\omega^+} \pi(x; z^L, z^U)\, dF(x). \tag{4}$$

Note, firstly, that the definition of P in equation 1 can be seen as the limiting case $z^U = \omega^+$ of the definition in equation 4. Secondly, P in equation (4) does not fulfil the traditional definitions of the focus and monotonicity axioms proposed by Sen in his seminal article on poverty measurement (Sen, 1976). A poverty index is said to comply with the focus axiom if the poverty level does not change when a non-poor person receives more of x. However for any individual with $x \in]z^L, z^U[$, there is always an increment $\kappa > 0$ such that $x + \kappa \geqslant z^U$, *i.e.* the individual falls into illfare. Likewise, the monotonicity axiom usually states that poverty does not increase whenever a poor person augments her x. Nevertheless in our setting we posit that increases above the upper poverty line z^U should not decrease poverty. These conflicts are not surprising as the focus and monotonicity axioms are usually defined for indices in the shape of equation (1). Since the focus and monotonicity axioms express simple and desirable properties, it is worth proposing new definitions for these axioms befitting our specific framework. Formally:

Axiom (FOC). $P_A(z^L, z^U) = P_B(z^L, z^U)$ *if distribution* B *is obtained from distribution* A *by adding* $\kappa \in \Re$ *to any observed value* $x \in]z^L, z^U[$ *such that* $x + \kappa \in]z^L, z^U[$.

Axiom (MON). $P_A(z^L, z^U) \leqslant P_B(z^L, z^U)$ *if distribution* B *is obtained from distribution* A *i) by substracting* $\kappa > 0$ *to any observed value* $x \in [\omega^-, z^L]$ *such that* $x - \kappa \in \Omega$*, or ii) by adding* $\kappa > 0$ *to any observed value* $x \in [z^U, \omega^+]$ *such that* $x + \kappa \in \Omega$.

Axioms FOC and MON are thus defined in order to preserve the spirit underlying their usual definitions. FOC assumes that a change in x for a non-poor person does not change poverty as long as the person remains non-poor. The monotonicity axiom is usually defined to state that movements towards the poverty line for a poor person do not increase poverty. That is exactly what axiom MON states. To elucidate that point, let us introduce the concepts of "loss" poverty and "excess" poverty. The former refers to an insufficient amount of a wellbeing attribute x, usually judged by comparing against the left-tail poverty line z^L. By contrast, "excess" poverty is the situation of an excessive, and detrimental, amount of a wellbeing attribute, or indicator, e.g. the BMI; which is determined by comparing x against the right-tail poverty line z^U. Then our monotonicity axiom states that both a decrease in x for a "loss" poor person, and an increase in x for an "excess" poor person do not decrease overall poverty.

We can now define the following class of non-monotone illfare indices:

$$\Pi^1(z^{L+}, z^{U-}) := \left\{ P \;\middle|\; \begin{array}{l} [z^{L+}, z^{U-}] \subseteq [z^L, z^U] \subset \Omega \\ \pi(z; z^L, z^U) = 0 \forall z \in \{z^L, z^U\} \\ \pi^{(1)}(x; z^L, z^U) \leqslant 0,\ \forall x \leqslant z^{L+},\ \text{and}\ \pi^{(1)}(x; z^L, z^U) \geqslant 0,\ \forall x \geqslant z^{U-} \end{array} \right\}, \tag{5}$$

where $\pi^{(1)}(x; z^L, z^U) := \frac{\partial \pi}{\partial x}$. Members from $\Pi^1(z^{L+}, z^{U-})$ fulfill FOC and MON as defined above. They also comply with the traditional anonymity, additivity, continuity and population invariance axioms. Anonymity states that x is the sole characteristic explaining why two individuals could exhibit differing values of π. Thus, other characteristics like age, household size, ethno-linguistic features, or gender, should not be considered when assessing poverty. Additivity means that overall social poverty is the sum of individual poverty measures, a property that is desirable within our framework in order to assess the relative contribution of "loss" and "excess" poverty to overall poverty. Continuity at the poverty line is the result of the second condition in (5), and is necessary to prevent small measurement errors from producing non-marginal variations in the estimated poverty level.[6] Finally, the population invariance principle states that replicating each member of the population the same number of times does not change the level of poverty, so that population of different size can be compared in terms of poverty. Fulfillment of this property requires the social poverty function to be an arithmetic average of the individual measures.

Interesting examples of $P \in \Pi_1(z^{L+}, z^{U-})$ are the following extensions of the traditional Watts's (1968) and Foster, Greer, and Thorbecke's (1984) poverty indices:

$$W_\beta(z^L, z^U) := \int_0^{z^L} \log \frac{z^L - \omega^-}{x - \omega^-} \, dF(x) + \beta \int_{z^U}^{\omega^+} \log \frac{\omega^+ - z^U}{\omega^+ - x} \, dF(x), \tag{6}$$

$$FGT_{\beta,\alpha_L,\alpha_U}(z^L, z^U) := \int_{\omega^-}^{z^L} \left(\frac{z^L - x}{z^L - \omega^-} \right)^{\alpha_L} dF(x) + \beta \int_{z^U}^{\omega^+} \left(\frac{x - z^U}{\omega^+ - z^U} \right)^{\alpha_U} dF(x), \tag{7}$$

with $\beta > 0$, $\alpha_L \geqslant 1$, and $\alpha_W \geqslant 1$. The family $FGT_{\beta,\alpha_L,\alpha_U}$ also includes the headcount index for $\alpha_L = \alpha_W = 0$. The headcount index ($\alpha_L = \alpha_W = 0$) is not a member of $\Pi_1(z^{L+}, z^{U-})$, as it is not continuous within the poverty domain; but provides useful information regarding the prevalence of poverty within the population. β is a weighing parameter that gives more emphasis on "loss" poverty for $\beta \in (0, 1)$ and on "excess" poverty for $\beta > 1$. The parameters α_L and α_U regulate the index's sensitivity to extreme forms of deprivation.

These indices are relative indices as the size of individual deprivations is normalized by the corresponding value for the maximum deprivation. Alternatively, one may use, for instance, the following absolute version of the $FGT_{\beta,\alpha_L,\alpha_U}$:

$$FGT^A_{\beta,\alpha_L,\alpha_U}(z^L, z^U) := \int_{\omega^-}^{z^L} \left(z^L - x\right)^{\alpha_L} dF(x) + \beta \int_{z^U}^{\omega^+} \left(x - z^U\right)^{\alpha_U} dF(x), \tag{8}$$

with $\alpha_L \geqslant 0$ and $\alpha_U \geqslant 0$.

In line with Sen (1976) we may prefer poverty indices to be sensitive to inequalities between the poor. Such distribution-sensitive indices are then supposed to comply with a transfer axiom that states that progressive transfers between two-poor individuals should decrease, or at least not increase, poverty. However, it is worth noting that, contrary to poverty indices of the type (1), Pigou-Dalton transfers within our framework have to be

[6] Note that continuity at the poverty line is not necessary for the design of first order stochastic conditions. Consequently, the conditions expressed below in Proposition 1 could also be applied to a broader class of poverty indices that may not respect continuity at the poverty line. On the other hand, continuity is desirable for second order dominance conditions. On this specific point, see for instance Araar and Duclos (2006).

considered over a non-convex set since the poverty domain is defined by the union of non-contiguous intervals. Consequently, we may consider three cases: *i)* when both people are "loss" poor; *ii)* when both are "excess" poor; and *iii)* when the two poor belong to different groups. The first two cases can be handled in the same manner as rank-preserving progressive transfers are in the traditional poverty literature (i.e. based on (1)). In the third case, a transfer from the "excess" poor to the "loss" poor means wellbeing improvements for both people, therefore it can be addressed using MON. Hence the apparent inability of our transfer axiom to deal with transfers between any pair of poor individuals is not a a matter of concern, since our poverty indices comply with MON.

The transfer axiom can thus be presented in the following manner:

Axiom (TRA). $P_A(z^L, z^U) \geqslant P_B(z^L, z^U)$ *if distribution* B *is obtained from distribution* A *by transferring* $\kappa > 0$ *from individual* i *to individual* j *such that* $\{x_i, x_j\} \subset [\omega^-, z^L]$ *or* $\{x_i, x_j\} \subset [z^U, \omega^+]$*, and* $|x_i - x_j| \geqslant |(x_i - \kappa) - (x_j + \kappa)|$.

Note that $W(z^L, z^U)$ complies with TRA while members from the class $FGT_{\alpha_L,\alpha_U}(z^L, z^U)$ respect this transfer axiom only for $\alpha_L > 1$ and $\alpha_U > 1$.

If we want poverty not to increase in the aftermath of Pigou-Dalton transfers, then we can consider the following class of indices satisfying TRA:

$$\Pi^2\left(z^{L+}, z^{U-}\right) := \left\{ P \in \Pi^1(z^{L+}, z^{U-}) \,\middle|\, \begin{array}{l} \pi^{(1)}\left(z; z^L, z^U\right) = 0 \forall z \in \{z^L, z^U\} \\ \pi^{(2)}\left(x; z^L, z^U\right) \geqslant 0,\ \forall x \in \Omega \end{array} \right\}, \tag{9}$$

where $\pi^{(2)}\left(x; z^L, z^U\right) := \frac{\partial^2 \pi}{(\partial x)^2}$. The first condition is basically a continuity assumption. The second condition in (9) captures the requirement regarding the sensitivity of the social poverty function to progressive transfers. In formal terms, the additivity of P associated with the second condition in (9) means that members from $\Pi^2\left(z^{L+}, z^{U-}\right)$ are S-convex in "loss" poverty values of x and also S-convex in "excess" poverty values of x. Both conditions mean finally that the marginal gain in the improvement of the situation of a poor person decreases and tends to zero as she moves closer to her deprivation line. It can be regarded as a desirable property as it rewards policy efforts focused on individuals experiencing severe "losses" or "excesses."

2.2 Partial orderings

The limited set of conditions expressed for the definition of the classes $\Pi^1\left(z^{L+}, z^{U-}\right)$ and $\Pi^2\left(z^{L+}, z^{U-}\right)$ leaves the door open for a wide variety of poverty indices; modified Watts and FGT indices are only suggestions of appropriate indices within our non-monotone framework. In the following paragraphs, we derive full robustness conditions for ordinal poverty comparisons based on stochastic dominance conditions; that is, results that do not hinge on specific poverty indices or poverty lines choices. We first propose a set of criteria for the class of poverty measures Π^1.

Proposition 1.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \forall P \in \Pi^1(z^{L+}, z^{U-}) \tag{10}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in [\omega^-, z^{L+}] \tag{11}$$
$$\textit{and} \quad F^A(x) \geqslant F^B(x) \quad \forall x \in [z^{U-}, \omega^+]. \tag{12}$$

Proof. See appendix A.1 ■

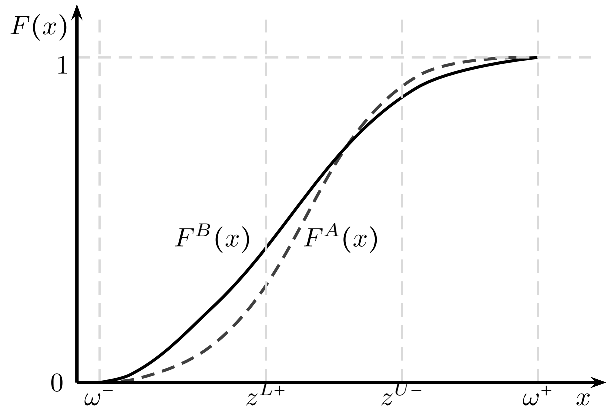


Figure 1: First order dominance

The first-order dominance relationship presented in Proposition 1 states that poverty in distribution A is not higher than in distribution B if the value of the "loss" poverty headcount index is never larger for distribution A for each value of the poverty line within the largest admissible "loss" poverty domain $[\omega^-, z^{L+}]$, and if the "excess" poverty headcount is never higher in A for each poverty line within the largest admissible "excess" poverty domain $[z^{U-}, \omega^+]$. Note that the "excess" poverty headcount is the survival function: $\overline{F}(z) := \Pr[x \geqslant z] = 1 - F(z)$. Hence condition (12) in Proposition 1 can alternatively be rendered: $\overline{F}^A(x) \leqslant \overline{F}^B(x) \quad \forall x \in [z^{U-}, \omega^+]$.

To illustrate numerically the conditions in Proposition 1, let us consider distributions $A := (1, 4, 6, 9, 12, 14)$ and $B := (1, 4, 7, 8, 13, 14)$, and assume $z^{L+} = 5$ and $z^{U-} = 10$. Using Proposition 1, it can easily be seen that distribution A never shows more poverty than distribution B for all indices in Π^1 and all pairs of poverty lines $\{z^L, z^U\} \notin (z^{L+}, z^{U-})$ since $F^A(x) = F^B(x)\ \forall x \in [\omega^-, 5] \cup [10, 12[\cup [13, \omega^+]$ but $F^A(x) > F^B(x)\ \forall x \in [12, 13[$. A similar situation is depicted by figure 1, which shows that the conditions from proposition 1 are fulfilled since distribution A's cdf is never above (below) B's for values of x lower (greater) than z^{L+} (z^{U-}).

Let $\boldsymbol{x}$ be a vector of values for the variable x and $\#(\boldsymbol{x})$ be the number of elements of $\boldsymbol{x}$. The following corollaries ensue directly from proposition 1:

Corollary 1. *There is a first-order stochastic dominance relationship between A and B $\forall P \in \Pi^1(z^{L+}, z^{U-})$ if $\exists \hat{\boldsymbol{x}} \in]z^{L+}, z^{U-}[^{\#(\hat{\boldsymbol{x}})}$ such that F^A and F^B cross only at the sole values of $\hat{\boldsymbol{x}}$ and $\#(\hat{\boldsymbol{x}})$ is an odd number.*

Proof. Obvious. ■

Corollary 2. *If $z^{L+} = z^{U-} = \tilde{z}$, distribution A dominates distribution B at the first order $\forall P \in \Pi^1(z^{L+}, z^{U-})$ if and only if F^A and F^B cross only once and at $\tilde{z}$.*

Proof. Obvious. ■

Here it is worth noting that Proposition 1 is reminiscent of famous results from the literature on risk (Rothschild and Stiglitz, 1970) and inequality (Atkinson, 1970) measurement as the distribution that shows more poverty also exhibits more weight at the tails of its distribution. However, Corollaries 1 and 2 show that dominance conditions are less restrictive since risk and inequality dominance conditions are defined for the distributions of the variable x after normalization with respect to the mean, or for distributions with the same mean. As a consequence, robust results can only be obtained if the cumulative distribution functions cross once and only at the mean. Considering our framework, dominance relationships can be observed with any odd number of crossings as long as they happen outside the poverty domain. In the case of a single crossing, Corollary 2 states that the crossing value is not necessarily the average value of x but can be any other value that is consistent with admissible definitions of the maximum poverty domain.

Proposition 1 only provides a partial ordering for any pair of distributions defined on the domain Ω. In other words, the results with empirical implementations of the test are likely to be non-conclusive for a significant portion of the performed comparisons as it is possible to observe crossings of the cumulative distribution functions within the poverty domain. Hence it can be useful to add restrictions regarding the behaviour of poverty indices in terms of their sensitivity to progressive transfers, and then focus on members of the subclass Π_2.

While the dominance conditions for class Π_1 (Proposition 1) only require using a single function, namely the cumulative distribution function, the conditions for subclass Π_2 entail manipulating two different functions that accumulate gaps from the boundaries of the domain of x. Let $G(z) := \int_{\omega^-}^{z} F(x)\,dx = \int_{\omega^-}^{z}(z-x)\,dF(x)$ and $\overline{G}(z) := \int_{z}^{\omega^+}\overline{F}(x)\,dt = \int_{z}^{\omega^+}(x-z)\,d\overline{F}(x)$. The function $G(z)$ is known in the literature on poverty and wellbeing dominance as the absolute poverty gap index, and gives the mean value of the censored gaps $\max\{0, z-x\}$ observed in the population. The function $\overline{G}(z)$ does not average losses but excesses with respect to the value z, that is $\max\{0, x-z\}$. More precisely it is the product of the average excesses observed in the population with respect to threshold z times the part of that population whose level of x is larger than z. Then we show:

Proposition 2.

$$P_A\left(z^L, z^U\right) \leqslant P_B\left(z^L, z^U\right) \forall P \in \Pi^2\left(z^{L+}, z^{U-}\right) \tag{13}$$

$$\textit{iff} \qquad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{14}$$

$$\textit{and} \qquad \overline{G}^A(x) \leqslant \overline{G}^B(x) \quad \forall x \in \left[z^{U-}, \omega^+\right]. \tag{15}$$

Proof. See appendix A.2 ■

The first part of the conditions presented in Proposition 2 is identical to the one suggested in Atkinson (1987) and Foster and Shorrocks (1988): for each value of x below z^{L+}: the value of the absolute poverty gap index should never be larger for population A than for population B. The second part considers the cumulative "excesses" and states that for

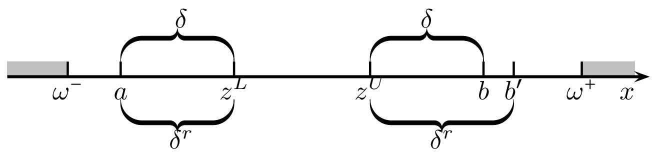


Figure 2: Comparability of the deprivations: absolute and relative gaps.

poverty not to be higher in population A, the value of the average excesses should be lower for population A than for population B for every value of x above the upper poverty line z^{U-}.

Finally, since we are dealing with sub-group additive poverty indices, we may distinguish two parts in the overall poverty level, that is the one corresponding to the presence of individuals within the bottom part of the poverty domain $[\omega^-, z^L]$ and the one corresponding to those people whose value of x is above the upper poverty line z^U. Overall poverty is consequently the sum of "loss" and "excess" poverty. Therefore we can focus on each group separately and then use only the corresponding condition in Propositions 1 and 2 to check whether a robust ordering can be obtained for the sole "loss" ("excess") poverty component when comparing two distributions. Using the example of distributions A and B in page 8, we can see that both populations show the same level of "loss" poverty but that "excess" poverty is robustly larger in population B.

3 The case of comparable deprivations

"Loss" and "excess" poverty may be due to different causes, and result in contrasted forms of wellbeing shortfalls. Yet we might feel sometimes that both types do not deserve the same attention when estimating overall poverty. However, no *a priori* ordering of the situation of a "loss" poor and an "excess" poor can be performed directly as both people exhibit different values for the attribute x. In order to enhance the comparability of the two poverty situations, it is thus useful to move from variable x to a common space. As in Fisher and Spencer (1992) and Lambert and Zoli (2012), it may be worth considering poverty indices defined with respect to distances (gaps) from the closest reference line for each individual, and then bring in additional assumptions regarding the relative size of poverty for individuals with different characteristics albeit showing the same gap.

3.1 Absolute gaps

Let $\delta \in \Re_+$ be defined as:

$$\delta := \begin{cases} z^L - x & \text{if } x \leqslant z^L, \\ 0 & \text{if } x \in]z^L, z^U[, \text{ and} \\ x - z^U & \text{if } x \geqslant z^U, \end{cases} \tag{16}$$

Figure 2 shows the situation of two individuals, one is a "loss" poor with $x = a$ and the other one is an "excess" poor with $x = b$. As the figure shows, both individuals exhibit the same absolute gap δ. That is why: $b = z^L + z^U - a$. However, if we assume that the situation of the "excess" poor cannot be regarded as severe as the situation of the "loss" poor, then we should obtain $\pi\left(a; z^L, z^U\right) \geqslant \pi\left(b; z^L, z^U\right)$. If this behaviour is deemed reasonable for every potential value of δ, that is, given $x \leqslant z^L$ for all $\{x, z^L + z^U - x\} \subset \Omega$, we can then consider the following subclass of poverty indices:

$$\tilde{\Pi}^1(z^{L+}, z^{U-}) := \left\{ P \;\middle|\; \begin{array}{l} P \in \Pi^1(z^{L+}, z^{U-}) \\ \left|\pi^{(1)}(x, z^L, z^U)\right| \geqslant \pi^{(1)}(z^L + z^U - x, z^L, z^U)\ \forall x \leqslant z^L \text{ s.t. } (z^L + z^U - x) \in \Omega \end{array} \right\}. \tag{17}$$

The first condition in (17) states that members from $\tilde{\Pi}^1(z^{L+}, z^{U-})$ comply with the properties of indices from $\Pi^1(z^{L+}, z^{U-})$. The second condition defines the specificity of these indices, stating that the marginal gain from improving the situation of an "excess" poor is never greater than the marginal gain for a "loss" poor with the same gap. It can easily be noted that, in conjunction with positing a zero poverty level at the deprivation lines, our additional assumption on the first-order derivatives of π is strictly equivalent to affirming that $\pi\left(x; z^L, z^U\right) \geqslant \pi\left(z^L + z^U - x; z^L, z^U\right)$. Members of $\tilde{\Pi}^1(z^{L+}, z^{U-})$ include, for instance, the indices $FGT^A_{\beta,\alpha_L,\alpha_U}(z^L, z^U)$ for which $\beta \in (0, 1)$ and $\alpha_L = \alpha_U$.

Considering different groups of poor people in a way that yields different individual poverty assessments for a given gap is not a new idea. Indeed, our framework is reminiscent of the literature on monetary poverty comparisons with differences in needs associated with particular attributes of individuals, e.g. their household sizes (Bourguignon, 1989, Atkinson, 1992, Jenkins and Lambert, 1993, Chambaz and Maurin, 1998, Duclos and Makdissi, 2005, Lambert and Zoli, 2012). These studies show that the ordering power of stochastic dominance procedures can be increased when simple assumptions are made about the difference between the individual poverty indices corresponding to two different groups. Here, we suggest that, in many cases, a similar assumption can be made regarding the situation of the "loss" and the "excess" poor.

It is worth stressing that, for a "loss" value a and an "excess" value b to be directly comparable, both should show the same distance δ from their respective poverty line. This point is important because stochastic dominance is often performed in order to check the robustness of poverty assessments to changes in poverty lines. However, when considering gap dominance relationships, each couple (z^L, z^U) defines all the pairwise comparable values a and b within the "loss" and "excess" poverty domains. For instance, increasing z^L by κ ($\kappa \in \Re_+$ with $\kappa < z^U - z^L$) while leaving z^U unchanged implies that the gap $\delta = x_2 - z^U$ does not make x_2 directly comparable with x_1 but with $x_1 + \kappa$. Consequently, results obtained when comparing distributions A and B with the vector of poverty lines (z^L, z^U) may not hold when using the vector $(z^L + \kappa, z^U)$ as the latter refers to different sets of pairwise comparable values of the wellbeing attribute.

On the other hand, if z^L is increased by a given quantity κ and z^U decreased by the same amount (with, of course, $2\kappa < z^U - z^L$), the value of the gap for a and b would raise by the same amount. Therefore the resulting gap $\delta - \kappa$ would still be associated with the same values of x, thereby leaving the correspondences between the "loss" poverty and "excess" poverty domains unchanged. With the assumption that a "loss" never yields less poverty

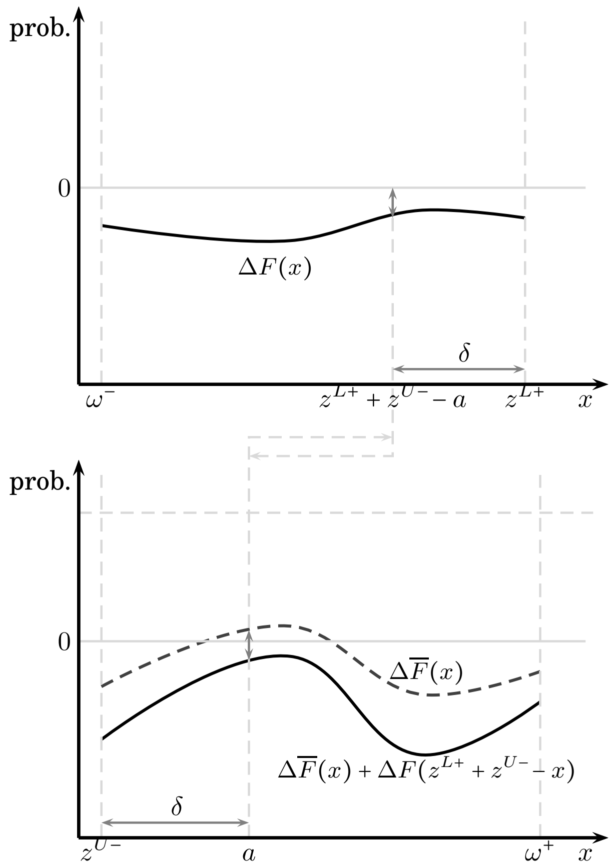


Figure 3: First order sequential gap dominance using Proposition 3.

than the corresponding "excess" given δ, one can consider the fulfillment of the following conditions in order to ensure ethically robust orderings for any members of the class of poverty indices $\tilde{\Pi}^1$:[7]

Proposition 3.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^1(z^{L+}, z^{U-}),\; z^{L+} - z^L = z^U - z^{U-} = \kappa \tag{18}$$

$$\textit{and} \quad \kappa \in \left[0, \min\{z^{L+} - \omega^-, \omega^+ - z^{U-}\}\right[\tag{19}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{20}$$

$$\textit{and} \quad \overline{F}^A(x) + F^A(z^{L+} + z^{U-} - x) \leqslant \overline{F}^B(x) + F^B(z^{L+} + z^{U-} - x) \quad \forall x \in [z^{U-}, \omega^+]. \tag{21}$$

Proof. See appendix B.1. ∎

Proposition 3 is a sequential dominance criterion in the spirit of those proposed in the aforementioned studies. First, condition (20) is the same as in Proposition 1 and states that the share of the population that experiences "loss" poverty, *i.e.* the neediest group, should be lower in population A than in B at each value of $x \leqslant z^{L+}$, for poverty to be lower

[7] A similar assumption is made in Lambert and Zoli (2012) for income poverty comparisons with group-specific poverty lines. As the authors consider gap-dominance relationships, they investigate the case of shifting all group-specific poverty lines up by the same amount.

in the former population. The second condition does not make any difference between "loss" and "excess" gaps since both are brought together for a comparison of the cdf of gaps for each possible value of δ within the poverty domain (expressed in terms of gaps). Figure 3 illustrates these conditions. An interesting feature of the subclass $\tilde{\Pi}^1(z^{L+}, z^{U-})$ is that a relatively worsening outlook regarding "excess" poverty can be compensated by relatively positive trends regarding the "loss" poor.

Let us illustrate that point with another example. Consider now distributions $A := (1,4,8,8,12)$ and $B := (1,2,7,7,11)$, still with $z^{L+} = 5$ and $z^{U-} = 10$. It can easily be seen that Proposition 1 does not hold since A exhibits less "loss" poverty than B but more "excess" poverty. However, if we suppose that a given gap δ yields more intense forms of poverty in the "loss" domain than in the "excess" domain, the two distributions can be ordered. Condition (20) is satisfied for each observed gap in the "loss" poverty domain. For the second condition, disregarding the nature of the gaps, we respectively obtain the following vectors of gaps $(0,0,1,2,4)$ and $(0,0,1,3,4)$ and it can then be seen that $\overline{F}^A(x) + F^A(5+10-x) = \overline{F}^B(x) + F^B(5+10-x)\ \forall x \in [10,12] \cup]13, \omega^+]$, but $\overline{F}^A(x) + F^A(5+10-x) < \overline{F}^B(x) + F^B(5+10-x)\ \forall x \in]12,13]$, so that condition (21) is also respected and we can conclude that A exhibits less poverty than B. It is also important to stress that the ordering is left intact if the lower and upper poverty lines are respectively decreased and raised by the same amount. For instance, if $z^L = z^{L+} - 1$ and $z^U = z^{U-} + 1$, we obtain the two vectors of gaps $(0,0,0,1,3)$ and $(0,0,0,2,3)$ and it can be seen that A still shows less poverty than distribution B whatever the precise functional form of P within $\tilde{\Pi}^1(z^{L+}, z^{U-})$.

It is worth noting that the sequential dominance conditions expressed in Proposition 3 differ from those proposed in the sequential dominance literature (a notable exception is Bourguignon, 1989) as the poverty domain for the neediest group is not necessarily larger than the one for the less needy group. Indeed, if $z^{L+} - \omega^- \leqslant \omega^+ - z^{U-}$, the size of the absolute gaps can be larger within the "excess" poverty domain than within the "loss" poverty domain, so that for values of $x \in]z^{L+} + z^{U-} - \omega^-, \omega^+]$ it is not possible for "loss" poverty situations to compensate for "excess" poverty situations in condition 21.

As with the class of poverty indices Π^1, we can also assume that indices from Π^2 are more averse to inequality between the poor at the bottom of the distribution than at its upper tail. We then consider the class $\tilde{\Pi}^2$ such that:

$$\tilde{\Pi}^2(z^{L+}, z^{U-}) := \left\{ P \,\middle|\, \begin{array}{l} P \in \tilde{\Pi}^1(z^{L+}, z^{U-}) \cap \Pi^2(z^{L+}, z^{U-}) \\ \pi^{(2)}(x, z^L, z^U) \geqslant \pi^{(2)}(z^L + z^U - x, z^L, z^U)\ \forall x \leqslant z^L \text{ s.t. } \{x, z^L + z^U - x\} \subset \Omega \end{array} \right\}. \tag{22}$$

The first condition in (22) states that members from $\tilde{\Pi}^2(z^{L+}, z^{U-})$ form a common subclass of both $\tilde{\Pi}^1(z^{L+}, z^{U-})$ and $\Pi^2(z^{L+}, z^{U-})$. The second line in (22) states that the marginal gains from improving the situation of a "loss" poor decrease more rapidly than for the "excess" poor. The corresponding dominance criterion for $\tilde{\Pi}^2(z^{L+}, z^{U-})$ is:

Proposition 4.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\ z^{L+} - z^L = z^U - z^{U-} = \kappa \tag{23}$$

$$\textit{and} \quad \kappa \in \left[0, \min\{z^{L+} - \omega^-, \omega^+ - z^{U-}\}\right[\tag{24}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{25}$$

$$\textit{and} \quad \overline{G}^A(x) + G^A(z^{L+} + z^{U-} - x) \leqslant \overline{G}^B(x) + G^B(z^{L+} + z^{U-} - x) \quad \forall x \in [z^{U-}, \omega^+]. \tag{26}$$

Proof. See appendix B.2. ■

While Propositions 3 and 4 allow for a large set of choices for the poverty lines (z^L, z^U), we may feel that the conditions linking z^L and z^U, given z^{L+} and z^{U-}, are too restrictive, since they do not make it possible to chose freely the vector of poverty lines within the set $[z^{L-}, z^{L+}] \times [z^{U-}, z^{U+}]$ of admissible pairs of poverty lines. If one desires to get such flexibility, it is then necessary to consider the following propositions:

Proposition 5.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^1(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{27}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{28}$$

$$\textit{and} \quad \overline{F}^A(x) + F^A(z^{L+} + z^{U-} - x) \leqslant \overline{F}^B(x) + F^B(z^{L+} + z^{U-} - x) \tag{29}$$

$$\forall x \in [z^{U-}, \omega^+],\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}].$$

Proposition 6.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{30}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{31}$$

$$\textit{and} \quad \overline{G}^A(x) + G^A(z^{L+} + z^{U-} - x) \leqslant \overline{G}^B(x) + G^B(z^{L+} + z^{U-} - x) \tag{32}$$

$$\forall x \in [z^{U-}, \omega^+],\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}].$$

Proof. See appendices B.1 and B.2. ■

While such conditions provide more robust conditions than those given by Propositions 3 and 4, it is easy to realize that they are computationally intensive. From a practical point of view, it is worth noting that, since Propositions 5 and 6 are generalizations of Propositions 3 and 4, respectively, the conditions in the former will never be met if those in the latter are not fulfilled. Hence checking first the easily implementable conditions (20) and (21), is advisable.

That said, conditions (29) and (32) can also be expressed in a different manner that renders their implementation more manageable, in the spirit of Bourguignon (1989). Let

$\varphi_1(x)$ be the maximum value of the difference $F^A(y) - F^B(y)$ for a given value of $x \in [z^{U-}, \omega^+]$ where y denotes the value of the wellbeing attribute that exhibits the same gap within the "loss" poverty domain as x does within the "excess" poverty domain, that is:

$$\varphi_1(x) = \max_{y \in \Lambda(x)} F^A(y) - F^B(y), \tag{33}$$

where $\Lambda(x) = \left[\max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+} - \max\{0, x - z^{U+}\}\right]$. In the same spirit, we define $\varphi_2(x)$ as:

$$\varphi_2(x) = \max_{y \in \Lambda(x)} \int_{\omega^-}^{y} F^A(t) - F^B(t)\, dt. \tag{34}$$

Propositions 5 and 6 can then be alternatively expressed as:

Proposition 7.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U)\ \forall P_\delta \in \tilde{\Pi}^1(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{35}$$

$$\textit{iff}\quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{36}$$

$$\textit{and}\quad \overline{F}^A(x) - \overline{F}^B(x) + \varphi_1(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{37}$$

Proposition 8.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U)\ \forall P_\delta \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{38}$$

$$\textit{iff}\quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{39}$$

$$\textit{and}\quad \overline{G}^A(x) - \overline{G}^B(x) + \varphi_2(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{40}$$

Proof. See appendix B.3. ■

Figure 4 illustrates Proposition 7. The upper part illustrates the first step of the procedure. The curve plots the difference $F^A(x) - F^B(x)$ over the maximum "loss" poverty domain. Condition (36) is fulfilled since the curve systematically returns negative values over the interval $[\omega^-, z^{L+}]$. Both the lower and upper panels are needed for the second step of the procedure. The dashed curve represents the difference $\overline{F}^A(x) - \overline{F}^B(x)$ over the maximum "excess" poverty domain. As condition (36) is respected, $\varphi_1(x)$ is non-positive and condition (37) will necessarily be satisfied when the dashed curve is below the horizontal line. So, condition (37) could possibly not be respected when the dashed curve is above the horizontal lines, that is for values of $x \in (u, v)$. Then for each value a within this interval, we first look at the corresponding interval $\Lambda(a)$ in the "loss" poverty domain and consider the values of $F^A(x) - F^B(x)$ for each value within $\Lambda(a)$. The largest values corresponds to $\varphi_1(a)$ and is added to $\overline{F}^A(x) - \overline{F}^B(x)$ in the lower panel. The continuous black curve in the lower part of Figure 4 thus plots $\overline{F}^A(x) - \overline{F}^B(x) + \varphi_1(x)$ for each value within the maximum "excess" poverty domain and it can be seen that condition (37) is fulfilled since the curve

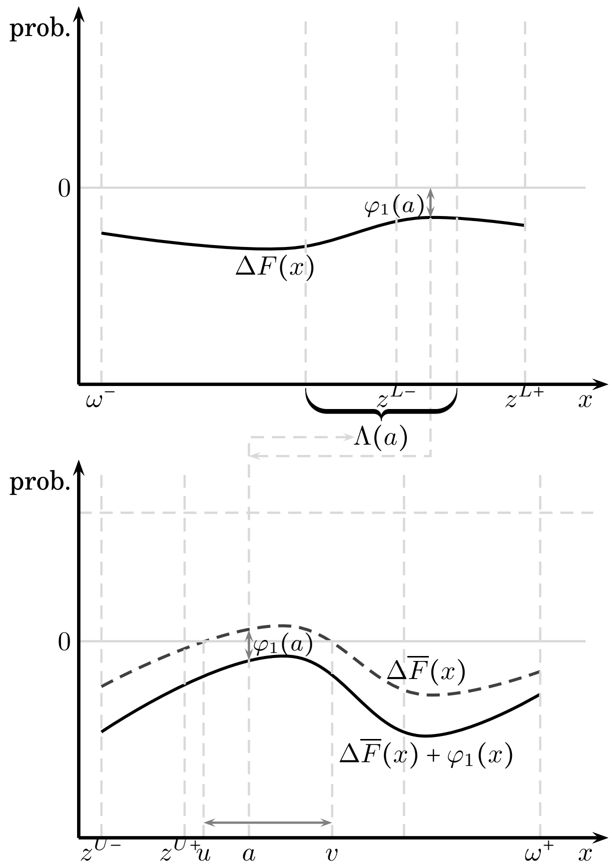


Figure 4: First order sequential gap dominance using Proposition 7.

is always below the zero horizontal line. Therefore we conclude that there is more poverty in distribution B than in distribution A, according to any members of $\tilde{\Pi}^1(z^{L+}, z^{U-})$.

We now illustrate the proposed algorithm with a simple example. Let $(\omega^-, z^{L-}, z^{L+}, z^{U-}, z^{U+}, \omega^+) = (0, 8, 10, 15, 20, 30)$, $A = (3, 9, 12, 12, 12, 12, 17, 18)$, and $B = (1, 1, 2, 8, 12, 12, 16, 24)$. We can observe that condition (11) is fulfilled $\forall x \in [0, 10]$, but (12) does not hold for $x \in]16, 17]$ so that Proposition 1 does not hold. Since condition (21) is met (Proposition 3 can thus be applied), it is worth considering condition (37). As $\overline{F}^A(x) - \overline{F}^B(x) > 0$ only for $x \in]16, 17]$ it is not necessary compute $\varphi_1(x)$ for values outside this interval. For values of x within $]16, 17]$ it can be checked that we have to look for the highest value of $F^A(x) - F^B(x)$ within $\bigcup_{x \in]16,17]} \Lambda(x) = \Lambda(17) = [5, 10[$. We then find $\left(\overline{F}^A(17) - \overline{F}^B(17)\right) + \varphi_1(17) = \frac{1}{8} - \frac{2}{8} < 0$. Condition (37) is thereby satisfied since $\Delta\overline{F}(x) + \varphi_1(x) \leqslant 0 \; \forall x \in]15, 30]$. Hence we can argue that poverty in population A is never above B according to any poverty index from $\tilde{\Pi}^1(z^{L+}, z^{U-})$ and pair of poverty lines within the subset $[8, 10] \times [15, 20]$.

Finally, note that the power of Propositions 7 and 8 depends heavily on the chosen values for the minimum and maximum poverty lines. In particular, as the probability of satisfying condition (37) depends on the width of $\Lambda(x)$, the ordering power of the two propositions should decrease as the ranges for z^L and z^U increase. For instance, in our last example, we observed $\Lambda(21) = [2, 9]$ for $z^L \in [8, 10]$ and $z^U \in [15, 20]$. With $z^L \in [9, 10]$ and $z^U \in [15, 17]$, $\Lambda(21)$ would have shrunk to $[3, 6]$, effectively decreasing the probability of obtaining $\overline{F}^A(21) - \overline{F}^B(21) + \varphi^1(21) > 0$.

3.2 Relative gaps

Up to now, we have considered social poverty indices whose individual indices are based on absolute deviations from the poverty lines. However, a usual practice is to quantify deprivations with relative gaps, e.g. as in the measures proposed in equations (6) and (7). That is, we can use δ^r such that:

$$\delta^r := \begin{cases} \frac{z^L - x}{z^L - \omega^-} & \text{if } x \leqslant z^L, \\ 0 & \text{if } x \in]z^L, z^U[, \text{ and} \\ \frac{x - z^U}{\omega^+ - z^U} & \text{if } x \geqslant z^U. \end{cases} \tag{41}$$

In principle, when $z^L - \omega^- \neq \omega^+ - z^U$, poverty assessments would not be affected by a change from absolute gaps to relative gaps. However, in other cases like the one in figure 2, such a change affects poverty orderings when additional assumptions are made regarding the relative contribution of "loss" and "excess" poverty to overall poverty. Using relative gaps δ^r, instead of absolute gaps δ, when performing the first-order and second-order dominance checks described in Proposition 1 and 2, does not change the results. Yet different results may ensue for Propositions 3 and 4 since relative gaps do not correspond to the same values of absolute gaps when $z^L - \omega^- \neq \omega^+ - z^U$. Moreover, dominance results with relative gaps are likely to be contingent upon the choices for the values of ω^- and/or ω^+.

If comparability of the two forms of poverty is based on relative gaps, then we must consider the following subclasses of poverty indices:

$$\tilde{\Pi}^1_r(z^{L+}, z^{U-}) := \left\{ P \;\middle|\; \begin{array}{l} P \in \Pi^1(z^{L+}, z^{U-}) \\ \left|\pi^{(1)}(x, z^L, z^U)\right| \geqslant \pi^{(1)}\left(z^U + \frac{z^L - x}{z^L - \omega^-}(\omega^+ - z^U), z^L, z^U\right) \;\; \forall x \leqslant z^L \end{array} \right\}. \tag{42}$$

$$\tilde{\Pi}^2_r(z^{L+}, z^{U-}) := \left\{ P \;\middle|\; \begin{array}{l} P \in \tilde{\Pi}^1(z^{L+}, z^{U-}) \cup \Pi^2(z^{L+}, z^{U-}) \\ \pi^{(2)}(x, z^L, z^U) \geqslant \pi^{(2)}\left(z^U + \frac{z^L - x}{z^L - \omega^-}(\omega^+ - z^U), z^L, z^U,\right) \;\; \forall x \leqslant z^L \end{array} \right\}. \tag{43}$$

The counterparts of Proposition 3 and 4 for relative gaps are then:

Proposition 9.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^1(z^{L+}, z^{U-}), \; \frac{z^{L+} - z^L}{z^{L+} - \omega^-} = \frac{z^U - z^{U-}}{\omega^+ - z^{U-}} = \kappa \in [0, 1[\tag{44}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{45}$$

$$\textit{and} \quad \overline{F}^A(x) + F^A\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right) \leqslant \overline{F}^B(x) + F^B\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right) \quad \forall x \in [z^{U-}, \omega^+]. \tag{46}$$

Proposition 10.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\ \frac{z^{L+} - z^L}{z^{L+} - \omega^-} = \frac{z^U - z^{U-}}{\omega^+ - z^{U-}} = \kappa \in [0, 1[\tag{47}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{48}$$

$$\textit{and} \quad \overline{G}^A(x) + G^A\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right) \leqslant \overline{G}^B(x) + G^B\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right)$$
$$\forall x \in [z^{U-}, \omega^+]. \tag{49}$$

Proof. See appendix B.1. ■

Finally, let $\varphi_k^r(x)$, $k = 1, 2$, be the counterpart of $\varphi^k(x)$ with relative gaps. In the case of $\varphi_1^r(x)$ we obtain:

$$\varphi_1^r(x) = \max_{y \in \Lambda^r(x)} F^A(y) - F^B(y), \tag{50}$$

where $\Lambda^r(x) = \left[z^{L-} + \frac{z^{U-} - x}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+} + \min\left\{0, \frac{z^{U+} - x}{\omega^+ - z^{U+}}(z^{L+} - \omega^-)\right\}\right]$. The analogues of Propositions 7 and 8 are:

Proposition 11.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U)\ \forall P \in \tilde{\Pi}_r^1(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{51}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{52}$$

$$\textit{and} \quad \overline{F}^A(x) - \overline{F}^B(x) + \varphi_1^r(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{53}$$

Proposition 12.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U)\ \forall P \in \tilde{\Pi}_r^2(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{54}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{55}$$

$$\textit{and} \quad \overline{G}^A(x) - \overline{G}^B(x) + \varphi_2^r(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{56}$$

4 Statistical inference

In empirical applications we estimate the following discrete counterpart of equation (4):

$$P(z^L, z^U) = \frac{1}{N} \sum_{n=1}^{N} \pi(x_n, z^L, z^U), \tag{57}$$

where N is the sample size and x_n is the value of x for individual n. Now, generally the functions π are likely to be different for “loss” and “excess” poverty, just as in the examples of (6) and (7). Hence we can write equation (57) as the sum of two distinct functions π, each multiplied by poverty identification functions:

$$
\begin{aligned}
P(z^L, z^U) &= \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n \leqslant z^L) + \pi(x_n, z^L, z^U)\mathbb{I}(x_n \geqslant z^U)] \\
&= \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n \leqslant z^L)] + \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n \geqslant z^U)], \qquad (58)
\end{aligned}
$$

where $\mathbb{I}(test)$ is an identification function returning 1 if $test$ is fulfilled and 0 otherwise. Now the standard error corresponding to expression (58) of P is going to depend on the the standard errors of the two averages on the right-hand side, i.e. $\hat{\sigma}_L$ and $\hat{\sigma}_U$, plus a negative covariance term. This covariance is negative because whenever $x_n \leqslant z^L$ then it is not the case that $x_n \geqslant z^U$, and viceversa. After some straightforward manipulations the variance of P is thus:

$$V(P) = \frac{\hat{\sigma}_L^2 + \hat{\sigma}_U^2 - 2P_L P_U}{N}, \qquad (59)$$

where:

$$P_L := \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n \leqslant z^L)], \qquad (60)$$

$$P_U := \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n \geqslant z^U)], \qquad (61)$$

$$\hat{\sigma}_L^2 := \frac{1}{N}\left(\sum_{n=1}^{N}\pi(x_n, z^L, z^U)^2\mathbb{I}(x_n \leqslant z^L)\right) - P_L^2, \qquad (62)$$

$$\hat{\sigma}_U^2 := \frac{1}{N}\left(\sum_{n=1}^{N}\pi(x_n, z^L, z^U)^2\mathbb{I}(x_n \geqslant z^U)\right) - P_U^2. \qquad (63)$$

The formulas can easily be adjusted to account for complex survey design (see for instance Deaton, 1997).

In order to test the stochastic dominance conditions derived above, we follow the testing procedures proposed in Kaur, Prakasa-Rao, and Singh (1994), Davidson and Duclos (2000) and Davidson and Duclos (2012) since they are based on rival hypotheses that make it possible to conclude in a statistically robust manner whether a distribution dominates another one for a given order of dominance. Basically, the test consists in a first step to oppose for each value of x within the poverty domain the following hypothesis:

$$\begin{cases} H_0 : \Delta S(x) = 0, \\ H_1 : \Delta S(x) < 0. \end{cases} \qquad (64)$$

where $\Delta S(x)$ is the considered criterion, for instance $\Delta S(x) = F^A(x) - F^B(x)$ in the case of Proposition 1. Non-dominance of distribution A over distribution B occurs when H_0 cannot be rejected. Since the functions used for the dominance criteria are basically linear combinations of averages, the hypotheses can be tested using a simple two-sample test. Since the test has to be performed over the whole poverty domain, it can be concluded that distribution A dominates distribution B in a statistically significant manner if H_0 is rejected for each value of x within the poverty domain at the chosen level of significance.

The test statistics for the whole procedure suggested by Kaur, Prakasa-Rao, and Singh (1994) is consequently:

$$t_{\max} = \max \left\{ \frac{\Delta \hat{S}(x)}{\sqrt{\hat{V}\big(S^A(x)\big) + \hat{V}\big(S^B(x)\big)}} \middle| x \in [\omega^-, z^{L+}] \cup [z^{U-}, \omega^+] \right\} \tag{65}$$

where $V\big(S^A(x)\big)$ is the variance of $S^A(x)$. Dominance is thus observed if $t_{\max}$ is less than the critical value of the standarized normal distribution corresponding to the chosen level of significance.

In spite of its appeal, the procedure is empirically not tractable unless distributions are censored at their tails as noted by Davidson and Duclos (2012). Indeed most observed distributions are likely to show $F(\omega^-) = 0$ or $\overline{F}(\omega^+) = 0$. Consequently it is highly probable to obtain $\Delta S(x) = 0$ so that estimating $t_{\max}$ systematically results in the non-rejection of H_0. As noted by Davidson and Duclos (2012), while censoring may *a priori* be at odd with the core axiomatic framework of poverty measurement, especially the strong versions of MON, there are valuable reasons for performing such censoring. From a practical point of view, censoring may be necessary as stochastic dominance procedures are highly sensitive to the presence of outliers: small measurement errors at the tails of the distribution may yield a non-dominance result though dominance should objectively be concluded. From an ethical point of view, it can be said that there are some thresholds at the two tails of Ω under and above which deprivation is total. For instance, consider two overweight persons with severe mobility impairment thereby exhibiting limited social interaction and high risk of premature death. If these two individuals are plainly identical except that the first one is 10kg lighter than the second one, hence resulting in a lower value of the BMI, we could reasonably argue that the BMI difference is not worth reflecting into even a marginal difference with respect to their individual poverty evaluation. Such individuals ought not to be dropped from the compared sample but to be treated as if they were exactly at the corresponding threshold of complete deprivation.

5 Empirical illustration: Health poverty in Bangladesh

5.1 Background, data and estimation details

Nutrition is a major concern in Bangladesh. Leaving aside the 1943 and 1974 famines that killed hundreds of thousands of people, a significant part of the Bangladeshi population suffers from inadequate dietary intakes. The country's share of under-five children with low weight-for-age is among the highest in the developing world (Stevens, Finucane, Paciorek, Flaxman, Whi 2012). For instance, nearly three out of five under-five children were underweight in 1990.[8] That figure fell to about 40% in 2009, a substantial but insufficient improvement toward the MDG objective of halving the share of underweight children between 1990 and 2015. Undernutrition is less significant for adults, but represents a challenge for policy-makers and multilateral development institutions (General Economics Division, 2012). Obesity is

[8] Source Unstats: http://mdgs.un.org/unsd/mdg/Data.aspx.

less documented than undernourishment, but Shafilque, Akhter, Stallkamp, de Pee, Panagides, and Bloem (2007) showed that Bangladesh experienced the same nutrition transition as the majority of developing countries; namely, the coexistence of both decreasing undernutrition and increasing obesity.

We compute poverty measures for BMI of Bangladeshi mothers between 16 and 49 years old, and for weight-for-age of Bangladeshi children (0 to 59 months old). The datasets are the Bangladesh Demographic and Health Surveys (DHS) for 1997, 2000, 2004 and 2007. The Bangladesh DHS have detailed health and anthropometric information for women in child-bearing age and their children, but not for men, hence our illustration focuses only on 15-49 years old mothers and under-five children. Table 1 shows the respective sample sizes for these two groups. The computations were performed using household weights and accounting for the clustered and stratified sampling design.[9]

Table 1: DHS sample sizes: Bangladeshi mothers and children, 1997-2007

Year	Mothers (16-49 years old)	Children (0-59 months old)
1997	5,914	5,600
2000	5,073	5,558
2004	5,114	7,055
2007	4,724	6,378

For mothers, our illfare evaluations are performed using the BMI with the standard underweight and overweight lines of 18.5 and 25 kilograms per square meter.[10] Values for ω^- and ω^+ are obtained from the minimum and maximum observed in the pooled samples; respectively: 12.06 and 57.94. For the children the z-scores of weight-for-age are computed using the WHO software.[11] The underweight and overweight lines are -2 and 2, corresponding to moderate underweight and moderate overweight. The weight-for-age values for ω^- and ω^+, respectively -6 and 5, are taken from the WHO, which regards them as biologically implausible.[12]

Finally we did not estimate other available anthropometric indicators for children due to conceptual problems. For instance, while a low height-for-age may reflect malnutrition,

[9] The 2007 survey does not have an explicit strata variable, but we generated it as the interaction between region and urban/rural area because that is how strata were defined in the previous surveys.

[10] As is well-known, the BMI is only a proxy of the health-status, whose use is justified on the strong correlation between high weight-for-height and obesity as measured by adiposity. However, occasionally, individuals may be found inside the poverty domain without showing any physiological, psychological, or social deprivations (on the limits of the BMI see, for instance, Burkhauser and Cawley, 2008). This is, for instance, the case of many high-level sportsmen whose lean body mass is generally high with respect to their height. World champions like the French judoka Teddy Riner or the All-Black Richie McCaw thus should be considered as moderately overweight considering the WHO classification (estimations based on weight and height figures reported on the English versions of Wikipedia) while not storing significant amounts of fat in their bodies. Of course, these are extreme cases that could easily be dropped from the health-poor population using Body Fat Percentage measures based on skinfold thickness. In our present illustration, we do not expect to find a significant number of muscular persons, like the aforementioned sportsmen, among our sample of Bangladeshi mothers and children. Therefore such a censoring procedure can be regarded as superfluous. The weight-for-age measure could be subjected to similar criticism, in theory. However we can reasonably assume that it is not relevant for under-five-year-old children.

[11] Available at: http://www.who.int/childgrowth/software/en/ (2011).

[12] See http://www.who.int/childgrowth/software/readme_stata.pdf.

a very high height-for-age does not reflect problems attributable to the family or economic environment. Rather it may reflect rare, if potentially detrimental, genetic endowments. Weight-for-height and BMI are not good indicators of health wellbeing among children because a badly malnourished child may be both too short and too thin for his/her age, thereby potentially attaining a deceitfully healthy value for indicators of weight by height.

5.2 Estimation results

5.2.1 Adult women

Table 2 shows the health-related illfare estimates for Bangldeshi women using BMI and members of the FGT family (equation 7). The top third of the table shows headcount indices treating both forms of illfare evenly, i.e. $FGT_{1,0,0}$. The results show a steady decrease in total illfare in Bangladesh between 1997 and 2007. Interestingly, this decrease has been led by a parallel decrease in “loss” illfare, i.e. related to malnutrition. By contrast, “excess” illfare has significantly increased during the same decade. However the overall result exhibits improvement since “loss” illfare in Bangladesh is a more prevalent problem. For instance, even in 2007, only 8.4% of women were overweight, whereas nearly 32% were undernourished. As Table 3 shows in its second-from-left column, the contribution of “loss” illfare to total illfare remains just below 80% in 2007, even though it has been constantly decreasing from 95% in 1997.

The middle third of table 2 shows the estimates for the poverty-gap indices, i.e. $FGT_{1,1,1}$. The results are qualitatively similar to those of the top third: a parallel decrease in the overall and “loss” poverty indices during the decade 1997-2007, negligibly offset by a mild increase in the “excess” poverty gap. Since the “excess” poverty-gap indices have low values to begin with, it is unsurprising that, as shown by the middle column of table 3, the contribution of “loss” poverty to overall poverty remains above 90% notwithstanding the declining trend. Combining these results with the previous ones, we can conclude that, while the overweight female population has increased in Bangladesh, it remains relatively close to the “excess” poverty line, on average.

Finally, the bottom third of table 2 shows the estimates for the square-poverty-gap indices, i.e. $FGT_{1,2,2}$. The trends of decline in overall poverty and “loss” poverty, this time measured by the square-poverty-gap indices, appear again. In tune with the apparent proximity of the “excess” poor to their respective poverty line, the proportion of “excess” poor is minimal. The contribution of “loss” poverty has also decreased but it remains high at 96%, according to table 3.

5.2.2 Children

Table 4 shows the poverty estimates for Bangldeshi children using weight-for-age and members of the FGT family (equation 7). As in table 2, the top third shows headcount indices, i.e. $FGT_{1,0,0}$. The results show a steady decrease in total illfare in Bangladesh between 1997 and 2007, which relents between 2000 and 2007. The decrease is led by a parallel decrease in “loss” illfare that is consistent with the results obtained by

Table 2: Nutrition-related illfaire (BMI): Bangladeshi women, 1997-2007.

Year	Total illfare	"Loss" illfare	"Excess" illfare
Headcount index			
1997	0.527	0.503	0.024
	[0.508, 0.546]	[0.484, 0.523]	[0.018, 0.029]
2000	0.477	0.432	0.045
	[0.460, 0.494]	[0.413, 0.452]	[0.037, 0.052]
2004	0.425	0.367	0.058
	[0.410, 0.439]	[0.351, 0.384]	[0.049, 0.67]
2007	0.402	0.319	0.084
	[0.385, 0.420]	[0.300, 0.337]	[0.072, 0.095]
Poverty gap index			
1997	0.123	0.120	0.003
	[0.116, 0.130]	[0.114, 0.127]	[0.002, 0.004]
2000	0.097	0.093	0.004
	[0.092, 0.102]	[0.088, 0.099]	[0.003, 0.005]
2004	0.080	0.076	0.005
	[0.076, 0.085]	[0.071, 0.080]	[0.004, 0.005]
2007	0.071	0.065	0.006
	[0.066, 0.076]	[0.060, 0.070]	[0.005, 0.007]
Squared poverty gap index			
1997	0.045	0.044	0.001
	[0.041, 0.048]	[0.040, 0.047]	[0.000, 0.002]
2000	0.031	0.030	0.001
	[0.029, 0.033]	[0.028, 0.032]	[0.000, 0.001]
2004	0.025	0.024	0.001
	[0.023, 0.027]	[0.022, 0.026]	[0.001, 0.001]
2007	0.022	0.021	0.001
	[0.019, 0.024]	[0.018, 0.023]	[0.001, 0.001]

Note: 95% confidence interval in parentheses.

Stevens, Finucane, Paciorek, Flaxman, White, Donner, and Ezzati (2012). By contrast, "excess" illfare has first decreased (between 1997 and 2000) and then increased (between 2000 and 2007) during the same decade. The overall result exhibits improvement since "loss" illfare in Bangladesh is a more prevalent problem also among children. Indeed, table 5 shows that undernourishment explains at least 99% of the overall headcount index. Thereupon the low values for "excess" poverty using $FGT_{1,1,1}$ and $FGT_{1,2,2}$ are unsurprising (see bottom two-thirds of rightmost in table 4 and respective contributions in table 5).

Both $FGT_{1,1,1}$ and $FGT_{1,2,2}$ have decreased for "loss" poverty among children (bottom two-thirds of middle column in table 4). Hence, given the small contributions for "excess" poverty, the decade 1997–2007 has witnessed improvement in the intensity of health-related poverty among children in Bangladesh.

Table 3: Contributions of "loss" illfare to total BMI illfare: Bangladeshi women, 1997-2007.

Year	$FGT_{1,0,0}$	$FGT_{1,1,1}$	$FGT_{1,2,2}$
1997	95%	98%	98%
2000	90%	96%	97%
2004	86%	94%	97%
2007	79%	91%	96%

5.3 Ethical robustness tests

5.3.1 Adult women

Figure 5 shows the point estimates of the criteria of Proposition 1 with respective confidence intervals, using $z^{L+} = 18.5$ and $z^{U-} = 25$. Criterion (11) is represented by the solid lines to the left of z^{L+}, whereas the alternative version of criterion (12) in terms of survival functions is represented by the solid lines to the right of z^{U-}. In every comparison, the most recent year takes the place of A (in the above propositions) and the other year plays the role of B (so the criteria are in terms of $A - B$). Consequently, under the hypothesis that the most recent distribution dominates the oldest one, *i.e.* nutrition-related illfare has declined during the period, we should observe the curves to be below zero for each value of the "loss" and "excess" poverty domains. With these details in mind, it is clear from the four panels of figure 5 that no robust conclusions can be made regarding overall changes in nutrition-related illfare considering all poverty indices from the class Π^1 and a wide array of poverty lines in the domain. Indeed while significant improvement are noticeable during the period 1997–2007 for underweight values of the BMI, hence showing a robust alleviation of the "loss" part of nutrition-related illfare, no conclusive result is obtained for the "excess" part. Although small, but statistically insignificant, progress is patent for large values of the BMI, the share of moderately overweight Bangladeshi mothers increased during the whole period. The same pattern appears for the subperiod 1997–2000, but not for the subperiods 2000–2004 and 2004–2007 where no robust conclusion can be made regarding undernutrition.

Resorting to the second-order dominance conditions of Proposition 2 does not yet yield robust orderings as the curves for condition (15) cross the zero horizontal line (cf figure 6). Criterion (14) is represented by the solid lines to the left of z^{L+}, whereas criterion (15) is represented by the solid lines to the right of z^{U-}. The other comparison settings are the same as in figure 5. Non-conclusive results within the "excess" domain are due to the limited improvements in obesity prevalence which cannot compensate for the relatively large increase in moderate overweight situations (BMI within the range $[25, 30]$). It is worth noting that had "excess" illfare been defined using $z^{U-} = 30$ as minimum threshold , i.e. considering only obesity, we would then have been able to observe robust orderings for the whole period and the first and second subperiods, albeit without statistical significance. Finally, we conclude that the "loss" component of nutrition-related illfare has declined during the subperiod 2004–2007 using distribution-sensitive poverty indices as the curve

Table 4: Nutrition-related illfare (weight-for-age): Bangladeshi children, 1997-2007.

Year	Total illfare	"Loss" illfare	"Excess" illfare
Headcount index			
1997	0.534	0.530	0.004
	[0.515, 0.553]	[0.511, 0.549]	[0.002, 0.006]
2000	0.419	0.417	0.002
	[0.401, 0.436]	[0.399, 0.434]	[0.001, 0.003]
2004	0.426	0.423	0.003
	[0.406, 0.446]	[0.403, 0.443]	[0.001, 0.004]
2007	0.425	0.420	0.004
	[0.407, 0.442]	[0.402, 0.438]	[0.002, 0.006]
Poverty gap index			
1997	0.140	0.138	0.001
	[0.132, 0.147]	[0.130, 0.146]	[0.001, 0.002]
2000	0.090	0.089	0.001
	[0.084, 0.096]	[0.084, 0.095]	[0.000, 0.001]
2004	0.088	0.087	0.001
	[0.082, 0.094]	[0.082, 0.093]	[0.000, 0.001]
2007	0.083	0.081	0.001
	[0.077, 0.088]	[0.076, 0.087]	[0.001, 0.002]
Squared poverty gap index			
1997	0.059	0.058	0.001
	[0.054, 0.064]	[0.053, 0.063]	[0.000, 0.001]
2000	0.032	0.032	0.000
	[0.029, 0.035]	[0.029, 0.035]	[0.000, 0.001]
2004	0.030	0.029	0.000
	[0.027, 0.032]	[0.027, 0.032]	[0.000, 0.001]
2007	0.027	0.027	0.001
	[0.025, 0.030]	[0.024, 0.029]	[0.000, 0.001]

Note: 95% confidence interval in parentheses.

on panel 6a is nowhere above the zero horizontal line for values of the BMI referring to undernourishment.

Results of these dominance checks show that no robust ordering regarding changes in overall nutrition-related illfare for Bangladeshi mothers can be obtained without imposing additional assumptions on the relative social costs of "loss" and "excess" illfare. Here, we assume that overweight is socially preferable to underweight, notably because, in the Bangladeshi context, underweight is generally associated with monetary poverty and consequently to lower access to good health services. This leaves unanswered the question of how we can make underweight and overweight comparable. In the present paper, we investigate two different comparability assumptions based on absolute and relative gaps with respect to deprivation thresholds.

Figure 7 present graphical implementations of the sequential dominance procedures described in Propositions 7 and 8 with $[z^{L-}, z^{L+}] = [18, 18.5]$ and $[z^{U-}, z^{U+}] = [25, 30]$.[13]

[13]Results for Propositions 3 and 4 are reported in appendix C.

Table 5: Contributions of "loss" illfare to total weight-for-age illfare: Bangladeshi children, 1997-2007.

Year	$FGT_{1,0,0}$	$FGT_{1,1,1}$	$FGT_{1,2,2}$
1997	99.3%	99.0%	98.6%
2000	99.6%	99.4%	99.1%
2004	99.4%	99.3%	99.2%
2007	99.0%	98.2%	97.1%

With the chosen intervals, it is worth stressing that compensation between the prevalence of overweight and underweight is limited to the range $[25, 36.44]$ considering conditions (37) and (40) as we assumed $\omega^- = 12.06$. Since the first conditions in our sequential dominance procedures are strictly the same as the one used for first and second-order dominance procedures, it is not necessary to look at the results for the subperiod 2000–2004, both at the first and second orders, and for the subperiod 2004-2007 at the first order, since we already know that no robust ordering can be obtained considering Propositions 3 to 8. For the 1997–2000 subperiod, the second-order sequential dominance procedure indicates that nutrition-related illfare has declined in a robust manner, but the 95% confidence interval shows that this result is not statistically significant. For each one of the remaining comparisons, our assumptions do not render robust orderings considering any deprivation thresholds within the aforementioned intervals and any illfare index within the classes $\tilde{\Pi}^1$ or $\tilde{\Pi}^2$.

Considering that a larger share of individuals exhibiting overweight can be compensated by a lower share of underweighed individuals showing the same relative gap (Propositions 11 and 12), we can conclude that illfare has robustly declined considering the population of Bangladeshi mothers during the period 1997–2007 for all poverty indices that are members of $\tilde{\Pi}^1_r$ and all sets of deprivation lines within the set $[18, 18.5] \times [25, 30]$ (Figure 8). Taking sampling variability into account does not alter this result. The same pattern of well-being improvement can be stressed for the subperiod 1997–2000. Finally, a robust comparison can be performed considering the subset of distribution-sensitive poverty indices for the subperiod 2004–2007, though this result is not statistically significant.

5.3.2 Children

Here we check whether the progress observed in table 4 considering nutrition-related illfare of under-five Bangladeshi children is robust to different deprivation lines and poverty indices within Π_1 and its subsets. Figure 9 shows that limited assumptions have to be made in order to conclude a robust decline of nutrition-related illfare during the whole period as both conditions in Proposition 1 are satisfied. The same conclusion holds for the subperiod 1997–2000 and both dominance results are statistically significant.

Considering the subperiod 2000–2004, the first-order dominance check does not warrant the conclusion that the fall in nutrition-related illfare observed in table 4 is robust to changes in the poverty indices within Π_1 unless the maximum value of the "loss" deprivation line is set a value of -3 corresponding to severe forms of underweight. With a value

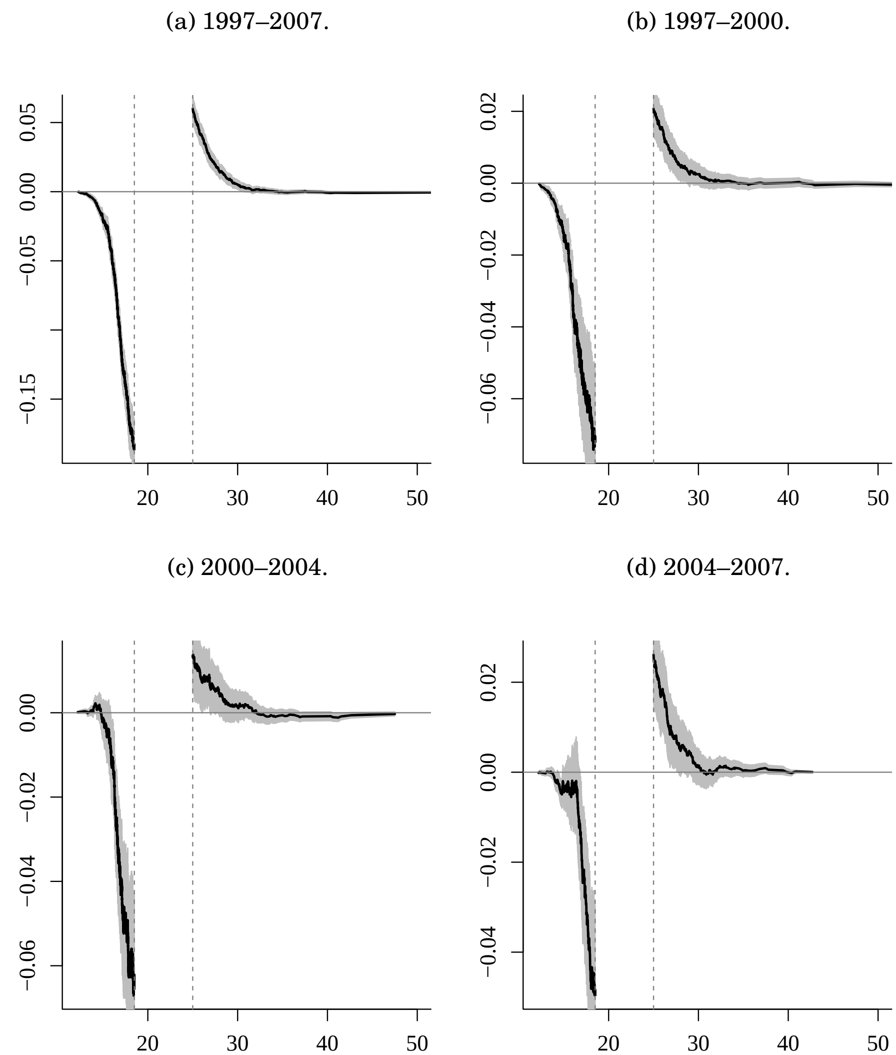


Figure 5: First-order dominance: BMI for 15-59 Bangladeshi mothers.

for z^{L+} set at -2 for our weight-for-age indicator, second-order dominance checks indicates that the decline is robust to a wide array of poverty lines in the domain, considering all poverty indices in family Π^2. Focusing on the "excess" domain, the first-order dominance procedure outlines a robust decline of "excess" illfare between 2000 and 2004. However, confidence intervals shows that these results are not statistically significant. The decline in the "excess" part of nutrition-related illfare is robust and statistically significant considering distribution-sensitive poverty indices.

Finally, no conclusive result can be obtained concerning the last subperiod as the dominance curves cross the zero horizontal line for z-score values within the "loss" domain at both the first and second order. Nevertheless, we can see that "excess" illfare has worsened during this period using members from Π^1, though the result is not statistically significant at the chosen confidence level.

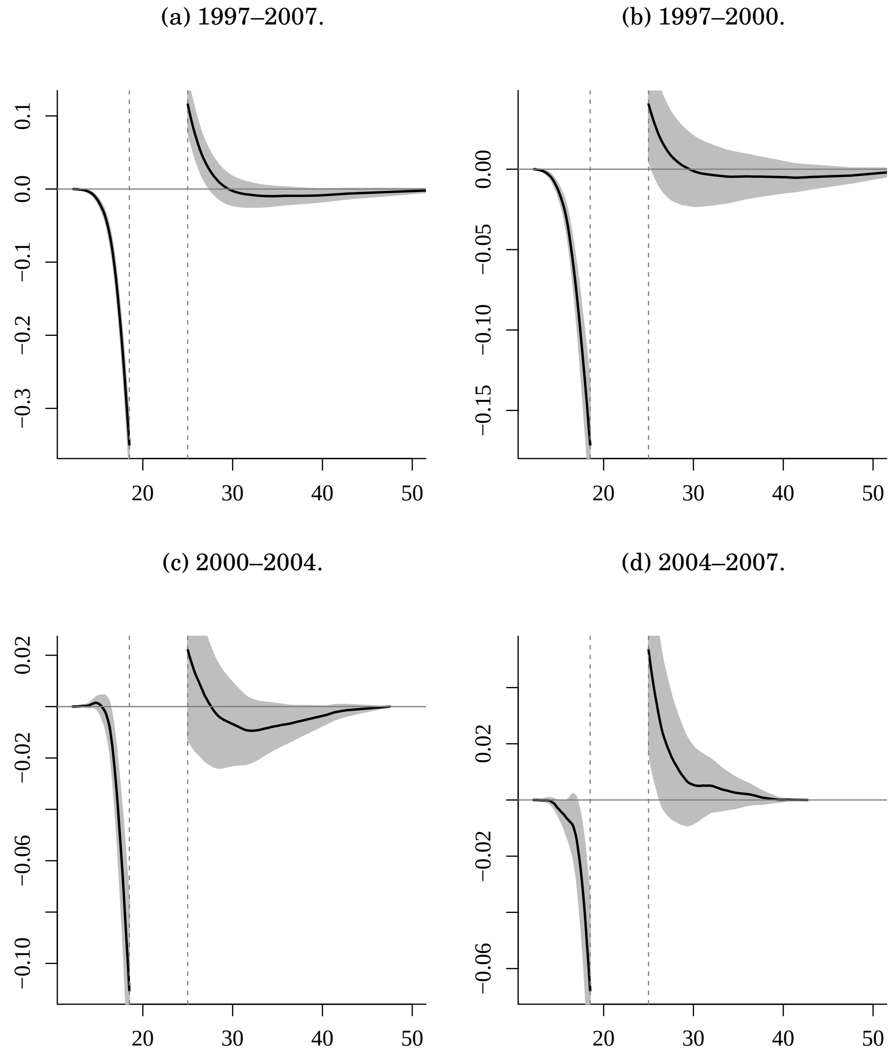


Figure 6: Second-order dominance: BMI for 15-59 Bangladeshi mothers.

6 Conclusion

Assessing human progress in health outcomes has a long history. The recent consensual recognition of poverty as a multidimensional phenomenon has prompted the use of poverty measurement tools to assess the extent of deprivation within the health dimension of well-being. However, contrary to traditional applications in monetary poverty, health indicators are likely to be related to wellbeing in a non-monotonic manner, so that individuals may suffer from either too low or too high levels of such variables. Providing a synthetic index for health-related illfare that can fully take into account the dual burden of, say, undernutrition and obesity, is thus a challenge that deserves consideration.

In the present paper, we proposed some alterations of traditional poverty measurement axioms in order to propose health-related illfare indices that are consistent with non-monotonic wellbeing relationships. Moreover, we provide dominance criteria to assess the ethical robustness of health-related illfare orderings, considering broad classes of illfare indices based on some reasonable assumptions and admissible ranges for the deprivation lines. Further developments should include the development of dominance technique when such non-monotonic relationships occur in a multidimensional framework, for instance when information on income, education or access to basic services are added to health variables in order to get a more comprehensive picture of illfare.

Finally, the usefulness of our indices and stochastic dominance tests is illustrated using the Bangladesh DHS datasets for the period 1997–2007. More specifically, nutrition-related illfare is assessed using the BMI for 16 to 49 year-old mothers, and z-scores of weight-for-age for under-five children. We show *inter alia* that the decline in nutrition-related illfare for young children during the period of interest is robust to a wide array of poverty lines in the domain and all poverty indices that comply with appropriate versions of the focus and monotonicity axioms. However recent trends show a robust increase in the social burden of child overweight. Regarding Bangladeshi mothers, we simultaneously observe an increase in overweight illfare and a decline in underweight illfare between 1997 and 2007. Concluding in a robust manner that overall nutrition-related illfare has fallen during the period comes at the cost of assuming that for a given relative gap with respect to the appropriate deprivation line, "loss" illfare should be regarded as more severe than the "excess" counterpart.

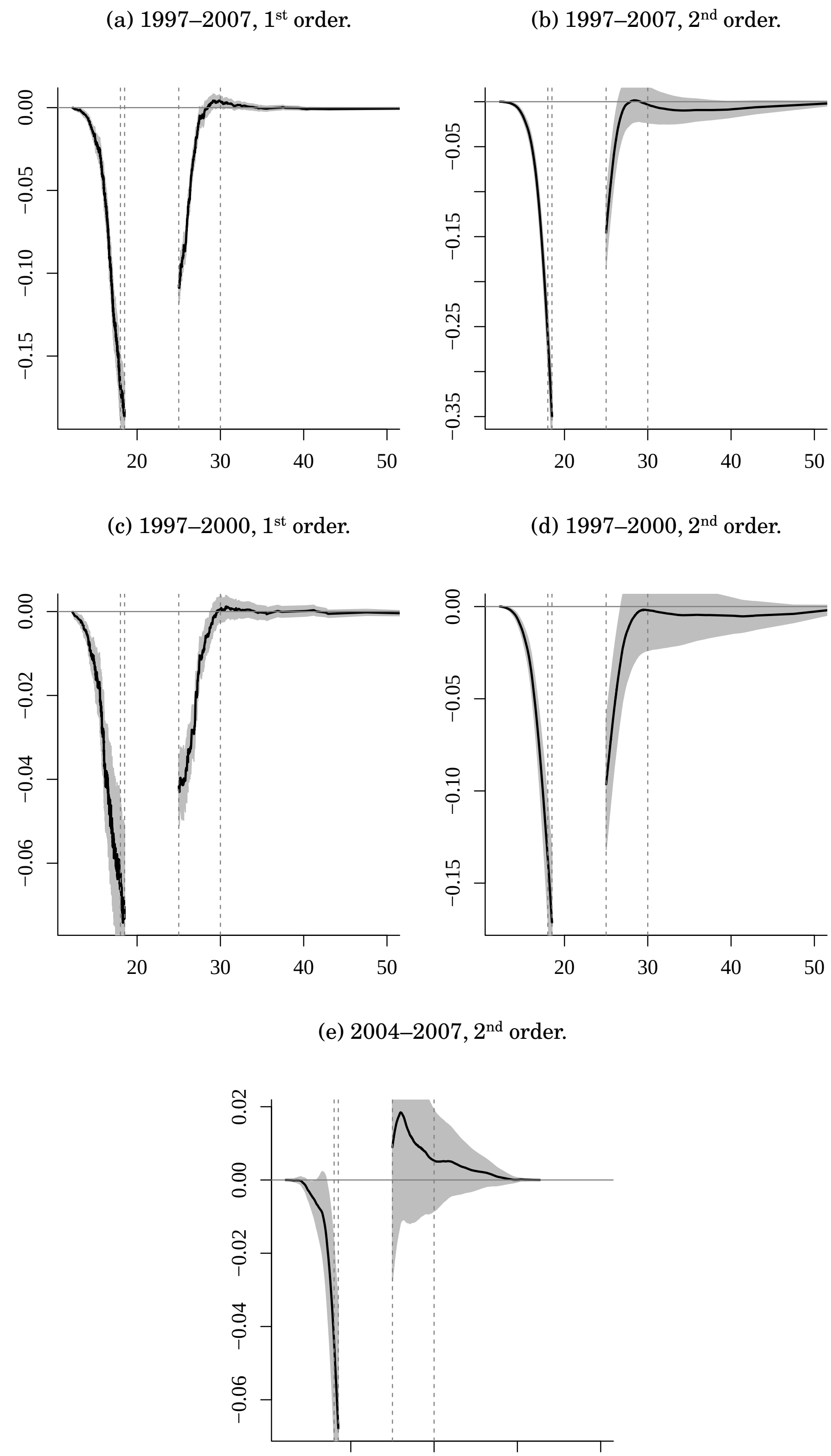
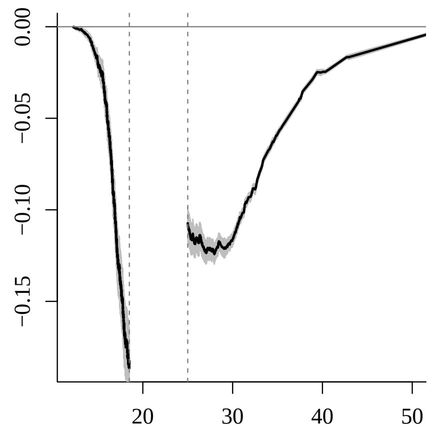
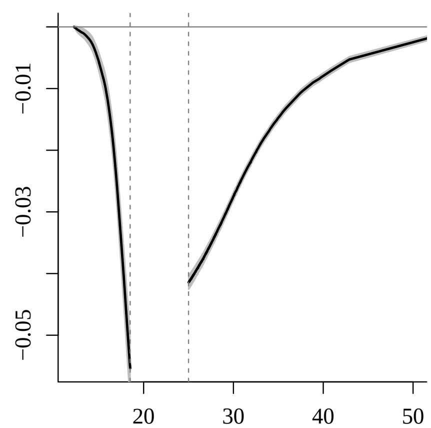


Figure 7: Sequential dominance for absolute gaps with independent deprivation lines: BMI for 15-59 Bangladeshi mothers.

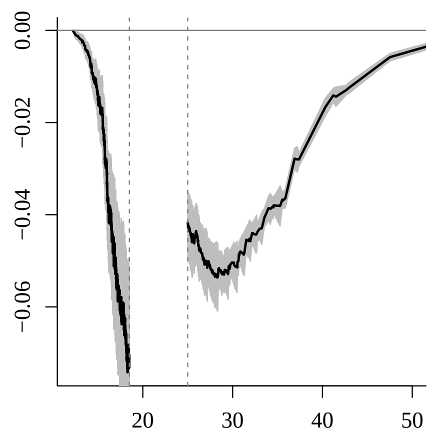
(a) 1997–2007, 1st order.



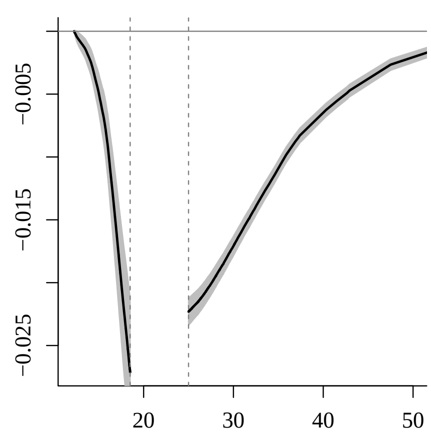
(b) 1997–2007, 2nd order.



(c) 1997–2000, 1st order.



(d) 1997–2000, 2nd order.



(e) 2004–2007, 2nd order.

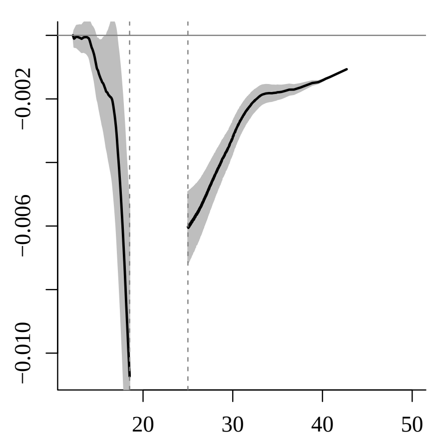


Figure 8: Sequential dominance for relative gaps with independent deprivation lines: BMI for 15-59 Bangladeshi mothers.

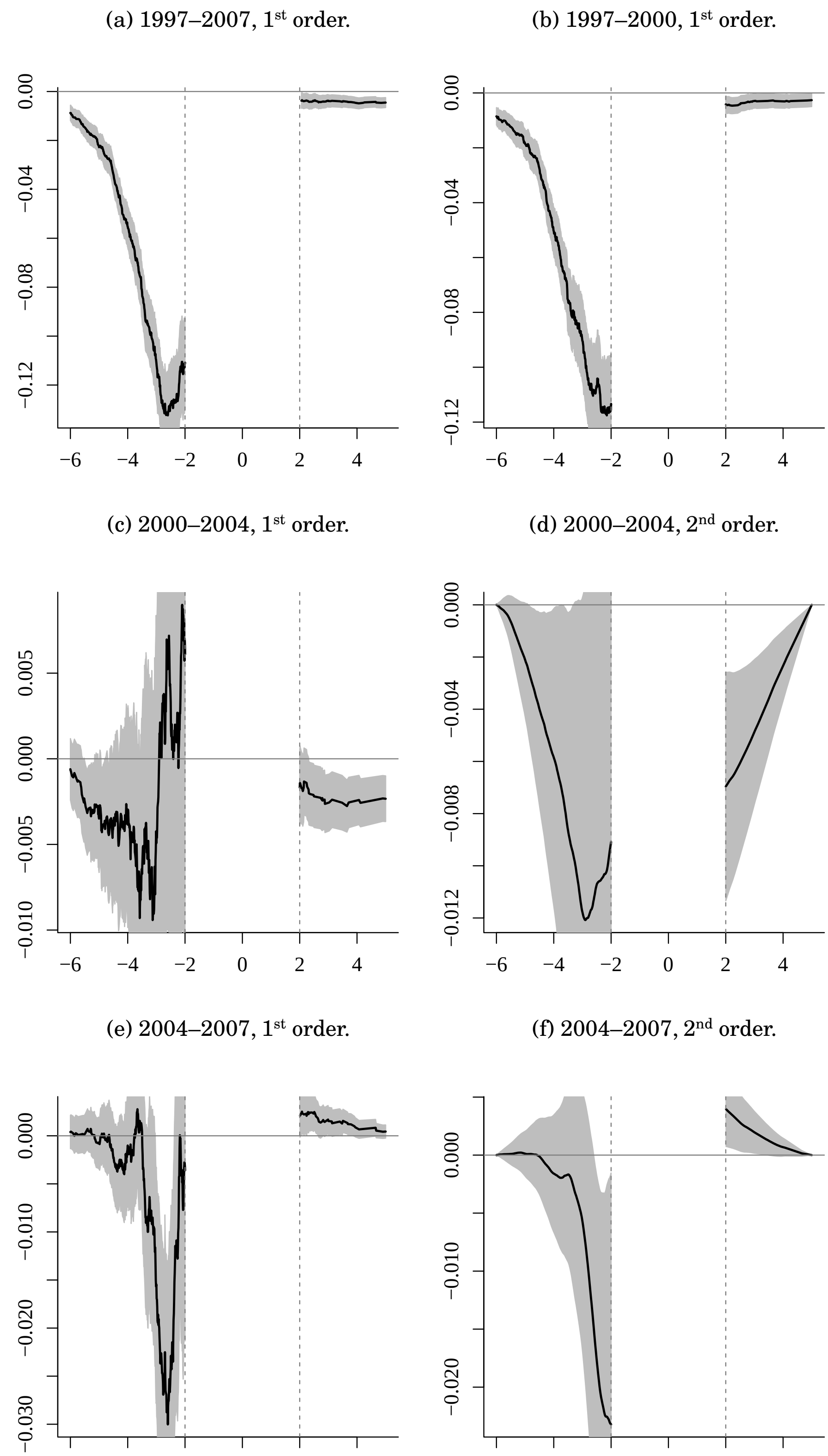


Figure 9: First and second-order dominance: Weight for age for under 5 Bangladeshi children.

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Appendices

A Proof of dominance conditions

A.1 Proposition 1

Let $\Delta P \coloneqq P_A - P_B$ be the difference between the statistics (e.g. P, or F) of populations A and B. Then note that equation (4) for the difference ΔP can be expressed as:

$$\Delta P(z^L, z^U) = \int_{\omega^-}^{z^L} \pi(x; z^L, z^U)\Delta f(x)\, dx + \int_{z^U}^{\omega^+} \pi(x; z^L, z^U)\Delta f(x)\, dx. \tag{66}$$

where $f : \Omega \to [0,1]$ is the density function. Integrating by parts each term in equation (66), we obtain:

$$\begin{aligned}\Delta P(z^L, z^U) = &\left[\pi(x; z^L, z^U)\Delta F(x)\right]_{\omega^-}^{z^L} - \int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx \\ &+ \left[\pi(x; z^L, z^U)\Delta F(x)\right]_{z^U}^{\omega^+} - \int_{z^U}^{\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx.\end{aligned} \tag{67}$$

Since $\pi(z^L; z^L, z^U) = \pi(z^U; z^L, z^U) = 0$ and $\Delta F(\omega^-) = \Delta F(\omega^+) = 0$, we obtain:

$$\Delta P(z^L, z^U) = -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx - \int_{z^U}^{\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx. \tag{68}$$

The rest of the proof follows by inspection.

A.2 Proposition 2

Noting that in univariate settings: $F(x) = 1 - \overline{F}(x)$ and therefore: $\Delta F(x) = -\Delta\overline{F}(x)$; we integrate equation (68) by parts, expressing the first right-hand side element in terms of F and the second right-hand side element in terms of $\overline{F}$. Keeping in mind that $\frac{\partial \overline{G}}{\partial x} = -\overline{F}(x)$, this yields:

$$\Delta P(z^L, z^U) = -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx - \int_{z^U}^{\omega^+} \pi^{(1)}(x; z^L, z^U)\left(-\Delta\overline{F}(x)\right)\, dx, \tag{69}$$

$$\begin{aligned}= &-\left[\pi^{(1)}(x; z^L, z^U)\Delta G(x)\right]_{\omega^-}^{z^L} + \int_{\omega^-}^{z^L} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\, dx \\ &-\left[\pi^{(1)}(x; z^L, z^U)\Delta\overline{G}(x)\right]_{z^U}^{\omega^+} + \int_{z^U}^{\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta\overline{G}(x)\, dx,\end{aligned} \tag{70}$$

$$= \int_{\omega^-}^{z^L} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\, dx + \int_{z^U}^{\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta\overline{G}(x)\, dx. \tag{71}$$

since $\pi^{(1)}(z^L; z^L, z^U) = \pi^{(1)}(z^U; z^L, z^U) = 0$ and $\Delta G(\omega^-) = \Delta\overline{G}(\omega^+) = 0$. The rest of the proof follows by inspection.

B Proof of sequential dominance conditions

B.1 Proof of Propositions 3 and 5

Considering $\pi^{(x)}(z^L + z^U - x; z^L, z^U) = -\pi^{(1)}(z^L + z^U - x; z^L, z^U)$ it can first be seen that (68) can be rewritten in the following way:

$$\begin{aligned}\Delta P(z^L, z^U) &= -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(x)}(z^L + z^U - x; z^L, z^U)\Delta \overline{F}(z^L + z^U - x)\,dx, \qquad (72)\\ &= -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\ &- \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta \overline{F}(z^L + z^U - x)\,dx. \qquad (73)\end{aligned}$$

In the case $z^{L+} - \omega^- \geqslant \omega^+ - z^{U-}$, equation (73) can be expressed as:

$$\begin{aligned}\Delta P(z^L, z^U) &= -\int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\ &- \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + (1-1)\pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta \overline{F}(z^L + z^U - x)\,dx, \qquad (74)\\ &= -\int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\ &- \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\left(\Delta \overline{F}(z^L + z^U - x) + \Delta F(x)\right)\,dx. \qquad (75)\end{aligned}$$

By assumption, $\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U) \leqslant 0\ \forall x \in [z^L + z^U - \omega^+, z^L]$. The rest of the proof follows by inspection.

In the case $z^{L+} - \omega^- \leqslant \omega^+ - z^{U-}$, equation (73) can be expressed as:

$$\begin{aligned}\Delta P(z^L, z^U) &= -\int_{\omega^-}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + (1-1)\pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta \overline{F}(z^L + z^U - x)\,dx \\ &+ \int_{\omega^-}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta \overline{F}(z^L + z^U - x)\,dx, \qquad (76)\\ &= -\int_{\omega^-}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta \overline{F}(z^L + z^U - x)\,dx \\ &+ \int_{\omega^-}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\left(\Delta \overline{F}(z^L + z^U - x) + \Delta F(x)\right)\,dx. \qquad (77)\end{aligned}$$

By assumption, $\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U) \leqslant 0\ \forall x \in [\omega^-, z^L]$. The rest of the proof follows by inspection.

B.2 Proof of Propositions 4 and 6

Considering members from $\tilde{\Pi}^2(z^{L+}, z^{U-})$, we first can rewrite equation (71) as:

$$\Delta P(z^L, z^U) = \int_{\omega^-}^{z^L} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\,dx + \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(2)}(z^L + z^U - x; z^L, z^U)\Delta\overline{G}(z^L + z^U - x)\,dx. \tag{78}$$

In the case $z^{L+} - \omega^- \geqslant \omega^+ - z^{U-}$, equation (78) can be expressed as:

$$\begin{aligned}\Delta P(z^L, z^U) &= \int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) + (1-1)\pi^{(2)}(z^L + z^U - x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(2)}(z^L + z^U - x; z^L, z^U)\Delta\overline{G}(z^L + z^U - x)\,dx, \end{aligned} \tag{79}$$

$$\begin{aligned} &= \int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) - \pi^{(2)}(z^L + z^U - x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(2)}(z^L + z^U - x; z^L, z^U)\left(\Delta\overline{G}(z^L + z^U - x) + \Delta G(x)\right)\,dx. \end{aligned} \tag{80}$$

By assumption, $\pi^{(2)}(x; z^L, z^U) + \pi^{(2)}(z^L + z^U - x; z^L, z^U) \geqslant 0\ \forall x \in [z^L + z^U - \omega^+, z^L]$. The rest of the proof follows by inspection.

In the case $z^{L+} - \omega^- \leqslant \omega^+ - z^{U-}$, equation (78) can be expressed as:

$$\begin{aligned}\Delta P(z^L, z^U) &= \int_{\omega^-}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) + (1-1)\pi^{(2)}(z^L + z^U - x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(2)}(z^L + z^U - x; z^L, z^U)\Delta\overline{G}(z^L + z^U - x)\,dx \\ &+ \int_{\omega^-}^{z^L} \pi^{(2)}(z^L + z^U - x; z^L, z^U)\Delta\overline{G}(z^L + z^U - x)\,dx, \end{aligned} \tag{81}$$

$$\begin{aligned} &= \int_{\omega^-}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) - \pi^{(2)}(z^L + z^U - x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(2)}(z^L + z^U - x; z^L, z^U)\Delta\overline{G}(z^L + z^U - x)\,dx \\ &+ \int_{\omega^-}^{z^L} \pi^{(2)}(z^L + z^U - x; z^L, z^U)\left(\Delta\overline{G}(z^L + z^U - x) + \Delta G(x)\right)\,dx. \end{aligned} \tag{82}$$

By assumption, $\pi^{(2)}(x; z^L, z^U) - \pi^{(2)}(z^L + z^U - x; z^L, z^U) \geqslant 0\ \forall x \in [\omega^-, z^L]$. The rest of the proof follows by inspection.

B.3 Proof of Propositions 7 and 8

The proof is inspired from Lambert and Zoli (2005).

Let first consider the case $x \in [z^{U-}, z^{U+}]$. It can then easily be seen that the largest potential value of delta is $\delta^+ = x - z^{U-}$ while the lowest is $\delta^- = 0$. For a given set of poverty lines z^L, z^U, the value y within the "loss" poverty domain that yields the same gap as x is $z^L - \delta$. Since $z^L \in [z^{L-}, z^{L+}]$, we then have $y \in [z^{L-} - \delta, z^{L+} - \delta]$ for a given value of δ. Taking into account that $\delta \in [0, x - z^{U-}]$, we then have $y \in [z^{L-} + z^{U-} - x, z^{L+}]$. Finally, as we should have $y \geqslant \omega^-$ but may observe $z^{L-} + z^{U-} - x < \omega^-$, we find $y \in \big[\max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+}\big]$.

Now, let have a look at the case $x \in [z^{U+}, \omega^+]$. Potential values of δ are then $\delta^+ = x - z^{U-}$ and $\delta^- = x - z^{U+}$. For a given value of x, taking the variability of z^L and z^U into account, we obtain $y \in [z^{L-} - \delta^+, z^{L+} - \delta^-] = [z^{L-} + z^{U-} - x, z^{L+} + z^{U+} - x]$. Once again, we have to observe $y \geqslant \omega^-$ but it is possible to have $z^{L-} + z^{U-} - x < \omega^-$, so the right interval for y is $\big[\max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+} + z^{U+} - x\big]$.

Bringing together the two cases, we get the general expression for the appropriate interval for y, that is $\Lambda(x) = \big[\max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+} - \max\{0, x - z^{U+}\}\big]$.

The rest of the proof is straightforward. Since by definition $\varphi^1(x)$ is the largest value of $F^A(t) - F^B(t)$ for $t \in \Lambda(x)$, we necessarily have $\overline{F}^A(x) - \overline{F}^B(x) + F^A(y) - F^B(y) \leqslant 0 \ \forall y \in \Lambda(x)$ if $\overline{F}^A(x) - \overline{F}^B(x) + \varphi^1(x) \leqslant 0$. The same line of reasoning yields Proposition 8.

B.4 Proof of Propositions 11 and 12

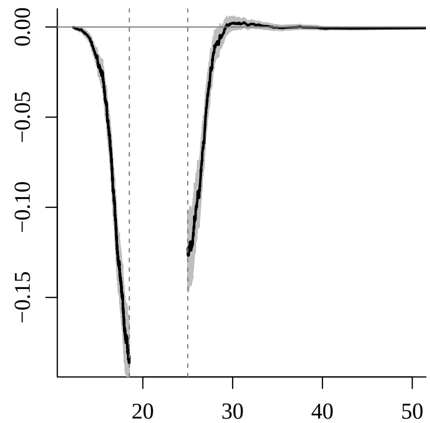
Let first consider the case $x \in [z^{U-}, z^{U+}]$. It can then easily be seen that the largest potential value of δ^r is $\delta^{r+} = \frac{x - z^{U-}}{\omega^+ - z^{U-}}$ (δ^r is a decreasing function of z^U) while the lowest is $\delta^{r-} = 0$. For a given set of poverty lines z^L, z^U, the value y within the "loss" poverty domain that yields the same relative gap as x is $z^L - \delta^r(z^L - \omega^-)$. Since $z^L \in [z^{L-}, z^{L+}]$, we then have $y \in [z^{L-} - \delta^r(z^{L-} - \omega^-), z^{L+} - \delta^r(z^{L+} - \omega^-)]$ for a given value of δ^r. Taking into account that $\delta^r \in \Big[0, \frac{x - z^{U-}}{\omega^+ - z^{U-}}\Big]$, we then have $y \in \Big[z^{L-} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+}\Big]$.

Now, let have a look at the case $x \in [z^{U+}, \omega^+]$. Potential values of δ are then $\delta^{r+} = \frac{x - z^{U-}}{\omega^+ - z^{U-}}$ and $\delta^{r-} = \frac{x - z^{U+}}{\omega^+ - z^{U+}}$. For a given value of x, taking the variability of z^L and z^U into account, we obtain $y \in [z^{L-} - \delta^{r+}(z^{L-} - \omega^-), z^{L+} - \delta^{r-}(z^{L+} - \omega^-)] = \Big[z^{L-} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+} - \frac{x - z^{U+}}{\omega^+ - z^{U+}}(z^{L+} - \omega^-)\Big]$.

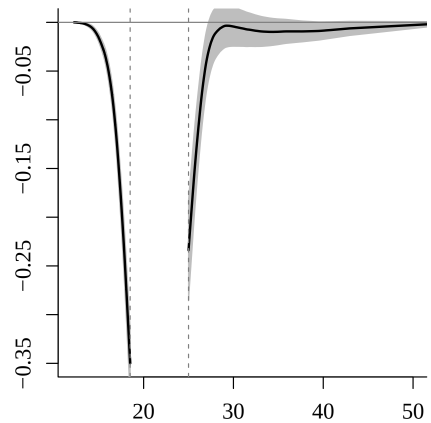
Bringing together the two cases, we get the general expression for the appropriate interval for y, that is $\Lambda^r(x) = \Big[z^{L-} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+} - \max\Big\{0, \frac{x - z^{U+}}{\omega^+ - z^{U+}}(z^{L+} - \omega^-)\Big\}\Big]$.

C Additional figures

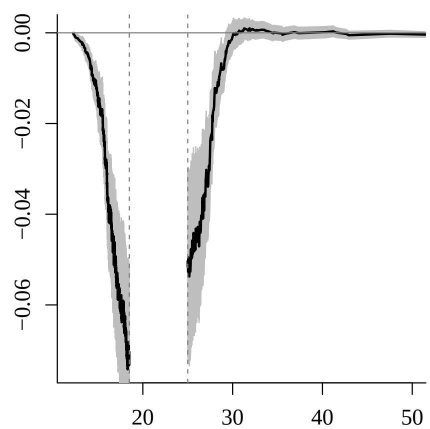
(a) 1997–2007, 1st order.



(b) 1997–2007, 2nd order.



(c) 1997–2000, 1st order.



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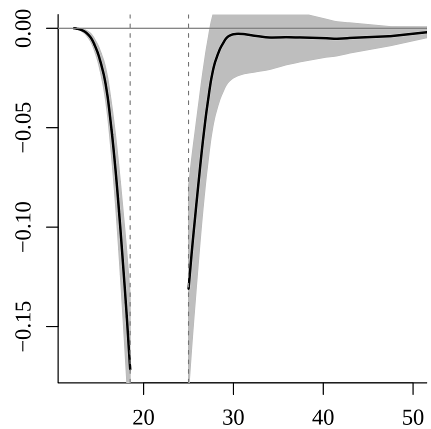

(e) 2004–2007, 2nd order.

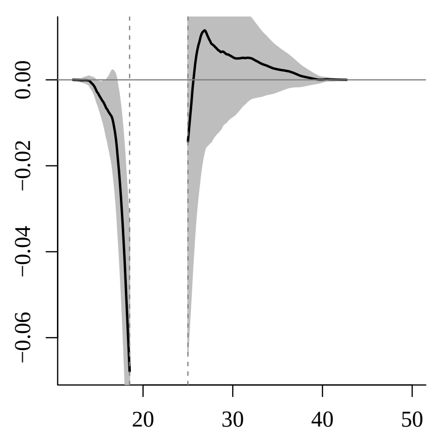

Figure 10: Sequential dominance for absolute gaps: BMI for 15-59 Bangladeshi mothers.

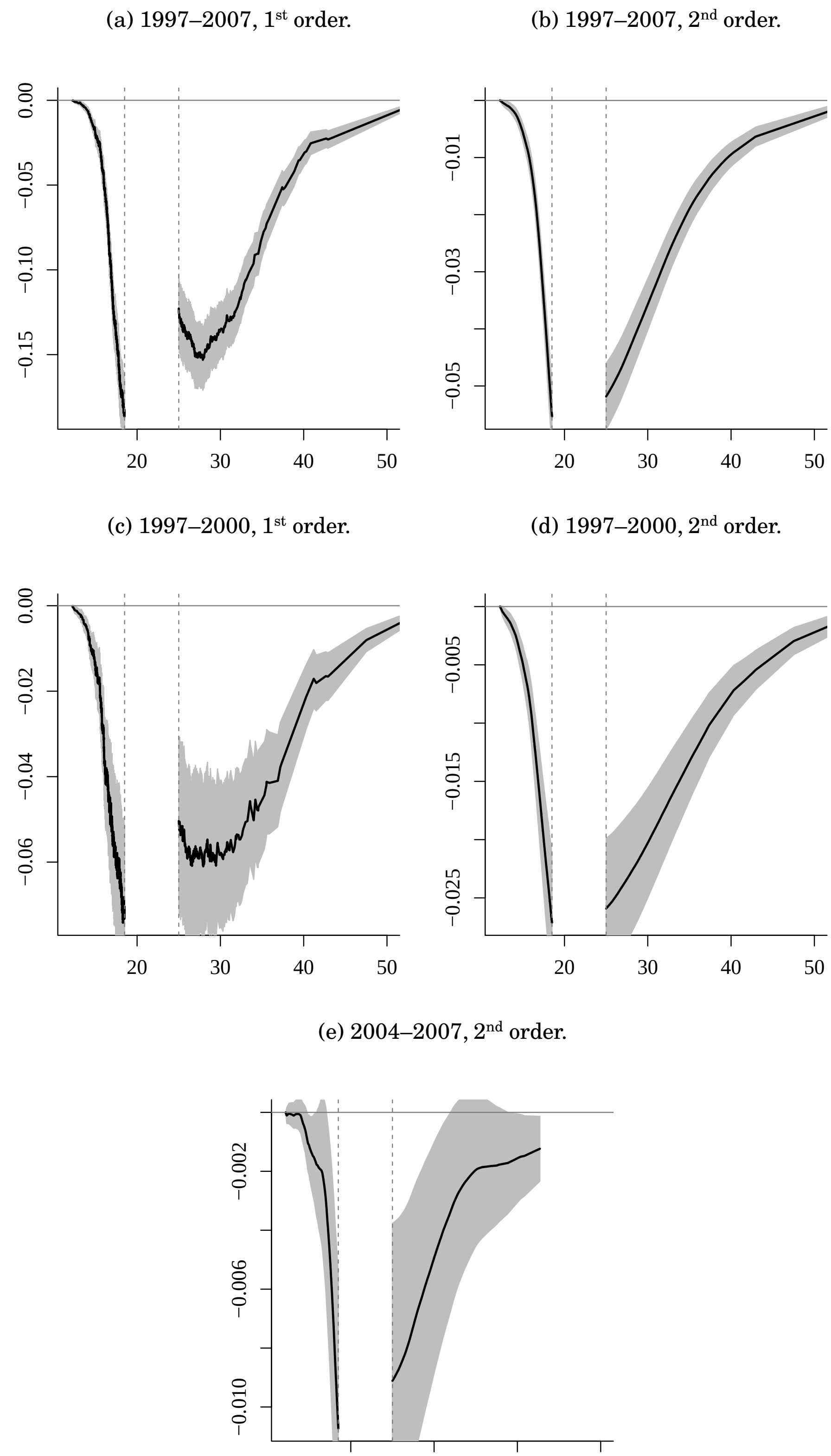

Figure 11: Sequential dominance for relative gaps: BMI for 15-59 Bangladeshi mothers.