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Notes on the Decomposition of Malmquist Productivity Indexes

W. Erwin Diewert and Kevin J. Fox

For additional information please contact:

Name: Kevin Fox
Affiliation: University of New South Wales, Australia

Email Address: K.Fox@unsw.edu.au

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W. Erwin Diewert[a,b] and Kevin J. Fox[b]

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Abstract

A paper by Caves, Christensen, Diewert (*Econometrica* 50,1393-1414, 1982) led to much interest in what they called the "Malmquist index"; a distance function method for representing technology in order to define families of input, output and productivity indexes. Since then there have been attempts to decompose the Malmquist productivity index into technical change, efficiency change and scale change components. Determining the appropriate way to do this has led to a healthy debate. This paper revisits this debate of how to decompose a Malmquist productivity index, with a focus on extracting technical progress, technical efficiency change, and returns to scale components.

a: University of British Columbia
b: University of New South Wales, School of Economics, Sydney 2052, Australia. K.Fox@unsw.edu.au

1. Introduction

The "Malmquist index", as defined by Caves, Christensen and Diewert (1982) (CCD), has attracted much interest in the literature on productivity analysis. This theoretical index is a distance function method for representing technology in order to define families of input, output and productivity indexes. CCD proposed a method for estimating a theoretical Malmquist productivity index for a firm using Törnqvist input and output indexes, augmented by exogenous estimates of local returns to scale; the assumptions they used were relaxed by Diewert and Fox (2010), which resulted in a variant of the original results.

Since the contribution of CCD, and particularly following Färe, Groskopf, Norris and Zhang (1994), there have been attempts to decompose the Malmquist productivity index into technical change, efficiency change and scale change components using the linear programming based "Data Envelopment Analysis" approach to implementing the Malmquist index. Determining the appropriate way to do this has led to a significant debate; see, for example, Ray and Desli (1997), Färe, Groskopf, Norris and Zhang (1994), Balk (2001), Lovell (2003) and Groskopf (2003) .

This paper revisits this debate of how to decompose a Malmquist productivity index, with a focus on extracting technical progress, technical efficiency change, and returns to scale components.

2. Malmquist Input Indexes

Caves, Christensen and Diewert (1982) (CCD) used the distance function method for representing a technology in order to define families of input, output and productivity indexes. The distance function was introduced into the economics literature in the consumer context by Malmquist (1953) and in the production context by Shephard (1953) (1970). The CCD definitions for Malmquist output and input indexes were generalized by Bjurek (1996) to cover applications of these indexes when estimates of best practice technologies are available. In this section and the following one, we give the basic theoretical definitions for the CCD-Bjurek *input* and *output indexes*. These input and output indexes are then used in order to define a family of Malmquist *productivity indexes*.

Let S^t be a *reference production possibilities set* for a production unit for periods t = 0,1. This reference technology could be determined via a data envelopment application[1] or could be estimated via econometric techniques. It represents the *best practice* or *efficient technology set* for period t. We assume that S^t is a nonempty closed subset of the nonnegative orthant in Euclidean M+N dimensional space. If (y,x) belongs to S^t, then the

[1] See Charnes, Cooper and Rhodes (1978) and Charnes and Cooper (1985) on early applications of DEA.

nonnegative vector of M outputs $y \equiv [y_1,...,y_M] \geq 0_M$ can be produced using the period t technology by the vector of N nonnegative inputs $x \equiv [x_1,...,x_N] \geq 0_N$.[2]

Using the period t reference technology set S^t and given a nonnegative, nonzero output vector $y > 0_M$ and a strictly positive input vector $x >> 0_N$, the period t *input distance function* D^t for periods t = 0,1 can be defined as follows:

(1) $D^t(y,x) \equiv \max_{\delta>0} \{\delta: (y,x/\delta)\in S^t\}$.

Thus given the nonnegative, nonzero vector of outputs y and the strictly positive vector of inputs x, $D^t(y,x)$ is the maximal amount that the input vector x can be deflated so that the deflated input vector $x/D^t(y,x)$ can produce the vector of outputs y using the period t technology S^t.

Instead of deflating the input vector x so that the resulting deflated vector is just big enough to produce the vector of outputs y, we could think of deflating the output vector so that the resulting deflated output vector is just producible by the input vector x. Thus given $y > 0_M$ and $x >> 0_N$ and the period t reference technology S^t, the period t *output distance function* d^t for periods t = 0,1 can be defined as follows:

(2) $d^t(y,x) \equiv \min_{\delta>0} \{\delta: (y/\delta,x)\in S^t\}$.

It is not immediately clear that the maximum in (1) or the minimum in (2) will exist. In fact, in order to obtain the existence of the functions D^t and d^t defined by (1) and (2), some restrictions on the production possibilities sets S^t are required (in addition to the assumption that S^t is a closed, nonempty subset of the nonnegative orthant). In the technical Appendix, we postulate a simple set of restrictions on the S^t which will guarantee the existence of these input and output distance functions.

Given a reference output vector $y > 0_M$ and two strictly positive input vectors $x^0 >> 0_N$ and $x^1 >> 0_N$, the input distance function $D^t(y,x)$ that corresponds to the period t reference technology S^t can be used to define the following family of *Malmquist input indexes*,[3] $Q(x^0,x^1,y,t)$:

(5) $Q(x^0,x^1,y,S^t) \equiv D^t(y,x^1)/D^t(y,x^0)$.

A value of the index greater than one implies that the input vector x^1 is larger than the input vector x^0, using y as a reference output and the period t best practice technology, S^t,

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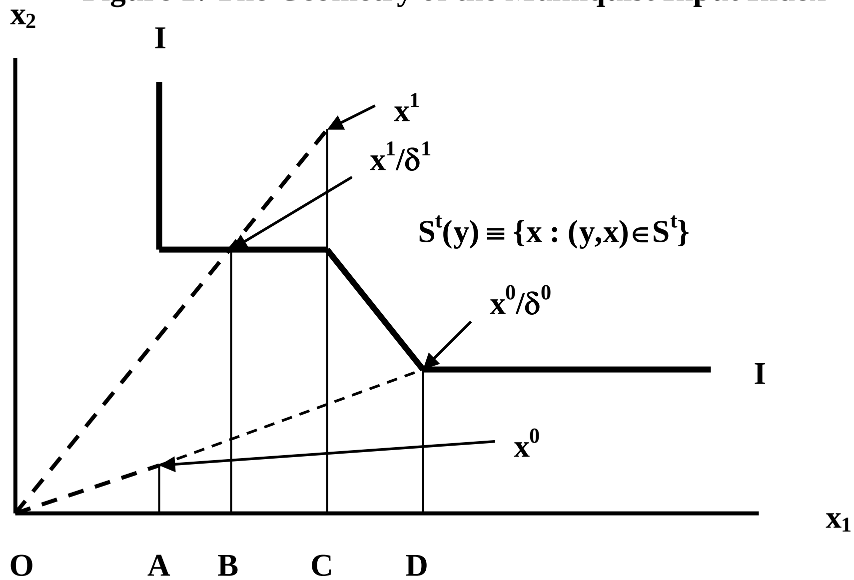
[2] Notation: $y \geq 0_M$ means each component of the vector y is nonnegative, $y >> 0_M$ means that each component is strictly positive, $y > 0_M$ means $y \geq 0_M$ but $y \neq 0_M$ and $p \cdot y$ denotes the inner product of the vectors p and y.

[3] The use of input distance functions to define input indexes can be traced back to Hicks (1961) and Moorsteen (1961). Fisher and Shell (1972; 51), Diewert (1980; 462) and Caves, Christensen and Diewert (1982; 1396) all used variants of this concept in the context of production theory. The basic idea of the index was developed in the consumer theory context by Malmquist (1953). The general definition of the input index given by (5) is due to Bjurek (1996; 307). [Check Balk 1998 and others]

as the reference technology. In the following sections, x^0 will be interpreted as the input vector that corresponds to a production unit that operates in period 0 and x^1 will be interpreted as the input vector that corresponds to a production unit that operates in period 1. If N = 1, so that there is only one input, then $Q(x_1^0,x_1^1,y,S^t)$ equals x_1^1/x_1^0.[4] The geometry of the Malmquist input index for two inputs is illustrated in Figure 1.

Figure 1: The Geometry of the Malmquist Input Index



Given a reference technology set S^t and a reference output vector $y > 0_M$, the set of inputs x that can produce the vector of outputs y is $S^t(y) \equiv \{x : (y,x)\in S^t\}$. In Figure 1, this set of feasible inputs lies on and above the kinked boundary line I-I. Note that the period 1 input vector $x^1 \equiv [x_1^1,x_2^1]$ lies in the interior of $S^t(y)$ while the period 0 input vector $x^0 \equiv [x_1^0,x_2^0]$ is exterior to $S^t(y)$. Define $\delta^0 \equiv D^t(y,x^0)$ so that x^0/δ^0 is on the boundary line I-I. It can be seen that δ^0 is less than one and δ^0 equals OA/OD, the distance OA divided by the distance OD. Define $\delta^1 \equiv D^t(y,x^1)$ so that x^1/δ^1 is on the boundary line I-I. It can be seen that δ^1 is greater than one and, δ^1 equals OC/OB. Thus the input index $Q(x^0,x^1)$ is equal to [OC/OB]/[OA/OD] = [OC/OB][OD/OA] where the distance ratios OC/OB and OD/OA are both greater than one in this case. It can be seen that if both input vectors x^0 and x^1 are on the frontier of the input production possibilities set $S^t(y)$ (i.e., they are both on the boundary line I-I), then $Q(x^0,x^1)$ equals one and the input vectors are regarded as having equivalent size. If x^0 is below the boundary line I-I and x^1 is on the boundary line or above it, then $Q(x^0,x^1)$ is greater than one and x^0 is regarded as being a smaller amount

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[4] Let N = 1 and let $y > 0_M$, $x^0 > 0$ and $x^1 > 0$. Let S^t satisfy the regularity conditions P1-P4 to be introduced below. Then it can be verified that $(x: (y,x)\in S^t)$ is the set $\{x : x \geq g(y) > 0\}$ where g(y) is the minimum amount of input required to produce the vector of outputs y using the technology set S^t. Thus $D^t(y,x^0) = \max_\delta \{\delta : (y,x^0/\delta)\in S^t\} = \max_\delta \{\delta: x^0/\delta \geq g(y)\} = \delta^0$ where $\delta^0 = x^0/g(y) > 0$. Similarly $D^t(y,x^1) = x^1/g(y) > 0$. Thus $Q(x^0,x^1,y,S^t) \equiv D^t(y,x^1)/D^t(y,x^0) = x^1/x^0$.

of aggregate input than the amount represented by x^1. This is the idea behind the Malmquist (1953) index, which was originally developed in the consumer context.

Now suppose that the strictly positive vector x^0 in Figure 1 were shifted down to the point A on the x_1 axis. It can be seen that in this case, $D^t(y,x^0)$ is not well defined; i.e., the x_1 axis never touches the input production possibilities set $S^t(y)$. Thus the restriction that the vectors x^0 and x^1 be strictly positive is required in order to ensure that the Malmquist input index is well defined. The example in Figure 1 is also consistent with the free disposability of inputs and this is another restriction on the technology we require in order for the input distance functions to be well defined.

We will now list our regularity conditions on the reference technology set S^t that will ensure that the input distance function $D^t(y,x)$ is well defined. Suppose that the reference technology set S^t satisfies the following regularity conditions:

P1. S is a nonempty closed subset of the nonnegative orthant in Euclidean M+N dimensional space.
P2. For every $y \geq 0_M$, there exists an $x \geq 0_N$ such that $(y,x) \in S$.
P3. $(y,x^1) \in S$, $x^2 \geq x^1$ implies $(y,x^2) \in S$.
P4. $y > 0_M$ implies that $(y,0_N) \notin S$.

Then in the Appendix, we show that $D^t(y,x)$ satisfies the following regularity conditions with respect to x over the positive orthant, $\Omega_N \equiv \{x: x >> 0_N\}$ in N dimensional space: for $y > 0_M$, $D^t(y,x)$ is *positive*, (positively) *linearly homogeneous*, *nondecreasing* (*increasing if all inputs increase*) and *continuous function* of x over Ω_N.

Let S^t satisfy properties P1-P4 and let $y > 0_M$. We now look at the axiomatic properties of $Q(x^0,x^1,y,S^t)$ defined by (5) above with respect to the two input vectors, x^0 and x^1. For brevity, we denote $Q(x^0,x^1,y,S^t)$ by $Q(x^0,x^1)$. Using the properties of the input distance function $D^t(y,x)$ listed in the paragraph above, it is reasonably straightforward to show that $Q(x^0,x^1)$ satisfies the following 12 properties for $x^0 = [x_1^0,...,x_N^0] >> 0_N$ and $x^1 = [x_1^1,...,x_N^1] >> 0_N$:

A1. *Identity*: $Q(x,x) = 1$; i.e., if the period 0 and 1 quantity vectors are equal to $x >> 0_N$, then the index is equal to unity.

A2: *Weak Monotonicity in Current Period Quantities*: $Q(x^0,x^1) \leq Q(x^0,x)$ if $x^1 < x$; i.e., if any period 1 quantity increases, then the quantity index increases or remains constant.

A3: *Strong Monotonicity in Current Period Quantities*: $Q(x^0,x^1) < Q(x^0,x)$ if $x^1 << x$; i.e., if all period 1 input quantities increase, then the quantity index increases.

A4: *Weak Monotonicity in Base Period Quantities*: $Q(x^0,x^1) \geq Q(x,x^1)$ if $x^0 < x$; i.e., if any period 0 quantity increases, then the quantity index decreases or remains constant.

A5: *Strong Monotonicity in Base Period Quantities*: $Q(x^0,x^1) > Q(x,x^1)$ if $x^0 << x$; i.e., if all period 0 input quantities increase, then the quantity index decreases.

A6: *Proportionality in Current Period Quantities*: $Q(x^0,\lambda x^1) = \lambda Q(x^0,x^1)$ if $\lambda > 0$; i.e., if all period 1 quantities are multiplied by the positive number λ, then the resulting quantity index is equal to the initial quantity index multiplied by λ.

A7: *Inverse Proportionality in Base Period Quantities*: $Q(\lambda x^0,x^1) = \lambda^{-1}Q(x^0,x^1)$ if $\lambda > 0$; i.e., if all period 0 quantities are multiplied by the positive number λ, then the resulting quantity index is equal to the initial quantity index multiplied by $1/\lambda$.

A8: *Mean Value Test*: $\min_n\{x_n^1/x_n^0: n = 1,...,N\} \le Q(x^0,x^1) \le \max_n\{x_n^1/x_n^0: n = 1,...,N\}$; i.e., the input quantity index lies between the smallest and largest quantity relatives.[5]

A9: *Time Reversal Test*: $Q(x^1,x^0) = 1/Q(x^0,x^1)$; i.e., if the data for periods 0 and 1 are interchanged, then the resulting quantity index should equal the reciprocal of the original quantity index.

A10: *Circularity*: $Q(x^0,x^1)Q(x^1,x^2) = Q(x^0,x^1)$; i.e., the quantity index going from period 0 to 1 times the quantity index going from period 1 to 2 equals the quantity index going from period 0 to 2 directly.

The circularity and identity tests imply the time reversal test; (just set $x^2 = x^0$). Thus the circularity test is essentially a strengthening of the time reversal test.

A11: *Commensurability*: $Q(\lambda_1x_1^0,...,\lambda_Nx_N^0;\lambda_1x_1^1,...,\lambda_Nx_N^1) = Q(x_1^0,...,x_N^0;x_1^1,...,x_N^1) = Q(x^0,x^1)$ for all $\lambda_1 > 0, ... , \lambda_N > 0$; i.e., if we change the units of measurement for each input, then the input quantity index remains unchanged.

A12: *Continuity*: $Q(x^0,x^1)$ is a jointly continuous function of x^0 and x^1 for $x^0 >> 0_N$ and $x^1 >> 0_N$.

O'Donnell (2009) considered many of the above axioms and some additional axioms for input quantity indexes. The above axioms are essentially the modification of the axioms used by Diewert (1992) for bilateral price indexes of the form $P(p^0,p^1,q^0,q^1)$ except that Q replaces P, x^0 and x^1 replace p^0 and p^1 and tests involving changes in q^0,q^1 are deleted.

Recall that (5) defines an entire family of Malmquist input quantity indexes, $Q(x^0,x^1,y,S^t)$; i.e., for each reference output vector $y > 0_M$ and for each reference technology set S^t , there is a possibly *different* input quantity index $Q(x^0,x^1,y,S^t)$. The question we now have to address is: if we are comparing the inputs of two different

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[5] Let $\beta \equiv \max_n \{x_n^1/x_n^0: n = 1,...,N\}$. Then $x^1 \le \beta x^0$ using the positivity of x^0. Thus $Q(x^0,x^1) \le Q(x^0,\beta x^0)$ (using A2) $= \beta Q(x^0,x^1)$ (using A6) $= \beta$ (using A1). Similarly, let $\alpha \equiv \min_n \{x_n^1/x_n^0: n = 1,...,N\}$. Then $x^1 \ge \alpha x^0$ using the positivity of x^0. Thus $Q(x^0,x^1) \ge Q(x^0,\alpha x^0)$ (using A2) $= \alpha Q(x^0,x^1)$ (using A6) $= \alpha$ (using A1). This proof follows that of Eichhorn (1978; 155) in the price index context.

production units who have the observed output and input vectors (y^0,x^0) and (y^1,x^1), what is an appropriate choice of the reference output vector y and the reference technology set S^t to insert into the definition of the Malmquist input quantity index $Q(x^0,x^1,y,S^t)$?

From the viewpoint of the period 0 production unit, the most appropriate choice of a reference output vector y would seem to be the actual output vector produced by the unit in period 0, which is y^0. Similarly, the most appropriate reference technology for the period 0 production unit would appear to be the period 0 best practice technology, S^0. Thus from the viewpoint of the period 0 production unit, the most appropriate input quantity index would appear to be the *Laspeyres type Malmquist input index*, $Q_L(x^0,x^1)$, defined as follows:

(6) $Q_L(x^0,x^1) \equiv Q(x^0,x^1,y^0,S^0) = D^0(y^0,x^1)/D^0(y^0,x^0)$.

From the viewpoint of the period 1 production unit, the most appropriate choice of a reference output vector y is the output vector produced by the unit in period 1, which is y^1. Similarly, the most appropriate reference technology for the period 1 production unit is the period 1 best practice technology, S^1. Thus from the viewpoint of the period 1 production unit, the most appropriate input quantity index is the *Paasche type Malmquist input index*, $Q_P(x^0,x^1)$, defined as follows:

(7) $Q_P(x^0,x^1) \equiv Q(x^0,x^1,y^1,S^1) = D^1(y^1,x^1)/D^1(y^1,x^0)$.

Since we have two separate relevant input quantity indexes[6] when comparing the relative size of the input vectors of two production units, it is natural to take a symmetric average of the two indexes defined by (6) and (7) in order to obtain a "final" measure of the relative magnitude of the input vector x^1 relative to x^0. But what form of average should we take? Caves, Christensen and Diewert (1982; 1397) found it convenient to take the geometric average of the above two indexes; i.e., define

(8) $Q_{CCD}(x^0,x^1) = [Q_L(x^0,x^1)Q_P(x^0,x^1)]^{1/2}$.

However, Caves, Christensen and Diewert chose the geometric average of the Laspeyres and Paasche type Malmquist input indexes because it led to an exact bilateral index number formula when they made various translog assumptions on the underlying technology. In our present context, we want to avoid the use of price information so we need another justification for taking the geometric mean of Q_L and Q_P as opposed to taking some other form of average.

In the present context, we should choose the form of average strategically so that the resulting index satisfies an important test or property. The important property that we will choose to focus on is the *Time Reversal Test*; see A9 above.

[6] The two input indexes defined by (6) and (7) were the ones that were introduced by Caves, Christensen and Diewert (1982; 1396). Diewert (1992; 235) also endorsed these two input indexes as being "natural" input indexes.

At this point, we need a bit of background information on the properties of averages or *means*. Let a and b be two positive numbers. Diewert (1993b; 361) defined a *symmetric mean* of a and b as a function m(a,b) that has the following properties:

(9) $m(a,a) = a$ for all $a > 0$ (mean property);
(10) $m(a,b) = m(b,a)$ for all $a > 0$, $b > 0$ (symmetry property);
(11) m(a,b) is a continuous function for $a > 0$, $b > 0$ (continuity property);
(12) m(a,b) is a strictly increasing function in each of its variables (increasingness property).

It can be shown that if m(a,b) satisfies the above properties, then it also satisfies the following property:[7]

(13) $\min\{a,b\} \leq m(a,b) \leq \max\{a,b\}$ (min-max property);

i.e., the mean of a and b, m(a,b), lies between the maximum and minimum of the numbers a and b. Since we have restricted the domain of definition of a and b to be positive numbers, it can be seen that an implication of (13) is that m also satisfies the following property:

(14) $m(a,b) > 0$ for all $a > 0$, $b > 0$ (positivity property).

If in addition, m satisfies the following property, then Diewert (1993b) defined m to be a *homogeneous symmetric mean*:

(15) $m(\lambda a,\lambda b) = \lambda m(a,b)$ for all $\lambda > 0$, $a > 0$, $b > 0$.

With the above material on homogeneous, symmetric means in hand, we can prove the following proposition:

Proposition 1: The CCD input quantity index $Q_{CCD}(x^0,x^1)$ defined by (8) above is the *only* index satisfying the Time Reversal Test A9 that is a homogeneous symmetric average of the Laspeyres and Paasche Malmquist input quantity indexes, Q_L and Q_P defined by (6) and (7).

Proof: Assume that the homogeneous mean function m satisfies the positivity and homogeneity properties, (14) and (15) above.

Let $x^0 \gg 0_N$ and $x^1 \gg 0_N$. Define $a \equiv Q_L(x^0,x^1) > 0$ and $b \equiv Q_P(x^0,x^1) > 0$. Looking at definitions (6) and (7), it can be seen that if we reverse the order of time:

(16) $Q_L(x^1,x^0) = 1/a = 1/Q_L(x^0,x^1)$; $Q_P(x^1,x^0) = 1/b = 1/Q_P(x^0,x^1)$.

Define the mean input quantity index Q using the function m as follows:

[7] To prove this, use the technique of proof used by Eichhorn and Voeller (1976; 10).

(17) $Q(x^0,x^1) \equiv m(Q_L(x^0,x^1),Q_P(x^0,x^1)) = m(a,b)$.

where we have used the definitions of the numbers a and b above. For Q to satisfy the time reversal test, the following equation must be satisfied:

(18) $Q(x^1,x^0) = m(Q_L(x^1,x^0),Q_P(x^1,x^0))$
$= m(a^{-1},b^{-1})$ using (16)
$= 1/Q(x^0,x^1)$
$= 1/m(a,b)$ using (17).

Using the positivity of a and b and property (14) for m, (18) can be rewritten as follows:

(19) $1 = m(a,b)m(b^{-1},a^{-1})$
$= am(1,b/a)a^{-1}m(a/b,1)$ using property (15) for m
$= m(1,x)m(x^{-1},1)$ letting $x \equiv b/a$
$= m(1,x)x^{-1}m(1,x)$ using property (15) for m.

Equation (19) can be rewritten as:

(20) $x = [m(1,x)]^2$.

Take the positive square root of both sides of (20) and obtain

(21) $m(1,x) = x^{1/2}$.

Using property (15) for m again, we have

(21) $m(a,b) = am(1,b/a)$
$= a[b/a]^{1/2}$ using (21)
$= a^{1/2}b^{1/2}$.

Now substitute (21) into (17) and we find that $Q(x^0,x^1) = Q_{CCD}(x^0,x^1)$. Q.E.D.

The above proof is a modification of a proof due to Diewert (1997; 138) in the price index context.

Using the mathematical properties of the input distance functions $D^0(y^0,x)$ and $D^1(y^1,x)$ with respect to the strictly positive input vector x, it is straightforward to establish the following Proposition:

Proposition 2: Let the technology sets S^0 and S^1 satisfy properties P1-P4 and let $y^0 > 0_M$ and $y^1 > 0_M$. Then the Caves, Christensen and Diewert Malmquist input quantity index $Q_{CCD}(x^0,x^1)$ defined by (8) above satisfies the axioms A1-A12 listed above for all $x^0 >> 0_N$ and $x^1 >> 0_N$.

The above Proposition implies that $Q_{CCD}(x^0,x^1)$ satisfies the circularity test A10 and this is true but note that this circularity test is conditional on only the two production possibility sets S^0 and S^1 and the two reference output vectors y^0 and y^1. Thus the index $Q_{CCD}(x^0,x^1)$ should be more properly denoted by $Q_{CCD}(x^0,x^1;y^0,y^1;S^0,S^1)$ and the circularity test that Q_{CCD} satisfies is the following one: for all $x^0 >> 0_N$, $x^1 >> 0_N$ and $x^2 >> 0_N$, we have:

(22) $Q_{CCD}(x^0,x^1;y^0,y^1;S^0,S^1)Q_{CCD}(x^1,x^2;y^0,y^1;S^0,S^1) = Q_{CCD}(x^0,x^2;y^0,y^1;S^0,S^1)$.

Thus there is a certain lack of symmetry in the index when input comparisons are made between three or more production units. Hence the CCD Malmquist input index is best suited for bilateral comparisons between a pair of production units (or the same production unit over two time periods) rather than multilateral comparisons between many production units.

We now turn our attention to Malmquist output indexes.

3. Malmquist Output Indexes

Given a strictly positive reference input vector $x >> 0_N$ and two nonnegative, nonzero output vectors $y^0 > 0_M$ and $y^1 > 0_M$, the output distance function $d^t(y,x)$ defined by (2) that corresponds to the period t reference technology S^t can be used to define the following family of *Malmquist output indexes,*[8] $q(y^0,y^1,x,S^t)$:

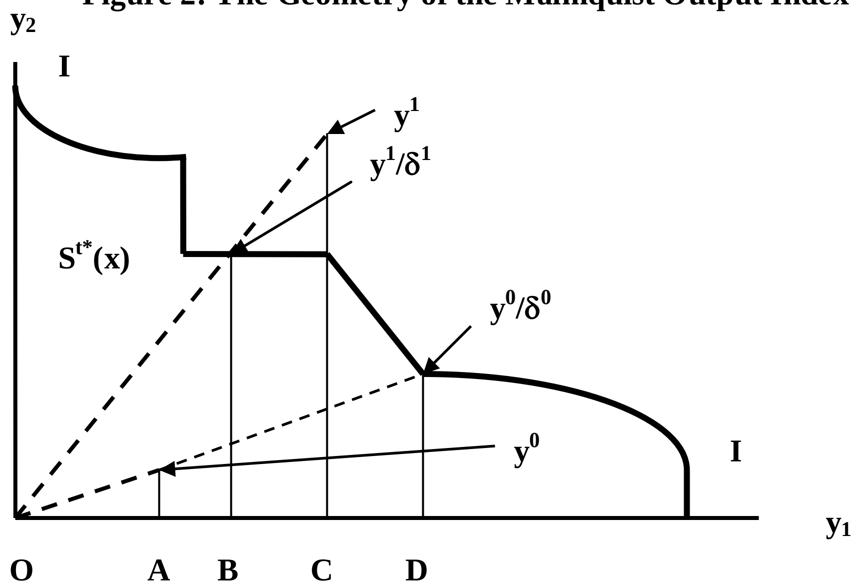
(23) $q(y^0,y^1,x,S^t) \equiv d^t(y^1,x)/d^t(y^0,x)$.

A value of the index greater than one implies that the output vector y^1 is larger than the output vector y^0, using x as a reference output and the period t best practice technology, S^t, as the reference technology. In the following sections, y^0 will be the output vector that corresponds to a production unit that operates in period 0 and y^1 will be the output vector that corresponds to a production unit that operates in period 1. If M = 1, so that there is only one output, then $q(y_1^0, y_1^1,x,S^t)$ equals y_1^1/y_1^0.[9] The geometry of the Malmquist output index for two inputs is illustrated in Figure 2.

[8] The general definition of the output index given by (23) is due to Bjurek (1996; 307). [Check Balk 1998 and others]

[9] Let M = 1 and let $x >> 0_N$, $y^0 > 0$ and $y^1 > 0$. Let S^t satisfy the regularity conditions P1 and P5-P7. Then it can be verified that $(y : (y,x)\in S^t)$ is the set $\{y : 0 \leq y \leq f(x)\}$ where $f(x) > 0$ is the maximum amount of the single output that can be produced by the strictly positive input vector x using the technology set S^t. Thus $d^t(y^0,x) = \min_\delta \{\delta: (y^0/\delta,x)\in S^t\} = \min_\delta \{\delta: y^0/\delta \leq f(x)\} = \delta^0$ where $\delta^0 = y^0/f(x) > 0$. Similarly $d^t(y^1,x) = y^1/f(x) > 0$. Thus $q(y^0,y^1,x,S^t) \equiv d^t(y^1,x)/d^t(y^0,x) = y^1/y^0$.

Figure 2: The Geometry of the Malmquist Output Index



Given a reference technology set S^t and a reference input vector $x >> 0_N$, the set of outputs y that can be produced by the vector of inputs x is $S^{t*}(x) \equiv \{y : (y,x) \in S^t\}$. In Figure 2, this set of feasible inputs is a subset of the nonnegative orthant and lies on and below the kinked boundary line I-I. Note that the period 1 output vector $y^1 \equiv [y_1^1, y_2^1]$ lies outside of $S^{t*}(x)$ while the period 0 output vector $y^0 \equiv [y_1^0, y_2^0]$ is in the interior of $S^{t*}(x)$. Define $\delta^0 \equiv d^t(y^0,x)$ so that y^0/δ^0 is on the boundary line I-I. It can be seen that δ^0 is less than one and δ^0 equals OA/OD, the distance OA divided by the distance OD. Define $\delta^1 \equiv d^t(y^1,x)$ so that y^1/δ^1 is on the boundary line I-I. It can be seen that δ^1 is greater than one and, δ^1 equals OC/OB. Thus the output index $q(x^0,x^1)$ is equal to [OC/OB]/[OA/OD] = [OC/OB][OD/OA] where the distance ratios OC/OB and OD/OA are both greater than one in this case. It can be seen that if both output vectors y^0 and y^1 are on the frontier of the input production possibilities set $S^{t*}(x)$ (i.e., they are both on the boundary line I-I), then $q(y^0,y^1)$ equals one and the output vectors are regarded as having equivalent size. If y^0 is below the boundary line I-I and y^1 is on the boundary line or above it, then $q(y^0,y^1)$ is greater than one and y^0 is regarded as being a smaller amount of aggregate output than the amount represented by y^1. Note that we do not require y^0 and y^1 to be strictly positive vectors in order for the output index to be well defined; we need only $y^0 > 0_M$ and $y^1 > 0_M$.

We will now list our regularity conditions on the reference technology set S^t that will ensure that the output distance function $d^t(y,x)$ is well defined. Suppose that the reference technology set S^t satisfies condition P1 listed in the previous section and the following three additional regularity conditions :

P5. $x \geq 0_N$, $(y,x) \in S$ implies $0_M \leq y \leq b(x)1_M$ where 1_M is a vector of ones of dimension M and $b(x) \geq 0$ is a finite nonnegative bound.
P6. $x >> 0_N$ implies that there exists $y >> 0_M$ such that $(y,x) \in S$.

P7. $(y^1,x)\in S$, $0_M \le y^0 \le y^1$ implies $(y^0,x)\in S$.

In the Appendix, we show that $d^t(y,x)$ satisfies the following regularity conditions with respect to y over the nonnegative orthant excluding the origin, $\Omega_M^* \equiv \{y: y > 0_N\}$: for $x >> 0_N$, $d^t(y,x)$ is *positive*, (positively) *linearly homogeneous*, *nondecreasing* (*increasing if all outputs increase*) and *continuous function* of y over Ω_M^*.

Let S^t satisfy properties P1 and P5-P7 and let $x >> 0_N$. We now look at the axiomatic properties of $q(y^0,y^1,x,S^t)$ defined by (23) above with respect to the two output vectors, y^0 and y^1. For brevity, we denote $q(y^0,y^1,x,S^t)$ by $q(y^0,y^1)$. Using the properties of the output distance function $d^t(y,x)$ listed in the paragraph above, it is straightforward to show that $q(y^0,y^1)$ satisfies the 12 properties A1-A12 listed in the previous section, where $Q(x^0,x^1)$ is replaced by $q(y^0,y^1)$ for $y^0 > 0_M$ and $y^1 > 0_M$.[10]

Recall that (23) defines an entire family of Malmquist output quantity indexes, $q(y^0,y^1,x,S^t)$; i.e., for each reference input vector $x >> 0_N$ and for each reference technology set S^t , there is a possibly *different* output quantity index $q(y^0,y^1,x,S^t)$. Thus we now have to address the same type of question that we addressed in the previous section: if we are comparing the outputs of two different production units who have the observed output and input vectors (y^0,x^0) and (y^1,x^1), what is an appropriate choice of the reference input vector x and the reference technology set t to insert into the definition of the Malmquist output quantity index $q(y^0,y^1,x,S^t)$?

From the viewpoint of the period 0 production unit, the most appropriate choice of a reference input vector x would seem to be the actual input vector used by the unit in period 0, which is x^0 (which we assume is a strictly positive vector). Similarly, the most appropriate reference technology for the period 0 production unit would appear to be the period 0 best practice technology, S^0. Thus from the viewpoint of the period 0 production unit, the most appropriate output quantity index would appear to be the *Laspeyres type Malmquist output index*, $q_L(y^0,y^1)$, defined as follows:

(24) $q_L(y^0,y^1) \equiv q(y^0,y^1,x^0,S^0) = d^0(y^1,x^0)/d^0(y^0,x^0)$.

From the viewpoint of the period 1 production unit, the most relevant choice of a reference input vector x is the input vector used by the unit in period 1, which is x^1 (where we assume $x^1 >> 0_N$). Similarly, the most relevant reference technology for the period 1 production unit is the period 1 best practice technology, S^1. Thus from the viewpoint of the period 1 production unit, the most appropriate output quantity index is the *Paasche type Malmquist output index*, $q_P(y^0,y^1)$, defined as follows:

(25) $q_P(y^0,y^1) \equiv q(y^0,y^1,x^1,S^1) = d^1(y^1,x^1)/d^1(y^0,x^1)$.

[10] The continuity property A12 holds only for strictly positive y vectors; i.e., we can establish the continuity of $q(y^0,y^1)$ for $y^0 >> 0_M$ and $y^1 >> 0_M$.

Since we have two separate relevant output quantity indexes[11] when comparing the relative size of the output vectors of two production units, it is natural to take a symmetric average of the two indexes defined by (24) and (25) in order to obtain a "final" measure of the relative magnitude of the output vector y^1 relative to y^0. But what form of average should we take? Caves, Christensen and Diewert (1982; 1401) found it convenient to take the geometric average of the above two indexes; i.e., define

(26) $q_{CCD}(y^0,y^1) = [q_L(y^0,y^1)q_P(y^0,y^1)]^{1/2}$.

The use of the geometric average of q_L and q_P instead of some other form of average can be justified if we want the average Malmquist output index to satisfy the time reversal test A9 (adapted to the output context); i.e., we can establish the following Proposition, using the same method of proof as was used in the proof of Proposition 2 in the previous section:

Proposition 3: The CCD output quantity index $q_{CCD}(y^0,y^1)$ defined by (26) above is the *only* index satisfying the (modified) Time Reversal Test A9 that is a homogeneous symmetric average of the Laspeyres and Paasche Malmquist output quantity indexes, q_L and q_P defined by (24) and (25).

The modified Tests A1-A11 simply replace the strictly positive input quantity vectors x^0 and x^1 by the nonnegative, nonzero output vectors $y^0 > 0_M$ and $y^1 > 0_M$. We denote the modified tests as B1-B11. For example, the first two modified tests are the following ones:

B1. *Identity*: $q(y,y) = 1$; i.e., if the period 0 and 1 quantity vectors are equal to $y > 0_M$, then the output quantity index is equal to unity.

B2: *Weak Monotonicity in Current Period Quantities*: $q(y^0,y^1) \leq q(y^0,y)$ if $y^0 > 0_M$ and $0_M < y^1 < y$; i.e., if any period 1 quantity increases, then the quantity index increases or remains constant.

However, the modified test A12 requires that the two output vectors y^0 and y^1 be strictly positive so that the test B12 is the following one:

B12: *Continuity*: $q(y^0,y^1)$ is a jointly continuous function of y^0 and y^1 for $y^0 >> 0_M$ and $y^1 >> 0_M$.

Assuming that $x >> 0_N$, using the mathematical properties of the output distance functions $d^0(y,x)$ and $d^1(y,x)$ with respect to the nonnegative, nonzero output vector y that are established in the Appendix, it is straightforward to prove the following Proposition:

Proposition 4: Let the technology sets S^0 and S^1 satisfy properties P1 and P5-P7 and let $x^0 >> 0_N$ and $x^1 >> 0_N$. Then the Caves, Christensen and Diewert Malmquist output

[11] The two output indexes defined by (24) and (25) were the ones that were introduced by Caves, Christensen and Diewert (1982; 1400).

quantity index $q_{CCD}(y^0,y^1)$ defined by (26) above satisfies the axioms B1-B11 for all $y^0 > 0_M$ and $y^1 > 0_M$ and B12 for all $y^0 >> 0_M$ and $y^1 >> 0_M$.

The above Proposition implies that $q_{CCD}(y^0,y^1)$ satisfies the circularity test B10 and this is true but note that this circularity test is conditional on only the two production possibility sets S^0 and S^1 and the two reference input vectors x^0 and x^1. Thus the index $q_{CCD}(y^0,y^1)$ is more properly denoted by $q_{CCD}(y^0,y^1,x^0,x^1;S^0,S^1)$ and the circularity test that q_{CCD} satisfies is the following one: for all $y^0 > 0_M$, $y^1 > 0_M$ and $y^2 > 0_M$, we have:

$$(27)\ q_{CCD}(y^0,y^1,x^0,x^1;S^0,S^1)q_{CCD}(y^1,y^2,x^0,x^1;S^0,S^1) = q_{CCD}(y^0,y^2,x^0,x^1;S^0,S^1).$$

Thus as was the case for the CCD input index, the CCD Malmquist output index is best suited for bilateral comparisons between a pair of production units (or the same production unit over two time periods) rather than multilateral comparisons between many production units.

We now turn our attention to productivity indexes.

4. Malmquist-Bjurek Productivity Indexes

Having defined families of input and output indexes using distance functions in the previous two sections, it is natural to define a *family of productivity indexes* as the ratio of a family of output indexes to a family of input indexes. Our goal is to compare the productivity of two production units, 0 and 1, in the same industry which have the observed output and input vectors (y^0,x^0) and (y^1,x^1) respectively. We assume that the input vectors are strictly positive, so that $x^0 >> 0_N$ and $x^1 >> 0_N$, and we assume that the output vectors are nonnegative but nonzero so that $y^0 > 0_M$ and $y^1 > 0_M$. Our definition of the productivity index will also require a nonnegative, nonzero *reference output vector* $y > 0_M$ and a strictly positive *reference input vector* $x >> 0_N$. Finally, the definition requires a *reference technology set* S^t which satisfy the regularity conditions P1-P7. Recall the family of Malmquist input indexes, $Q(x^0,x^1,y,t)$ defined by (5) above, and the family of Malmquist output indexes, $q(y^0,y^1,x,t)$ defined by (23) above. These two families of indexes can be used in order to define the following family of *Malmquist-Bjurek productivity indexes*:

$$(28)\ \Pi(x^0,x^1,y^0,y^1,x,y,S^t) \equiv q(y^0,y^1,x,S^t)/Q(x^0,x^1,y,S^t)$$
$$= [d^t(y^1,x)/d^t(y^0,x)/[D^t(y,x^1)/D^t(y,x^0)].$$

If $\Pi(x^0,x^1,y^0,y^1,x,y,S^t)$ is greater (less) than one, we say that production unit 1 is more (less) productive than production unit 0; if $\Pi(x^0,x^1,y^0,y^1,x,y,S^t)$ equals one, then the units have equal levels of productivity. At this level of generality, the index defined by (28) is due to Bjurek (1996; 308). Special cases of this type of index were described by Hicks (1961; 22) (1981; 256), Moorsteen (1961; 462) and Diewert (1992; 240). The mathematical properties of Π with respect to x^0,x^1,y^0,y^1 are of course determined by the mathematical properties of the input index $Q(x^0,x^1,y,t)$ with respect to x^0 and x^1 and the mathematical properties of the output index $q(y^0,y^1,x,t)$ with respect to y^0 and y^1; see sections 2 and 3 above for these properties. It can be verified that if $N = 1$ and $M = 1$ so that there is only one input and one output, then the Bjurek productivity index collapses to $[y^1/y^0]/[x^1/x^0]$, which is also equal to $[y^1/x^1]/[y^0/x^0]$, the growth in Total Factor Productivity going from the production unit 0 inputs and outputs to the production unit 1 inputs and outputs. Thus if $\Pi(x^0,x^1,y^0,y^1,x,y,S^t)$ is greater than one, production unit 1 can produce more aggregate output per unit aggregate input than production unit 0.

As usual, when faced with a family of indexes, we need to determine which member of the family should be chosen for empirical applications. Again following the lead of Caves, Christensen and Diewert (1982) and Bjurek (1996; 310), it is natural to pick the two members of the family of indexes defined by (28) that are of most relevance to the two production units being compared. The most relevant productivity comparison for unit 0 is the Laspeyres version of (28), which is Π_L defined below by (29), where we pick the reference output and input vectors, y and x, to be the observed vectors for unit 0, y^0 and x^0, and we pick the reference technology set S^t to be S^0, the best practice technology set for production unit 0. Thus define the *Bjurek-Laspeyres productivity index* between production units 0 and 1 as:

$$(29)\quad \Pi_L(x^0,x^1,y^0,y^1) \equiv q(y^0,y^1,x^0,S^0)/Q(x^0,x^1,y^0,S^0)$$
$$= [d^0(y^1,x^0)/d^0(y^0,x^0)]/[D^0(y^0,x^1)/D^0(y^0,x^0)].$$

Similarly, the most relevant productivity comparison for unit 1 is the Paasche version of (28), Π_P defined below by (30), where we pick the reference output and input vectors, y and x, to be the observed vectors for unit 1, y^1 and x^1, and we pick the reference technology set S^t to be S^1, the best practice technology set for production unit 1. Thus define the *Bjurek-Paasche productivity index* between production units 0 and 1 as:

$$(30)\quad \Pi_P(x^0,x^1,y^0,y^1) \equiv q(y^0,y^1,x^1,S^1)/Q(x^0,x^1,y^1,S^1)$$
$$= [d^1(y^1,x^1)/d^1(y^0,x^1)]/[D^1(y^1,x^1)/D^1(y^1,x^0)].$$

Finally, Bjurek (1996; 310-311) suggested that a good productivity index would result (that remedied some of the problems with existing productivity indexes) if we took the geometric mean of the indexes defined by (29) and (30). Thus we define the *Bjurek productivity index* as follows:

$$(31)\quad \Pi_B(x^0,x^1,y^0,y^1) \equiv [\Pi_L(x^0,x^1,y^0,y^1)\Pi_P(x^0,x^1,y^0,y^1)]^{1/2}$$
$$= \{[d^0(y^1,x^0)/d^0(y^0,x^0)][D^0(y^0,x^0)/D^0(y^0,x^1)][d^1(y^1,x^1)/d^1(y^0,x^1)][D^1(y^1,x^0)/D^1(y^1,x^1)]\}^{1/2}.$$

Thus the Bjurek productivity index is the product of two sets of output distance function ratios times two sets of input distance function ratios—a rather complicated function.

When comparing the productivity levels of two production units, it is very useful to have the productivity measure satisfy the time reversal test; i.e., if we have a productivity measure $\Pi(x^0,x^1,y^0,y^1)$ that compares the productivity level of production unit 1, characterized by the input-output data (x^1,y^1), with the productivity level of production unit 0, characterized by the input-output data (x^1,y^1), then the comparison should not depend materially on which unit is being compared to which; i.e., it would be desirable if the productivity measure satisfied the following *time reversal test*:

(32) $\Pi(x^1,x^0,y^1,y^0) = 1/\Pi(x^0,x^1,y^0,y^1)$.

It is straightforward to establish the following counterpart to Proposition 3 above:

Proposition 5: The Bjurek productivity index $\Pi_B(x^0,x^1,y^0,y^1)$ defined by (31) above is the *only* productivity index satisfying the Time Reversal Test (32) that is a homogeneous symmetric average of the Laspeyres and Paasche productivity indexes, $\Pi_L(x^0,x^1,y^0,y^1)$ and $\Pi_P(x^0,x^1,y^0,y^1)$ defined by (29) and (30).

Our goal in this paper is to decompose (31) into the product of readily interpreted explanatory factors; namely changes in the technical efficiency of the production units, technical progress due to a change in the reference best practice technology set from S^0 to S^1 and a measure of returns to scale. Hence in the following sections, we will consider some definitions for these explanatory variables based on distance function representations.

5. Radial Measures of Technical Efficiency

Our measures of technical efficiency for the two production units being compared are conventional Debreu (1951) Farrell (1957; 254) radial measure of efficiency loss except that we use output measures of loss rather than the input oriented measures they used.

We suppose that there are best practice technology sets S^0 and S^1 (satisfying properties P1 and P5-P7) that are relevant for production units 0 and 1; i.e., the observed output and input vector for unit 0 belongs to the period 0 best practice technology set S^0 and the observed output and input vector for unit 1 belongs to the period 1 best practice technology set S^1 but that these observed vectors are not necessarily on the frontiers of these best practice sets. We assume that $y^t > 0_M$ and $x^t >> 0_N$ for t = 0,1 and we also assume that:

(33) $(y^0,x^0)\in S^0$; $(y^1,x^1)\in S^1$.

For production units t = 0,1, the *output technical efficiency* of unit t, ε^t, is defined as follows:

(34) $\varepsilon^t \equiv d^t(y^t,x^t) \equiv \min_\delta \{\delta: (y^t/\delta,x^t)\in S^t\} \leq 1$

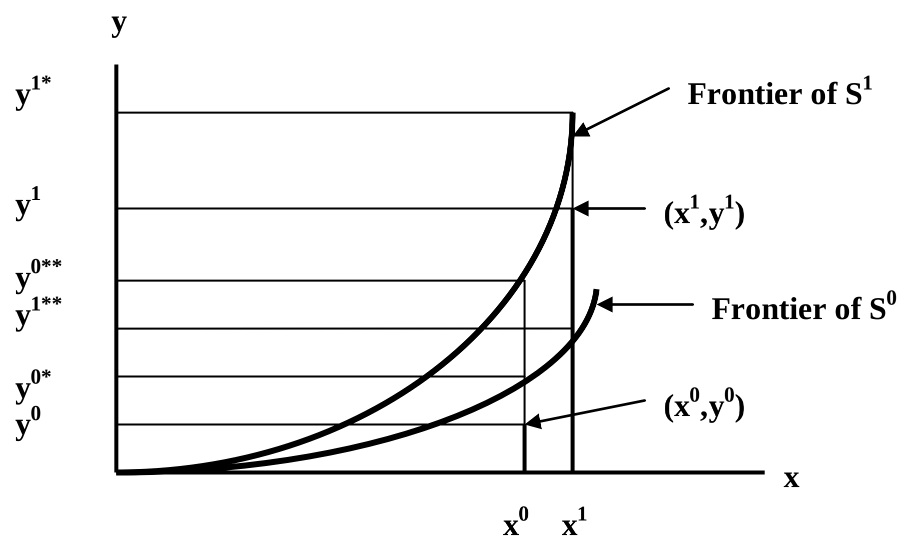
where the inequalities in (34) follow from assumptions (33) using a feasibility argument.[12]

[12] Our regularity conditions also imply that $\varepsilon^0 > 0$ and $\varepsilon^1 > 0$.

If $\varepsilon^0 = 1$, then production unit 0 is regarded as being efficient since the point (y^0,x^0) is on the frontier of the period 0 best practice production possibilities set. Similarly, if $\varepsilon^1 = 1$, then production unit 1 is regarded as being efficient. Alternatively, if $\varepsilon^0 < 1$, then production unit 0 is clearly not efficient since an efficient period 0 producer could produce the output vector y^0/ε^0 which is strictly greater than y^0 for all positive components of y^0, using the same input vector x^0. The amount that ε^0 is less than one is a quantitative indicator of the inefficiency of production unit 0.[13]

For the case of a single output and a single input, the technical efficiency measures can be illustrated in Figure 3.

Figure 3: Decomposition Factors for the One Output One Input Case



The observed input for unit 0 is x^0 and the corresponding amount of output produced is y^0. Note that this point lies below the frontier of the period 0 best practice technology set, S^0. The best practice technology can produce $y^{0*} > y^0$ units of output, using the same amount of input x^0. Thus the technical efficiency of production unit 0 is $\varepsilon^0 \equiv d^0(y^0,x^0) \equiv \min_\delta \{\delta : (y^0/\delta,x^0)\in S^0\} = \delta^0 = y^0/y^{0*} < 1$. Similarly, the observed input for production unit 1 is x^1 and the corresponding amount of output produced is y^1. This point lies below the frontier of the period 1 best practice technology set, S^1. The best practice technology can produce $y^{1*} > y^1$ units of output, using the same amount of input x^1. Thus the technical efficiency of production unit 1 is $\varepsilon^1 \equiv d^1(y^1,x^1) \equiv \min_\delta \{\delta: (y^1/\delta,x^1)\in S^1\} = \delta^1 = y^1/y^{1*} < 1$.

[13] The problem with this radial measure of inefficiency is that we could have ε^0 or ε^1 equal to one (indicating that production unit 0 or 1 is efficient) but in fact, production need not be completely efficient. This problem can be illustrated using Figure 2 where it can be seen that y^1/δ^1 is on the frontier of the reference production possibilities set S^t but it is clear that y^1/δ^1 is not completely efficient since we could use the same reference input vector to produce a greater amount of output 1 without reducing the production of output 2. This problem and possible solutions are discussed in depth by Russell and Schworm (2009) (2010). In the present paper, we will work with the rather weak measures of technical efficiency defined by (34) for the sake of simplicity but this limitation of our analysis should be kept in mind.

We now turn our attention to defining measures of technical change.

6. The Measurement of Technical Change

In this section, we want to use output distance functions in order to construct measures indicating by how much the reference technology changes going from period 0 to 1.[14]

Let the reference technology sets S^0 and S^1 satisfy properties P1 and P5-P7. Assume that the reference input vector x is strictly positive and that the reference output vector is nonnegative and nonzero. Then the two output distance functions $d^0(y,x)$ and $d^1(y,x)$ are well defined by (2) above and we can use these functions to define the following family of *Malmquist output based technical change measures*:

(35) $\tau(y,x,S^0,S^1) \equiv d^0(y,x)/d^1(y,x)$.

Recall that the set of outputs y that are producible by the input vector x using the period t technology set S^t was denoted by $S^t(x) \equiv \{y: (y,x)\in S^t\}$ for t = 0,1. It turns out that $\tau(y,x,S^0,S^1)$ defined by (35) is a radial measure of how much bigger (or smaller) the set $S^1(x)$ is relative to $S^0(x)$; i.e., if $\tau(y,x,S^0,S^1) > 1$, then $S^0(x)$ is a strict subset of $S^1(x)$ and if if $\tau(y,x,S^0,S^1) < 1$, then $S^1(x)$ is a strict subset of $S^0(x)$ as the proof of the following proposition will show. Thus if $\tau(y,x,S^0,S^1)$ is greater (less) than one, then we have technological progress (regress) in the best practice technology going from period 0 to 1. Note also that if we reverse the role of time, then we obtain the reciprocal of the original measure of technical change; i.e., $\tau(y,x,S^1,S^0) =1/\tau(y,x,S^0,S^1)$.

Proposition 6: Let $x >> 0_N$ and $y > 0_M$ and suppose that the reference technology sets S^0 and S^1 satisfy properties P1 and P5-P7. Suppose that $S^0(x)$ is a subset of $S^1(x)$ so that the best practice technology does not suffer from technical regress at the reference input vector x. Then $\tau(y,x,S^0,S^1) \geq 1$. Conversely, suppose that $S^1(x)$ is a subset of $S^0(x)$. Then $\tau(y,x,S^0,S^1) \leq 1$.

Proof: Let $x >> 0_N$ and $y > 0_M$ and suppose that $S^0(x) \subset S^1(x)$. Using definition (35), we have:

$$
\begin{aligned}
(36)\ \tau(y,x,S^0,S^1) &\equiv d^0(y,x)/d^1(y,x) \\
&= \min_\delta \{\delta: (y/\delta,x)\in S^0\}/\min_\delta \{\delta: (y/\delta,x)\in S^1\} \\
&= \delta^0/\delta^1
\end{aligned}
$$

where $(y/\delta^0,x)\in S^0$ and $(y/\delta^1,x)\in S^1$ and $\delta^0 > 0$, $\delta^1 > 0$. Note that $y/\delta^0 \in S^0(x)$ and since $S^0(x) \subset S^1(x)$, it can be seen that $y/\delta^0 \in S^1(x)$ and hence, δ^0 is feasible for the minimization problem $\min_\delta \{\delta: (y/\delta,x)\in S^1\} = \delta^1$. Thus $0 < \delta^1 \leq \delta^0$ and $\tau(y,x,S^0,S^1) \geq 1$. The second half of the Proposition follows in an analogous manner. Q.E.D.

[14] In the context of cross sectional comparisons of efficiency, we want to compare the best practice technology set in region 0 with the corresponding best practice set in region 1.

Unfortunately, the technical progress measure defined by (35) is not completely satisfactory as soon as there is more than one output. The problem is that in the many outputs case, we could have an outward shift in the reference production possibilities set going from period 0 to 1 that affects only a subset of the outputs and this can lead to a technical progress score of unity (indicating no technical progress) but in fact, there has been some technical progress. This is the same type of difficulty that we face when using radial measures of technical efficiency; see Russell and Schworm (2009) (2010) for a thorough discussion of these difficulties. In the present paper, we will ignore these difficult problems.

As usual, it is useful to choose particular cases of the general measures of technical progress defined by (35) that are most relevant to the production units being compared. Thus it is natural to choose as the reference output and input vectors, y and x, the observed output and input vectors for production units 0 and 1. This leads to the following *Laspeyres and Paasche type measures of technical progress*:

(37) $\tau_L \equiv \tau(y^0,x^0,S^0,S^1) \equiv d^0(y^0,x^0)/d^1(y^0,x^0)$;
(38) $\tau_P \equiv \tau(y^1,x^1,S^0,S^1) \equiv d^0(y^1,x^1)/d^1(y^1,x^1)$.

These measures can be illustrated in the one output, one input case using Figure 3 above.

We start by analyzing the Laspeyres type measure of technical progress defined by (37) above. Note that (x^0,y^0) lies below the period 0 best practice frontier. We need to hold x^0 constant and increase y^0 to y^{0*} so that the resulting input and output combination, (x^0,y^{0*}) lies on the period 0 best practice frontier. The distance $d^0(y^0,x^0) \equiv \delta^{0*}$ will deflate y^0 onto the period 0 frontier; i.e., we have $y^0/\delta^{0*} = y^{0*}$ so that $\delta^{0*} = y^0/y^{0*}$. Next we need to hold x^0 constant and increase y^0 to y^{0**} so that the resulting input and output combination, (x^0,y^{0**}) lies on the period 1 best practice frontier. The distance $d^1(y^0,x^0) \equiv \delta^{0**}$ will deflate y^0 onto the period 1 frontier; i.e., we have $y^0/\delta^{0**} = y^{0**}$ so that $\delta^{0**} = y^0/y^{0**}$. Thus we have $\tau_L = d^0(y^0,x^0)/d^1(y^0,x^0) = \delta^{0*}/\delta^{0**} = [y^0/y^{0*}]/[y^0/y^{0**}] = y^{0**}/y^{0*}$ and it can be seen that this is a perfectly sensible proportional measure of the increase in output that is producible by the best practice technology going from period 0 to 1, using x^0 as the reference amount of input.

The analysis of the Paasche type measure of technical progress defined by (38) above proceeds in a similar manner. Note that (x^1,y^1) lies below the period 1 best practice frontier. We need to hold x^1 constant and increase y^1 to y^{1*} so that the resulting input and output combination, (x^1,y^{1*}) lies on the period 1 best practice frontier. The distance $d^1(y^1,x^1) \equiv \delta^{1*}$ will deflate y^1 onto the period 1 frontier; i.e., we have $y^1/\delta^{1*} = y^{1*}$ so that $\delta^{1*} = y^1/y^{1*}$. Next we need to hold x^1 constant and increase y^1 to y^{1**} so that the resulting input and output combination, (x^1,y^{1**}) lies on the period 0 best practice frontier. The distance $d^0(y^1,x^1) \equiv \delta^{1**}$ will deflate y^1 onto the period 0 frontier; i.e., we have $y^1/\delta^{1**} = y^{1**}$ so that $\delta^{1**} = y^1/y^{1**}$. Thus we have $\tau_P = d^0(y^1,x^1)/d^1(y^1,x^1) = \delta^{1**}/\delta^{1*} = [y^1/y^{1**}]/[y^1/y^{1*}] = y^{1*}/y^{1**}$ and it can be seen that this is a reasonable proportional measure of the increase in output that is producible by the best practice technology going from period 0 to 1, using x^1 as the reference amount of input.

We would like a measure of technical change that is a symmetric average of the Laspeyres and Paasche measures, τ_L and τ_P defined above by (37) and (38). As usual, taking a geometric average often has nice properties. Thus define a *Fisher* (1922) *type measure of technical change* τ_F as the geometric average of τ_L and τ_P:

(39) $\tau_F \equiv [\tau_L\tau_P]^{1/2}$.

Note that both τ_L and τ_P satisfy a time reversal test; i.e., we have:

(40) $\tau_L = \tau(y^0,x^0,S^0,S^1) = 1/\tau(y^0,x^0,S^1,S^0)$;
(41) $\tau_P = \tau(y^1,x^1,S^0,S^1) = 1/\tau(y^1,x^1,S^1,S^0)$.

It can be seen that there is a counterpart to Proposition 3 above in the present context: the only measure of technical change that is a homogeneous symmetric average of τ_L and τ_P and also satisfies the time reversal test is the Fisher measure of technical change τ_F defined by (39).

Our final factor for the explanation of productivity change between two production units is returns to scale and we now turn to a discussion of possible measures of returns to scale.

7. Global Measures of Returns to Scale

The period 0 best practice technology will exhibit increasing returns to scale if increases in the rate of growth of inputs lead to a proportionally greater rate of growth in outputs for input-output combinations on the frontier of S^0. This concept can be illustrated in the case of one output and one input by using Figure 3. All of the Malmquist input indexes in the case of one input will be equal to x^1/x^0. For our measure of output growth, we cannot use the observed output growth ratio y^1/y^0 because the points (x^0,y^0) and (x^1,y^1) are not on the frontier of S^0. However, the points (x^0,y^{0*}) and (x^1,y^{1**}) are on the frontier of S^0 and it can be seen that our desired measure of period 0 efficient output growth (which corresponds to the input growth rate of x^1/x^0) is y^{1**}/ y^{0*}. Thus in the case of one output and one input, our *Laspeyres type measure of returns to scale*, ρ_L, is defined to be $[y^{1**}/y^{0*}]/[x^1/x^0]$. Note that y^{1**} is y^1 divided by the output distance $\delta^{1**} \equiv d^0(y^1,x^1)$ so that $(y^{1**}/\delta^{1**},x^1)$ is on the frontier of S^0. Note also that y^{0*} is y^0 divided by the output distance $\delta^{0*} \equiv d^0(y^0,x^0)$ so that $(y^{0*}/\delta^{0*},x^0)$ is also on the frontier of S^0. We will use these output distance functions to project the observed output vectors y^0 and y^1 onto the frontier of the period 0 best practice technology in the general case of many outputs and many inputs. Thus assume S^0 satisfies P1 and P5-P7, $y^0 > 0_M$, $y^1 > 0_M$, $x^0 >> 0_N$ and $x^1 >> 0_N$. Define the *projections* of y^0 and y^1 onto the efficient period 0 best practice frontier S^0, y^{0*} and y^{1**}, as follows:

(42) $y^{0*} \equiv y^0/d^0(y^0,x^0)$; $y^{1**} \equiv y^1/d^0(y^1,x^1)$.

Our *Laspeyres type measure of returns to scale*, ρ_L, is defined to be the Laspeyres type Malmquist output index, $q_L(y^{0*},y^{1**})$, defined by (24) above, divided by the Laspeyres type Malmquist input index, $Q_L(x^0,x^1)$, defined by (6) above:[15]

(43) $\rho_L \equiv [d^0(y^{1**},x^0)/d^0(y^{0*},x^0)]/[D^0(y^0,x^1)/D^0(y^0,x^0)]$
$= [d^0(y^0,x^0)/d^0(y^1,x^1)][d^0(y^1,x^0)/d^0(y^0,x^0)]/[D^0(y^0,x^1)/D^0(y^0,x^0)]$
using (42) and the linear homogeneity of $d^0(y,x^0)$ in y
$= [d^0(y^0,x^0)/d^0(y^1,x^1)]\Pi_L(x^0,x^1,y^0,y^1)$

where $\Pi_L(x^0,x^1,y^0,y^1)$ is the *Bjurek-Laspeyres productivity index* between production units 0 and 1 defined by (29) above.

Naturally, there is a companion Paasche type measure of returns to scale that is determined by the period 1 best practice technology set S^1. This companion concept can again be illustrated in the case of one output and one input by using Figure 3. Again, all of the Malmquist input indexes in the case of one input will be equal to x^1/x^0. For our measure of output growth on the frontier of the set S^1, we cannot use the observed output growth ratio y^1/y^0 because the points (x^0,y^0) and (x^1,y^1) are not on the frontier of S^1. However, the points (x^0,y^{0**}) and (x^1,y^{1*}) are on the frontier of S^1 and it can be seen that our desired measure of period 1 efficient output growth (which corresponds to the input growth rate of x^1/x^0) is $y^{1*}/\ y^{0**}$. Thus in the case of one output and one input, our *Paasche type measure of returns to scale*, ρ_P, is defined to be $[y^{1*}/y^{0**}]/[x^1/x^0]$. Note that y^{1*} is y^1 divided by the output distance $\delta^{1*} \equiv d^1(y^1,x^1)$ so that $(y^{1*}/\delta^{1*},x^1)$ is on the frontier of S^1. Note also that y^{0**} is y^0 divided by the output distance $\delta^{0**} \equiv d^1(y^0,x^0)$ so that $(y^{0**}/\delta^{0**},x^0)$ is also on the frontier of S^1. We will use these output distance functions to project the observed output vectors y^0 and y^1 onto the frontier of the period 1 best practice technology in the general case of many outputs and many inputs. Thus assume S^1 satisfies P1 and P5-P7, $y^0 > 0_M$, $y^1 > 0_M$, $x^0 >> 0_N$ and $x^1 >> 0_N$. Define the *projections* of y^0 and y^1 onto the efficient period 1 best practice frontier S^1, y^{0**} and y^{1*}, as follows:

(44) $y^{0**} \equiv y^0/d^1(y^0,x^0)\ ;\ y^{1*} \equiv y^1/d^1(y^1,x^1)$.

Our *Paasche type measure of returns to scale*, ρ_P, is defined to be the Paasche type Malmquist output index, $q_P(y^{0*},y^{1**})$, defined by (25) above, divided by the Paasche type Malmquist input index, $Q_P(x^0,x^1)$, defined by (7) above:

(45) $\rho_P \equiv [d^1(y^{1*},x^1)/d^0(y^{0**},x^1)]/[D^1(y^1,x^1)/D^1(y^1,x^0)]$
$= [d^1(y^0,x^0)/d^1(y^1,x^1)][d^1(y^1,x^1)/d^1(y^0,x^1)]/[D^1(y^1,x^1)/D^1(y^1,x^0)]$
using (44) and the linear homogeneity of $d^1(y,x^1)$ in y
$= [d^1(y^0,x^0)/d^1(y^1,x^1)]\Pi_P(x^0,x^1,y^0,y^1)$

[15] Let N = 1 and M = 1. Then using the linear homogeneity properties of $d^0(y,x^0)$ in y and the linear homogeneity properties of $D^0(y^0,x)$ in x, it can be seen from the first equation in (43) that $\rho_L = [y^{1**}/y^{0*}]/[x^1/x^0]$; see Figure 3 for the geometric interpretation of this measure of returns to scale.

where $\Pi_P(x^0,x^1,y^0,y^1)$ is the *Bjurek-Paasche productivity index* between production units 0 and 1 defined by (30) above.

Note that our measures of returns to scale have a "global" nature to them; i.e., they look at the average rate of growth of aggregate output between the two production units divided by the corresponding rate of growth of aggregate input where the output vectors are scaled to be on the efficient frontiers (on the frontier of S^0 for the Laspeyres measure and on the frontier of S^1 for the Paasche measure). These measures of returns to scale are different from the local measures of returns to scale introduced by Caves, Christensen and Diewert (1982), which relied on differentiability of the production surfaces. In our present approach, it is not necessary to make any differentiability assumptions.

If we are in the one input, one output case and the period 0 production possibilities set is a closed convex cone, then we have a best practice period 0 production function that is linear and hence, it exhibits constant returns to scale. In this case, we would like our Laspeyres measure of returns to scale, ρ_L defined by (43), to equal one. Using the definitions of d^0 and D^0 and the geometry exhibited in Figure 3, it is straightforward to verify that ρ_L equals 1. Similarly, in the one output, one input case with the frontier of the period 1 technology equal to a linear production function through the origin, then it is straightforward to verify that ρ_P equals 1.[16]

Now consider the case of many outputs and many inputs and suppose that the period 0 best practice technology S^0 is a cone[17] (in addition to satisfying the regularity conditions P1-P7). It would be ideal if our Laspeyres type measure of returns to scale turned out to equal unity in this case but this need not be the case. The problem is that the output and input aggregates are formed in a more or less independent manner and *mix effects* can cause the output growth rate to differ from the input growth rate. However, it is possible to show that if the observed outputs and inputs of the two production units are proportional to each other, then ρ_L equals 1. Similarly, if S^1 is a cone and the output and input vectors are proportional, then ρ_P equals 1.

Proposition 7: Suppose the period 0 and 1 best practice production possibility sets S^0 and S^1 satisfy the regularity conditions P1-P7 and in addition, S^0 and S^1 are cones. Let $x^0 >> 0_N$ and $y^0 > 0_M$. Suppose in addition, that the two production units being compared have *proportional output and input vectors*; i.e., there exist $\alpha > 0$ and $\beta > 0$ such that

(46) $x^1 = \alpha x^0$; $y^1 = \beta y^0$.

Then the Laspeyres and Paasche type measures of returns to scale defined by (43) and (45) are equal to one; i.e., we have $\rho_L = 1$ and $\rho_P = 1$.

Proof: Using the definition of the output distance function $d^0(y^0,x^0)$, we have the existence of a positive δ^0 such that

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[16] Proposition 7 below can be used to formally establish these results for the M = 1 and N = 1 case.

[17] S^0 is a cone if and only if $(y,x)\in S^0$ and $\lambda > 0$ implies $(\lambda y,\lambda x)\in S^0$.

(47) $d^0(y^0,x^0) \equiv \min_\delta \{\delta: (y^0/\delta,x^0)\in S^0\} = \delta^0 > 0$.

From (47), it can be seen that the largest positive multiple λ of y^0, λy^0, which is such that $(\lambda y^0,x^0)\in S^0$, is $\lambda = 1/\delta^0$. Similarly, using the definition of the output distance function $d^0(y^1,x^1)$, we have the existence of a positive δ^1 such that

(48) $d^0(y^1,x^1) \equiv \min_\delta \{\delta: (y^1/\delta,x^1)\in S^0\} = \delta^1 > 0$.

From (48), it can be seen that the largest positive multiple λ of y^1, λy^1, which is such that $(\lambda y^1,x^1)\in S^0$, is $\lambda = 1/\delta^1$.

Using the linear homogeneity property of $d^0(y,x^0)$ in y and assumptions (46), we have

(49) $d^0(y^1,x^0)/d^0(y^0,x^0) = d^0(\beta y^0,x^0)/d^0(y^0,x^0) = \beta > 0$.

Using the linear homogeneity property of $D^0(y^0,x)$ in x and assumptions (46), we have

(50) $D^0(y^0,x^1)/D^0(y^0,x^0) = D^0(y^0, \alpha x^0)/D^0(y^0,x^0) = \alpha > 0$.

Using (47)-(50) and definition (43) for ρ_L, we have:

(51) $\rho_L = [\delta^0/\delta^1][\beta/\alpha]$.

Using (48), it can be seen that $(y^1/\delta^1,x^1)\in S^0$. Thus using (46), we have $(\beta y^0/\delta^1,\alpha x^0)\in S^0$. Since S^0 is a cone and $\alpha > 0$, we must have $(\beta y^0/\alpha\delta^1,x^0)\in S^0$ as well. Thus $\beta/\alpha\delta^1$ is a feasible solution for the minimization problem in (47) and so we must have

(52) $\delta^0 \leq \beta/\alpha\delta^1$.

Using (47), it can be seen that $(y^0/\delta^0,x^0)\in S^0$. Thus using (46), $(y^1/\beta\delta^0,x^1/\alpha)\in S^0$. Since S^0 is a cone and $\alpha^{-1} > 0$, we must have $(\alpha y^1/\beta\delta^0,x^1)\in S^0$ as well. Thus $\alpha/\beta\delta^0$ is a feasible solution for the minimization problem in (48) and so

(53) $\delta^1 \leq \alpha/\beta\delta^0$.

The inequalities (52) and (53) imply that $\rho_L = 1$ using (51). The proof that shows that ρ_P equals 1 is similar to the above proof. Q.E.D.

What happens to our measures of returns to scale if we compare unit 0 to unit 1 instead of comparing unit 1 to unit 0? Denote our original Laspeyres measure of returns to scale ρ_L as $\rho_L(1/0)$ and our original Paasche measure of returns to scale ρ_P as $\rho_P(1/0)$. Now reverse the role of time and interchange the data of the two units and interchange the reference best practice technology sets S^0 and S^1. Denote the resulting Laspeyres and Paasche type measures of returns to scale by $\rho_L(0/1)$ and $\rho_P(0/1)$. It can be shown that these new measures of returns to scale are related to the old measures in the following way:[18]

(54) $\rho_L(0/1) \equiv [d^1(y^1,x^1)/d^1(y^0,x^0)][d^1(y^0,x^1)/d^1(y^1,x^1)]/[D^1(y^1,x^0)/D^1(y^1,x^1)] = 1/\rho_P(1/0)$;
(55) $\rho_P(0/1) \equiv [d^0(y^1,x^1)/d^0(y^0,x^0)][d^0(y^0,x^0)/d^0(y^1,x^0)]/[D^0(y^0,x^0)/D^0(y^0,x^1)] = 1/\rho_L(1/0)$.

Thus when we reverse the basis for comparing the two production units, the new Laspeyres type measure of returns to scale is equal to the reciprocal of the old Paasche type measure and the new Paasche type measure is equal to the reciprocal of the old Laspeyres type measure. The relations (54) and (55) suggest (as usual) that if we want a single measure of returns to scale that is a symmetric, homogeneous mean of ρ_L and ρ_P that is invariant to the way we compare the two production units, then taking the geometric mean of ρ_L and ρ_P leads to a "best" measure of returns to scale in the present context. Thus we define a *Fisher type measure of best practice returns to scale* ρ_F as the geometric average of ρ_L and ρ_P:

(56) $\rho_F \equiv [\rho_L \rho_P]^{1/2}$.

8. The Decomposition of Malmquist Productivity Indexes into Explanatory Factors

In this section, we assume that the best practice production possibilities sets S^0 and S^1 satisfy the regularity conditions P1-P7. Our goal is to compare the productivity of two production units where the observed input vector for unit t is $x^t >> 0_N$ and the observed output vector for unit t is $y^t > 0_M$ for t = 0,1.

Recall that the Bjurek-Laspeyres productivity index between units 0 and 1 was $\Pi_L(x^0,x^1,y^0,y^1)$ defined by (29) above. Equation (43) in the previous section can be manipulated to give us the following exact expression for this productivity index:

(57) $\Pi_L(x^0,x^1,y^0,y^1) = [d^0(y^1,x^1)/d^0(y^0,x^0)]\rho_L$
$= [d^1(y^1,x^1)/d^0(y^0,x^0)][d^0(y^1,x^1)/d^1(y^1,x^1)]\rho_L$
$= [\varepsilon^1/\varepsilon^0]\ \tau_P\ \rho_L$

where the unit t technical efficiency measures ε^t are defined by (34), the Paasche measure of technical progress τ_P is defined by (38) and the Laspeyres measure of returns to scale ρ_L is defined by (43). Thus we have an *exact decomposition* of the Bjurek-Laspeyres productivity measure between units 0 and 1 into the product of the relative efficiency ratio $\varepsilon^1/\varepsilon^0$ times the Paasche measure of technical change between the two best practice technologies τ_P times the Laspeyres measure of returns to scale for the period 0 best practice technology ρ_L.

[18] For the new measures, use the old definitions but interchange 0 and 1 everywhere in the old definitions.

In a similar fashion, recall that the Bjurek-Paasche productivity index between units 0 and 1 was $\Pi_P(x^0,x^1,y^0,y^1)$ defined by (30) above. Equation (45) in the previous section can be manipulated to give us the following exact expression for this productivity index:

$$(58)\ \Pi_P(x^0,x^1,y^0,y^1) = [d^1(y^1,x^1)/d^1(y^0,x^0)]\rho_P$$
$$= [d^1(y^1,x^1)/d^0(y^0,x^0)][d^0(y^0,x^0)/d^1(y^0,x^0)]\rho_P$$
$$= [\varepsilon^1/\varepsilon^0]\ \tau_L\ \rho_P$$

where the unit t technical efficiency measures ε^t are defined by (34), the Laspeyres measure of technical progress τ_L is defined by (37) and the Paasche measure of returns to scale ρ_P is defined by (45). Thus we have an *exact decomposition* of the Bjurek-Paasche productivity measure between units 0 and 1 into the product of the relative efficiency ratio $\varepsilon^1/\varepsilon^0$ times the Laspeyres measure of technical change between the two best practice technologies τ_L times the Paasche measure of returns to scale for the period 1 best practice technology ρ_P.[19]

Recall that Bjurek's recommended productivity index, $\Pi_B(x^0,x^1,y^0,y^1)$ defined by (31), was the geometric mean of the above two productivity indexes. Using (57) and (58), we have the following exact decomposition of the Bjurek productivity index:

$$(59)\ \Pi_B(x^0,x^1,y^0,y^1) \equiv [\Pi_L(x^0,x^1,y^0,y^1)\Pi_P(x^0,x^1,y^0,y^1)]^{1/2}$$
$$= [\varepsilon^1/\varepsilon^0]\ \tau_F\ \rho_F$$

where τ_F is the geometric mean of τ_L and τ_P and ρ_F is the geometric mean of ρ_L and ρ_P. The exact productivity decomposition given by (59) is our preferred decomposition of the Bjurek-Malmquist productivity index into explanatory factors.

[19] Of course, a knowledge of the best practice technology sets S^0 and S^1 is required in order to be able to implement the decompositions (57) and (58).

Appendix: **Regularity Conditions on the Reference Technology and Properties of Distance Functions**

Recall definitions (1) and (2) in the main text which defined the input and output distance functions, $D^t(y,x)$ and $d^t(y,x)$, which corresponded to the reference technology set S^t. In this Appendix, we will place restrictions on the sets S^t which are sufficient to ensure that the maximum in definition (1) and the minimum in definition (2) exist and are finite, provided that the output vector y is nonnegative and nonzero and the input vectors x is strictly positive.[20]

In order to simplify the notation, we will drop the superscript t in what follows. We assume that the production possibilities set S is given and for $y > 0_M$ and $x >> 0_N$, the *input distance function* D and the *output distance function* d are defined as follows:

(A1) $D(y,x) \equiv \max_{\delta>0} \{\delta: (y,x/\delta)\in S\}$.
(A2) $d(y,x) \equiv \min_{\delta>0} \{\delta: (y/\delta,x)\in S\}$.

Consider the following *four properties for S*:

P1. S is a nonempty closed subset of the nonnegative orthant in Euclidean M+N dimensional space.

P2. For every $y \geq 0_M$, there exists an $x \geq 0_N$ such that $(y,x)\in S$.

The interpretation of P2 is that every finite output vector y is producible by a finite input vector x.

P3. $(y,x^0)\in S$, $x^1 \geq x^0$ implies $(y,x^1)\in S$.

Thus if S satisfies P3, then there is free disposability of inputs.

P4. $y > 0_M$ implies that $(y,0_N)\notin S$.

The interpretation of P4 is that zero amounts of all inputs cannot produce a positive output; i.e., there is no free lunch in production.

Diewert and Fox (2010) showed that if $y > 0_M$ and $x >> 0_N$ and S satisfies Properties P1-P4, then the input distance function D(y,x) is well defined as the maximum in (A1) with $D(y,x) > 0$.

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[20] Our regularity conditions are a weakening of the ones used by Diewert and Fox (2010), which in turn are a variant of the conditions used by Färe and Primont (1995). For discussions on regularity conditions, see Balk (1998) (2003), Coelli, Rao and Battese (1997), Färe and Lovell (1978) and Färe, Grosskopf and Lovell (1985).

In order to show that the output distance function d(y,x) defined by (A2) is well defined as a positive minimum, we will require an additional three properties for S:

P5. $x \geq 0_N$, $(y,x) \in S$ implies $0_M \leq y \leq b(x)1_M$ where 1_M is a vector of ones of dimension M and $b(x) \geq 0$ is a finite nonnegative bound.

The interpretation of P5 is: bounded inputs imply bounded outputs.

P6. $x >> 0_N$ implies that there exists $y >> 0_M$ such that $(y,x) \in S$.

Thus the technology is such that every strictly positive input vector can produce a strictly positive vector of outputs.

P7. $(y^1,x) \in S$, $0_M \leq y^0 \leq y^1$ implies $(y^0,x) \in S$.

Thus if the input vector x can produce the output vector y^1 and y^0 is equal to or less than y^1, then x can also produce the smaller vector of outputs, y^0 (free disposability of outputs).

Diewert and Fox (2010) showed that if $y >> 0_M$ and $x >> 0_N$ and S satisfies properties P1 and P5-P7, then the output distance function d(y,x) is well defined as the minimum in (A2) with $d(y,x) > 0$.[21]

Note that the above results did not require any convexity assumptions on the technology set S.

Since the two distance functions are the basic building blocks for the Malmquist input and output indexes, it is useful to develop their mathematical properties. Variants of the following Propositions are well known in the literature but our regularity conditions are a bit weaker than the conditions used by others. [Kevin: could you check this assertion?]

Proposition 8: Suppose the production possibilities set S satisfies Properties P1-P4 listed above. Suppose the reference output vector y satisfies $y > 0_M$ and define the positive orthant in N dimensional Euclidean space by $\Omega_N \equiv \{x : x >> 0_N\}$. Then the input distance function D(y,x) defined by (A1) above is (i) well defined and positive, (ii) nondecreasing, (iii) positively linearly homogeneous, (iv) increasing if all inputs increase and (v) continuous in x over Ω_N.

Proof of (i): Follows from Proposition 2 in Diewert and Fox (2010).

Proof of (ii): Let $y > 0_M$ and $0_N << x^0 < x^1$. Using definition (A1) and property (i), we have the existence of a positive scalar δ^0 such that:

$$\text{(A3)}\quad D(y,x^0) \equiv \max_{\delta>0} \{\delta : (y,x^0/\delta) \in S\} = \delta^0 > 0$$

[21] However, their result can be strengthened: their assumption that $y >> 0_M$ can be relaxed to $y > 0_M$ as will be shown below.

where $(y,x^0/\delta^0)\in S$. Since $x^1 > x^0$ and $\delta^0 > 0$,

(A4) $x^1/\delta^0 > x^0/\delta^0$.

Since $(y,x^0/\delta^0)\in S$ and (A4) holds, Property P3 implies that

(A5) $(y,x^1/\delta^0)\in S$.

Thus

(A6) $D(y,x^1) \equiv \max_{\delta>0} \{\delta: (y,x^1/\delta)\in S\}$
$\geq \delta^0$ since by (A5) δ^0 is feasible for the maximization problem
$= D(y,x^0)$ using (A3).

Proof of (iii): Let $y > 0_M$, $x >> 0_N$ and $\lambda > 0$. Then using property (i) and definition (A1), we have the existence of a positive scalar δ^* such that

(A7) $D(y,x) \equiv \max_{\delta>0} \{\delta: (y,x/\delta)\in S\} = \delta^* > 0$.

Thus $(y,x/\delta^*)\in S$ and since $\lambda > 0$, we also have

(A8) $(y,\lambda x/\lambda\delta^*)\in S$.

Now calculate the value of the input distance function $D(y,\lambda x)$:

(A9) $D(y,\lambda x) \equiv \max_{\varepsilon>0} \{\varepsilon: (y,\lambda x/\varepsilon)\in S\}= \varepsilon^* \equiv \lambda\delta^{**} \geq \lambda\delta^* = \lambda D(y,x)$

where the inequality follows from the feasibility of $\lambda\delta^*$ for the maximization problem in (A9); see (A8). Note that we have defined $\delta^{**} \equiv \varepsilon^*/\lambda$.

Suppose the strict inequality in (A9) holds. Then δ^{**} is such that

(A10) $D(y,\lambda x) = \lambda\delta^{**} > \lambda\delta^*$.

The equality in (A10) implies that $(y,\lambda x/\lambda\delta^{**})\in S$. But then we also have

(A11) $(y,x/\delta^{**})\in S$.

From (A7), we have

(A12) $\delta^* = \max_{\delta>0} \{\delta : (y,x/\delta)\in S\}$
$\geq \delta^{**}$ since by (A11), δ^{**} is feasible for the maximization problem
$> \delta^*$ using $\lambda > 0$ and (A10).

But (A12) is a contradiction and thus our *supposition* is false and property (iii) follows.

Proof of (iv): Let $0_N << x^0 << x^1$. Then there exists a scalar $\lambda > 1$ such that

(A13) $\lambda x^0 \leq x^1$.

Since $\lambda > 1$, we have:

(A14) $D(y,x^0) < \lambda D(y,x^0)$	since $D(y,x^0) > 0$ and $\lambda > 1$
$= D(y,\lambda x^0)$	using the linear homogeneity of D
$\leq D(y,x^1)$	using (A13) and weak monotonicity.

Proof of (v): Let $x^0 >> 0_N$ and choose $\alpha > 0$ small enough so that $x^0 - \alpha x^0 = (1 - \alpha)x^0 >> 0_N$. Define the *hyperblock* in N dimensional space of size α that is centered around x^0 as follows:

(A15) $H(x^0,\alpha) \equiv \{x: (1 - \alpha)x^0 \leq x \leq (1 + \alpha)x^0\}$.

Note that $H(x^0,\alpha)$ is a subset of the positive orthant and x^0 is in the interior of $H(x^0,\alpha)$. Using the definition of $D(y,x^0)$, there exists a $\delta^0 > 0$ such that $D(y,x^0) = \delta^0$ and $(y,x^0/\delta^0)$. Using the linear homogeneity property of D(y,x) in x, we have:

(A16) $D(y,(1 - \alpha)x^0) = (1 - \alpha)D(y,x^0) = (1 - \alpha)\delta^0$;
(A17) $D(y,(1 + \alpha)x^0) = (1 + \alpha)D(y,x^0) = (1 + \alpha)\delta^0$.

Using the weak monotonicity property of D(y,x) in x and (A16) and (A17), it can be seen that for all $x \in H(x^0,\alpha)$, we have:

(A18) $(1 - \alpha)\delta^0 = D(y,(1 - \alpha)x^0) \leq D(y,x) \leq D(y,(1 + \alpha)x^0) = (1 + \alpha)\delta^0$.

The inequalities in (A18) are sufficient to imply the continuity of D(y,x) at the point x^0.
Q.E.D.

We turn to the analysis of the properties of the output distance function, d(y,x), in y for fixed $x >> 0_N$.

Proposition 9: Suppose the production possibilities set S satisfies Properties P1 and P5-P7 listed above. Suppose the reference input vector x satisfies $x >> 0_N$ and define the nonnegative orthant in M dimensional Euclidean space, excluding the origin, by $\Omega_M^* \equiv \{y : y > 0_M\}$. Then the output distance function d(y,x) defined by (A2) above is (i) well defined and positive for $y \in \Omega_M^*$, (ii) nondecreasing, (iii) positively linearly homogeneous, (iv) increasing if all outputs increase and (v) continuous in y over the interior of Ω_M^*; i.e., D(y,x) is continuous in y over the set of strictly positive y.

Proof of (i): Let $y > 0_M$ and $x >> 0_N$. Since $x >> 0_N$, by P6, there exists a $y^* >> 0_M$ such that $(y^*,x)\in S$. Since y^* is strictly positive and y is nonnegative but nonzero, there exists $\delta^* > 0$ large enough so that $y/\delta^* \leq y^*$. Using P7, we see that $(y/\delta^*,x)\in S$ and thus we have a feasible solution for the minimization problem in (A2). From definition (A2), we want to make $\delta \geq 0$ as small as possible such that $(y/\delta,x)\in S$. However, we cannot make $\delta > 0$ but arbitrarily close to 0 and have $(y/\delta,x)$ belong to S because this would contradict property P5. Using property P1, we see that a finite positive minimum for the minimization problem in (A2) exists.

Proof of (ii): Let $x >> 0_N$ and $0_M < y^0 \leq y^1$. Using definition (A2) and property (i), we have the existence of a positive scalar δ^1 such that:

(A13) $d(y^1,x) \equiv \min_{\delta>0} \{\delta: (y^1/\delta,x)\in S\} = \delta^1 > 0$

where $(y^1/\delta^1,x)\in S$. Since $y^0 \leq y^1$ and $\delta^1 > 0$,

(A14) $y^0/\delta^1 \leq y^1/\delta^1$.

Since $(y^1/\delta^1,x)\in S$ and (A14) holds, Property P7 implies that

(A15) $(y^0/\delta^1,x)\in S$.

Thus

(A16) $D(y^0,x) \equiv \min_{\delta>0} \{\delta: (y^0/\delta,x)\in S\}$
$\leq \delta^1$ since by (A15) δ^1 is feasible for the minimization problem
$= D(y^1,x)$ using (A13).

Proofs of (iii), (iv) and (v): Analogous to the proofs of (iii), (iv) and (v) in the previous Proposition. Q.E.D.

Note that we can only establish the continuity of d(y,x) in y over the *positive orthant* in M space; our regularity conditions are not strong enough to rule out discontinuities at the boundary of the positive orthant. Q.E.D.

The regularity conditions on S listed above do not include any convexity assumptions. If we are willing to make some convexity assumptions on the reference technology, then we can deduce some additional properties for the output and input distance functions. Thus we consider the following two additional regularity conditions on S:[22]

P8: For each $y > 0_M$, the input possibilities set $S(y) \equiv \{x: (y,x)\in S\}$ is a convex set.
P9: For each $x >> 0_N$, the output possibilities set $S^*(x) \equiv (y: (y,x)\in S)$ is a convex set.

[22] Note that the convexity assumptions P8 and P9 do not rule out increasing returns to scale for the reference technology S. These types of convexity assumptions are relevant if the reference technology S is generated by a DEA exercise.

Proposition 10: Suppose the technology set S satisfies properties P1-P4 and P8. Then for each $y > 0_M$, the input distance function $D(y,x)$ is a *concave function* of x over the positive orthant Ω_N.

Proof: Let $y > 0_M$. $x^1 >> 0_N$. $x^2 >> 0_N$ and $0 < \lambda < 1$. From Proposition 8, $D(y,x)$ is positive, monotonic and linearly homogeneous in x for $x \in \Omega_N$. We first show that $D(y,x)$ is a quasiconcave function of x over Ω_N. Let $D(y,x^1) = \delta^1 > 0$, $D(y,x^2) = \delta^2 > 0$ and without loss of generality, assume:

(A17) $0 < D(y,x^1) = \delta^1 \leq \delta^2 = D(y,x^2)$.

Using (A17) and the linear homogeneity property of $D(y,x)$ in x, we have:

(A18) $D(y,x^1/\delta^1) = 1$; $D(y,x^2/\delta^2) = 1$

and hence $(y,x^1/\delta^1) \in S$ and $(y,x^2/\delta^2) \in S$. Hence using property P8, we have

(A19) $(y,\lambda[x^1/\delta^1] + (1-\lambda)[x^2/\delta^2]) \in S$.

Thus using definition (A1):

(A20) $D(y,\lambda[x^1/\delta^1] + (1-\lambda)[x^2/\delta^2]) \equiv \max_{\delta>0} \{\delta : (y,(\lambda[x^1/\delta^1] + (1-\lambda)[x^2/\delta^2])/\delta) \in S\} \geq 1$

since by (A19), $\delta = 1$ is feasible for the maximization problem in (A20). Thus we have

(A21) $1 \leq D(y,\lambda[x^1/\delta^1] + (1-\lambda)[x^2/\delta^2])$
$\leq D(y,\lambda[x^1/\delta^1] + (1-\lambda)[x^2/\delta^1])$ using (A17) and property (ii) in Proposition 8
$= [\delta^1]^{-1} D(y,\lambda x^1 + (1-\lambda)x^2)$ using property (iii) in Proposition 8.

But (A21) and (A17) shows that

(A22) $D(y,\lambda x^1 + (1-\lambda)x^2) \geq \min \{D(y,x^1), D(y,x^2)\}$

which establishes the quasiconcavity of $D(y,x)$ in x over Ω_N. We now establish the concavity of $D(y,x)$ with respect to x over Ω_N. Recall the definitions and inequalities in (A17). Define α as follows:

(A23) $\alpha \equiv D(y,x^1)/D(y,x^2) = \delta^1/\delta^2 \leq 1$.

Therefore $D(y,x^1) = \alpha D(y,x^2) = D(y,\alpha x^2)$ using the linear homogeneity of $D(y,x)$ in x. Thus for all μ such that $0 \leq \mu \leq 1$, we have

(A24) $\min\{D(y,x^1), D(y,\alpha x^2)\} = D(y,x^1) \leq D(y,\mu x^1 + (1-\mu)\alpha x^2)$

where the inequality follows using the quasiconcavity of D(y,x) in x. Now look for a $\beta > 0$ and a μ between 0 and 1 such that

(A25) $\mu x^1 + (1-\mu)\alpha x^2 = \beta[\lambda x^1 + (1-\lambda)x^2]$.

The β and μ solution to (A25) is

(A26) $\beta = \alpha/[1 - \lambda + \lambda\alpha] > 0$ and $\mu = \alpha\lambda/[1 - \lambda + \alpha\lambda]$

where μ lies between 0 and 1. Using (A24), we have

(A27) $D(y,x^1) \leq D(y,\mu x^1 + (1-\mu)\alpha x^2)$
$= D(y, \beta[\lambda x^1 + (1-\lambda)x^2])$ using (A25)
$= \beta D(y, \lambda x^1 + (1-\lambda)x^2)$ using the linear homogeneity of D(y,x) in x.

(A27) can be rewritten as

(A28) $D(y, \lambda x^1 + (1-\lambda)x^2) \geq \beta^{-1} D(y,x^1)$
$= [1 - \lambda + \lambda\alpha]\alpha^{-1} D(y,x^1)$ using (A26)
$= \lambda D(y,x^1) + [1 - \lambda]\alpha^{-1} D(y,x^1)$
$= \lambda D(y,x^1) + [1 - \lambda]D(y,x^2)$ using (A23)

which establishes the concavity of D(y,x) over Ω_N. Q.E.D.

The fact that a positive, quasiconcave and linearly homogeneous function is also concave was first established by Berge (1963). The above method of proof for this result was used by Diewert (1993c; 141).

Proposition 11: Suppose the technology set S satisfies properties P1,P5-P7 and P9. Then for each $x >> 0_N$, the output distance function d(y,x) is a *convex function* of y over the nonnegative orthant less the origin Ω_M^*.

Proof: The proof is a straightforward modification of the proof of Proposition 10. Q.E.D.

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