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A Class of Distribution and Association Sensitive Multidimensional Welfare Indices

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A Class of Distribution and Association Sensitive Multidimensional Welfare Indices

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Abstract

This paper axiomatically characterizes a class of two-parameter generalized mean social welfare indices having two or more dimensions of well-being. These indices, under appropriate parametric restrictions, are sensitive to two distinct forms of interpersonal inequality. The first form of inequality is concerned with the dispersion of each dimensional achievement across the population. The second is concerned with the association or correlation across dimensions, reflecting the observation that the correlation of individual components of well-being across dimensions is relevant for the social welfare evaluation. It is shown that many existing multidimensional welfare indices are closely related to this new class.

JEL Classification: O12, D63, I31
Keywords: Social welfare measurement, multidimensional inequality, multidimensional association, equally distributed equivalent, generalized mean

1 Introduction

Measuring social welfare has always been a challenging task for economic theorists and policy analysts. It is now widely believed that economic affluence, often measured in terms of income, should not be used as the only indicator of social welfare because it completely ignores the importance of various other attributes of well-being, such as education and health. Motivated by the basic needs approach and later by the capability approach, several multidimensional indices of social welfare, poverty, and inequality have been proposed by social scientists in the past two decades. Recently, a number of governments have also been interested in evaluating social welfare and poverty from a multidimensional perspective.[1]

In this article, we are concerned with the evaluation of social welfare when there are two or more attributes of well-being. To have a common basis for comparison across different

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[1] A recent commission appointed by French President Nicolas Sarkozy recommends using a multidimensional definition of well-being (Stiglitz *et al.*, 2009, p. 14). The government of Mexico is in the process of developing a multidimensional poverty index. In 2002, the government of India started identifying families below the poverty line using a multidimensional survey. For other examples, see Alkire and Sarwar (2009).

societies, we suppose that the set of attributes is fixed. However, to allow for comparisons across societies with different set of individuals, we define our indices for all population sizes. For a society, we summarize the achievement of every individual in every attribute by an achievement matrix. A social welfare index is defined as a real-valued function on the set of possible achievement matrices. We propose a new class of multidimensional social welfare indices and characterize them axiomatically. Indices in this class are constructed in two stages. First, an overall achievement score is obtained for each individual by aggregating over the different attributes of well-being and then these scores are aggregated across individuals. In each stage of this aggregation, we use a generalized mean, which is characterized by a single parameter. Therefore, we refer to our new two-stage welfare indices as the class of *two-parameter generalized mean social welfare indices*.[2] The class includes several indices proposed in the literature, such as those of Foster *et al.* (2005) and Decancq and Ooghe (2009), as special cases. Indices in our class are particularly amenable for empirical applications because of their simple functional form. Seth (2009) has used this new class of indices to critically evaluate the Human Development Index.

A satisfactory index of social welfare should be sensitive to the inequality in the distributions of the attributes of well-being (Atkinson, 1970; Foster and Sen, 1997). Aside from its direct concern, inequality may well have negative indirect effects on social welfare. For example, high levels of inequality can lead to political instability, tensions among ethnic groups, increase in crime rates, and feelings of deprivation among the members of society.

When there are multiple attributes of well-being, there are two distinct forms of inequality. The first is concerned with the dispersion across the individual achievements of each attribute (Kolm, 1977) and the second is concerned with the correlation — or more precisely, association — across attributes (Atkinson and Bourguignon, 1982). The first form of inequality is *distribution sensitive inequality* and the second is *association sensitive inequality*. Many multidimensional indices of social welfare, inequality, or poverty, such as the Human Development Index, Human Poverty Index, and various physical quality of life indices, are insensitive to either of these forms of inequality, whereas others, such as those proposed by Hicks (1997), Foster *et al.* (2005), Gajdos and Weymark (2005), and Alkire and Foster (2008) only take account of distribution sensitive inequality. There have also been a small number of multidimensional indices proposed that take account of both kinds of inequality. See, for example, Tsui (1995, 1999, 2002), Bourguignon (1999), Bourguignon and Chakravarty (2003), Decancq and Lugo (2009), and Decancq and Ooghe (2009).

This article is most closely related to that of Foster *et al.* (2005). They also constructed a class of welfare indices by applying a two-stage aggregation procedure in which a generalized mean is used in each stage. However, they used the same generalized mean parameter in both stages. Using a single parameter is quite restrictive because it is then not possible for their indices to be association sensitive. By using two parameters, our indices can be both distribution and association sensitive.

Bourguignon (1999) has also proposed a two-parameter class of indices, albeit in the context of measuring inequality. Each of Bourguignon's indices is a monotonic transform of one of our indices. However, Bourguignon does not provide an axiomatic characterization of

[2] Since writing the first version of this article, we have learned that Kockläuner (2006) has proposed a similar class of indices for measuring poverty and has discussed some of its properties. See also Kockläuner (2008).

his class. Furthermore, as discussed below, the value of his welfare indices can respond to a change in the inequality aversion parameter in a way that may be counter-intuitive.

The rest of this article is organized as follows. In Section 2, we introduce our basic definitions and notation. In Section 3, we define and discuss the class of two-parameter generalized mean social welfare indices. In Section 4, we introduce the non-distributional axioms and use them to characterize the class defined in Section 3. We then, in Section 5, introduce our inequality aversion axioms and characterize the subclasses of our class of indices that satisfy them. We discuss related social welfare indices in Section 6. In the concluding section, we discuss possible extensions of our analysis and provide some concluding remarks.

2 Preliminaries

The set of attributes of well-being is $\mathbf{D} = \{1, \ldots, D\}$, where $D \subset \mathbb{N}$ is the number of attributes.[3] Throughout the analysis, D is assumed to be fixed with $D \geq 2$. The attributes, for example, could be income, years of education, and an index of health status. Alternatively, the attributes of well-being could be incomes in different time periods or states of nature. The former would be appropriate for studying income inequality over time, whereas the latter would be appropriate for studying income inequality under uncertainty (Ben Porath *et al.*, 1997). The set of individuals is $\mathbf{N} = \{1, \ldots, N\}$. We let the population size vary, so N can be any integer in $\mathbb{N}$.

The quantity of an attribute obtained by an individual is referred to as an *achievement.* An *achievement matrix* for a population of size N is a matrix $H \in \mathbb{R}^{ND}_{++}$, whose nd^{th} entry is the achievement h_{nd} of attribute d by person n. The n^{th} row $h_{n\cdot}$ of H is the vector listing the achievements of all D attributes by person n. The d^{th} column $h_{\cdot d}$ of H is the vector listing the achievements of all N individuals for attribute d. Let $\mathcal{H}_N$ denote the set of all possible achievement matrices of population size N and let $\mathcal{H} = \cup_{\mathbf{N} \subset \mathbb{N}} \mathcal{H}_N$ be the set of all possible achievement matrices.

A *social welfare index* is a function $W : \mathcal{H} \to \mathbb{R}$. The social welfare associated with the achievement matrix $H \in \mathcal{H}_N$ is at least as large as the social welfare associated with the achievement matrix $H' \in \mathcal{H}_{N'}$ if and only if $W(H) \geq W(H')$. The matrices H and H' could be for societies with different sets of individuals, as would be the case when making comparisons between different countries or regions. Of course, if they are the achievement matrices for a single society, then N must equal N'.

We employ the following operations on vectors and matrices. For all $M \in \mathbb{N}$ and all $x, y \in \mathbb{R}^M$, the *join* of x and y is $(x \vee y) = (\max(x_1, y_1), \ldots, \max(x_M, y_M))$ and the *meet* of x and y is $(x \wedge y) = (\min(x_1, y_1), \ldots, \min(x_M, y_M))$. For all $r, M \in \mathbb{N}$ and all $z \in \mathbb{R}^M$, the *r-replication* of z is the vector $[z]_r = (z, \ldots, z) \in \mathbb{R}^{rM}$ in which z has been replicated r times. Similarly, for all $r, L, M \in \mathbb{N}$ and all $Y \in \mathbb{R}^{LM}$, the *r-replication* of Y is the matrix $[Y]_r \in \mathbb{R}^{L'M}$ in which the rows of Y have been replicated r times, where $L' = r \cdot L$.

The following special vectors and matrices are used in the subsequent discussion. The M vector whose components are all equal to 1 is $\mathbf{1}_M$. Similarly, the $L \times M$ matrix $\mathbf{1}_{LM}$ is

[3] We use the following standard notation. The set $\mathbb{N}$ is the set of positive integers. The Euclidean k-space is $\mathbb{R}^k$ and it's non-negative and positive orthants are $\mathbb{R}^k_+$ and $\mathbb{R}^k_{++}$, respectively. It is sometime convenient to think of a $j \times k$ real-valued matrix as being a vector in $\mathbb{R}^{jk}$. The N-dimensional simplex is $S^D = \{x \in \mathbb{R}^{D+1}_+ \mid \sum_{i=1}^{D+1} x_i = 1\}$. The interior of S^D is denoted by $Int(S^D)$.

the matrix with a 1 in every entry. The M vector whose components are all equal to $1/M$ is ξ_M.

3 A Class of Indices

The class of social welfare indices that is introduced here is defined using generalized means. For vectors in $\mathbb{R}^M_{++}$, for all $\gamma \in \mathbb{R}$, and all $a \in \mathbb{R}^M_{+}$, the *generalized mean of order* γ for the weight vector $a \in S^{M-1}$ is the function $\mu^M_\gamma(\cdot; a)$ on $\mathbb{R}^M_{++}$ defined by setting, for all $x \in \mathbb{R}^M_{++}$,

$$\mu^M_\gamma(x;a) = \begin{cases} \left[\sum_{m=1}^M a_m x_m^\gamma\right]^{1/\gamma} & \text{if } \gamma \neq 0 \\ \prod_{m=1}^M x_m^{a_m} & \text{if } \gamma = 0 \end{cases}. \quad (1)$$

The parameter γ determines the curvature of the level surfaces of μ^M_γ. For $\gamma = 1$, a generalized mean is simply a weighted arithmetic mean. It is a weighted geometric mean and a weighted harmonic mean for $\gamma = 0$ and $\gamma = -1$, respectively. As $\gamma \to \infty$, $\mu^M_\gamma(x;a) \to \max_{m\in M}\{x_m\}$, and as $\gamma \to -\infty$, $\mu^M_\gamma(x;a) \to \min_{m\in M}\{x_m\}$.[4] Of particular interest are generalized means in which all attributes receive the same weight. That is, in (1), the weight vector a is equal to ξ_M. Note that a generalized mean is twice differentiable.

It is common in the literature on multidimensional social welfare and inequality to construct an overall index in two stages. This can be done by either (i) first aggregating across individuals for each attribute and then aggregating across attributes or (ii) first aggregating across attributes for each individual and then aggregating across individuals. Following Pattanaik *et al.* (2007), the former method is called *column-first two-stage aggregation* and the latter is called *row-first two-stage aggregation*. Pattanaik *et al.* (2007, Propositions 1 and 2) have shown that the column-first procedure completely ignores interactions across dimensions, which is important if the index is to be association sensitive. Thus, here, we only consider the row-first procedure. In the first stage, achievements are aggregated to obtain an individual's *overall achievement score*. For a population size of $N \in \mathbb{N}$, the overall achievement score for individual n is obtained by applying an aggregation function $Q^N_n : \mathbb{R}^D_{++} \to \mathbb{R}$ for all n in $\mathbf{N}$. Then, in the second stage, these scores are aggregated using a function $\Phi_N : \mathbb{R}^N \to \mathbb{R}$. Formally, the *row-first two-stage aggregation* method can be defined as follows.

Row-First Two-Stage Aggregation For every $\mathbf{N} \subset \mathbb{N}$ and every n in $\mathbf{N}$, there exist functions $\Phi_N : \mathbb{R}^N \to \mathbb{R}$ and $Q^N_n : \mathbb{R}^D_{++} \to \mathbb{R}$ such that for all $H \in \mathcal{H}_N$, the social welfare index W can be written as

$$W(H) = \Phi_N(Q^N_1(h_{1\cdot}), \ldots, Q^N_N(h_{N\cdot})). \quad (2)$$

The indices we propose use generalized means for each stage of the aggregation. For every choice of the parameters α and β in $\mathbb{R}$ and every weight vector a in $Int\left(S^{D-1}\right)$, the *two-parameter generalized mean social welfare index* $W(\cdot;\alpha,\beta,a)$ is defined by setting

$$W(H;\alpha,\beta,a) = \mu^N_\alpha(\mu^D_\beta(h_{1\cdot};a), \ldots, \mu^D_\beta(h_{N\cdot};a); \xi_N). \quad (3)$$

[4]We require that γ be in $\mathbb{R}$ and thereby exclude the limiting cases of $\gamma = \infty$ and $\gamma = -\infty$.

for every $\mathbf{N} \subset \mathbb{N}$ and every $H \in \mathcal{H}_N$. Note that (3) is obtained from (2) by setting $\Phi_N(\cdot) = \mu_\alpha^N(\cdot;\xi_N)$ and $Q_n^N(\cdot) = \mu_\alpha^D(\cdot;a)$ for all $\mathbf{N} \subset \mathbb{N}$ and for all n in $\mathbf{N}$. Intuitively, the index is a generalized mean of generalized means. Let $\mathcal{G}$ denote the set of all two-parameter generalized mean social welfare indices.

Following Atkinson (1970), $\mu_\alpha^N(x)$ is referred to as the *equally distributed equivalent overall achievement*, where x is the vector of overall achievements. The parameter α measures society's aversion towards inter-personal inequality in these achievements. That is, α measures the degree to which one individual's overall achievement is substitutable for a second individual's overall achievement in the social welfare index W. Similarly, the parameter β measures the degree of substitutability across the dimensions of well-being of any individual.

In defining the class of indices $\mathcal{G}$, we have not required that they be either distribution or association sensitive. As we shall show, such sensitivity can be achieved by placing restrictions on the parameters that define these indices. In the subsequent sections, under the maintained assumption that we use row-first aggregation, we shall provide an axiomatic characterization of the class of all two-parameter generalized mean social welfare indices, as well as characterizations of the sub-classes that satisfy distribution sensitivity, association sensitivity, or both of these properties together.

4 Non-Distributional Axioms

In this section, we axiomatically characterize the two-parameter class of generalized mean social welfare indices $\mathcal{G}$ given our assumption that the index is constructed using row-first aggregation, that is, assuming that the social welfare index W has the form in (2). The axioms that we employ are standard in the literature. Furthermore, none of the axioms considered in this section take into account distributional or associational concerns.

The first axiom requires the value of social welfare index to change continuously with a change in the achievement of any person in any dimension.

Continuity (CONT) For every $\mathbf{N} \subset \mathbb{N}$, W is continuous on $\mathbb{R}_{++}^{ND}$.

The next axiom imposes convenient normalizations on the aggregation function Q and the social welfare index W. If an individual has the same achievement in all dimensions, then the overall achievement is equal to this value. Moreover, if everybody has the same overall achievements, then the value of the social welfare index is equal to this common value.

Normalization (NORM) For every $\mathbf{N} \subset \mathbb{N}$, every $\zeta > 0$, and every $H \in \mathcal{H}_N$ such that $H = \zeta 1_{ND}$,

$$Q_n^N(h_{n\cdot}) = \zeta \ \forall n \in \mathbf{N} \quad \text{and} \quad W(H) = \zeta.$$

The social welfare index can be thought of as being a representation of a social preference on the set of achievement matrices. We assume that this preference is homothetic. A preference is *homothetic* if whenever two achievement matrices for the same population are socially indifferent, then so are the achievement matrices obtained by proportionally scaling both of them. By assuming that this preference is homothetic, we are implicitly assuming that we are concerned with relative inequality; that is, there is no change in inequality if an achievement matrix is proportionally scaled.[5]

Homotheticity (HOM) For every $\mathbf{N} \subset \mathbb{N}$, every $\delta > 0$, and every $H, H' \in \mathcal{H}_N$,

$$W(H') = W(H) \Leftrightarrow W(\delta H') = W(\delta H).$$

We assume that the identities of individuals are not ethically significant. This is accomplished by requiring the social welfare index to be symmetric in the sense that the index is invariant with respect to permutations of the individual achievement vectors.

Anonymity (ANON) For every $\mathbf{N} \subset \mathbb{N}$, for every $H, H' \in \mathcal{H}_N$, and for every permutation matrix $P \in \mathbb{R}_+^{NN}$ such that $H' = PH$,

$$W(H') = W(H).$$ [6]

The preceding axioms do not place any restrictions on the value of the index for achievement matrices for societies with different population sizes. We assume that if an achievement matrix is replicated an arbitrary number of times, then the value of the social welfare index is unchanged. Thus, social welfare is being measured in per capita terms.

Population Replication Invariance (POPRI) For every $r \in \mathbb{N}$ and every $H, H' \in \mathcal{H}$ such that $H' = [H]_r$,

$$W(H') = W(H).$$

We assume that each attribute of well-being contributes positively to social welfare. It is, therefore, natural to assume that the value of the social welfare function increases if the value of some attribute for some individual increases with no decrease in the value of any attribute for any individual.

Monotonicity (MON) For every $\mathbf{N} \subset \mathbb{N}$ and every $H, H' \in \mathcal{H}_N$ such that $H' \geq H$ and $H' \neq H$,

$$W(H') > W(H).$$

The restriction of the social welfare index to achievement matrices in $\mathcal{H}_N$ provides an index of social welfare for any group of size N. We assume that social welfare increases if the social welfare of a subgroup of the society increases, while that of the rest of the population is unchanged. This increase in subgroup social welfare may be accompanied by both increases and decreases in achievements of individuals in the subgroup. Our monotonicity axiom does not apply to such comparisons.

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[5] Tsui (1995) introduced a stronger version of homotheticity axiom called *ratio scale invariance*, which has also been used by Decancq and Ooghe (2009). However, this axiom has been questioned by Bourguignon (1999, p. 479). For a related discussion, see Weymark (2006, p. 311).

[6] A permutation matrix is a square matrix with each row and column having exactly one element equal to one and the rest equal to zero.

Subgroup Consistency (SUBCON) For every $N_1, N_2, N \in \mathbb{N}$ such that $N_1 + N_2 = N$, every $H_1, H_1' \in \mathcal{H}_{N_1}$, and every $H_2, H_2' \in \mathcal{H}_{N_2}$, if $W(H_1') > W(H_1)$ and $W(H_2') = W(H_2)$, then $W(H_1', H_2') > W(H_1, H_2)$.

It is common in empirical analysis for an individual's overall achievement score to be obtained by taking a weighted sum of his achievements in each dimension. These weights could measure the relative importance of the different achievements. See, for example, Decancq and Lugo (2008). Alternatively, they can be used to convert the units for each dimension into a common scale. Suppose that the set of achievements $\mathbf{D}$ is partitioned into two disjoint subsets $\mathbf{D}_1$ and $\mathbf{D}_2$. For given values of the achievements in $\mathbf{D}_2$, the aggregation function Q_n^N for person n in a row-first two-stage aggregation procedure defines a conditional ordering of achievement vectors for the attributes in $\mathbf{D}_1$. When fixed weights are used to aggregate the attributes in $\mathbf{D}$, this conditional ordering is independent of the values in $\mathbf{D}_2$. We do not assume *a prori* that fixed weights are used in this aggregation. However, we do assume that for every partition of $\mathbf{D}$ into disjoint subsets $\mathbf{D}_1$ and $\mathbf{D}_2$, the aggregation function Q_n^N defines a conditional ordering of achievement vectors for the attributes in $\mathbf{D}_1$ that is independent of the values of the attributes in $\mathbf{D}_2$. That is, Q_n^N is assumed to be *completely strictly separable.* More precisely, we assume that Q_n^N is additively separable for all n in $\mathbf{N}$.[7]

Additive Separability (ADDSEP) For every $\mathbf{N} \subset \mathbb{N}$ and every $n \in \mathbf{N}$, the aggregation function Q_n^N can be written as

$$Q_n^N(h_{n\cdot}) = U_n\left(V_1^n(h_{n1}) + \cdots + V_D^n(h_{nD})\right) \tag{4}$$

for all $h_{n\cdot} \in \mathbb{R}_{++}^D$, where $U_n : \mathbb{R} \to \mathbb{R}$ is a continuous and increasing function, and $V_d^n : \mathbb{R}_{++} \to \mathbb{R}$ is a continuous function for all d in $\mathbf{D}$.

For row-first two-stage aggregation, Theorem 1 shows that the non-distributional axioms introduced in this section characterize the set of two-parameter generalized mean social welfare indices $\mathcal{G}$.

Theorem 1 *An index $W : \mathcal{H} \to \mathbb{R}$ is a two-parameter generalized mean social welfare index if and only if W is obtained using row-first two-stage aggregation and satisfies CONT, NORM, HOM, ANON, POPRI, MON, SUBCON, and ADDSEP.*

Proof. See Appendix A. ■

5 Inequality Sensitivity Axioms

In this section, we introduce axioms that are concerned with the sensitivity of the social welfare indices to the two forms of inequality described above. First, we introduce a distribution sensitivity axiom that ensures that the social welfare index takes account of the spread

[7] Additive separability of Q_n^N is equivalent to complete strict separability if $D \geq 3$. However, for $D = 2$, additive separability is a somewhat stronger assumption than complete strict separability. See Blackorby *et al.* (1978, Section 4.4).

of the multidimensional distribution and we then characterize the subclass of the class of two-parameter generalized mean social welfare indices $\mathcal{G}$ that satisfies this axiom. Next, we introduce two alternative association sensitivity axioms and we characterize the subclasses of $\mathcal{G}$ that satisfy each of these axioms. Finally, we characterize the subclasses of $\mathcal{G}$ that satisfy both of our distribution sensitivity axiom and one of our association sensitive axioms.

Distribution Sensitive Inequality

Distributional sensitivity of the social welfare index W is obtained by requiring that the value of the index increases if an achievement matrix is subjected to a common smoothing. For every $\mathbf{N} \subset \mathbb{N} \setminus \{1\}$ and every $H', H \in \mathcal{H}_N$, H' is obtained from H by a *common smoothing* if there exists a bistochastic matrix B such that $H' = BH$ and H' is not a permutation of H.[8] Note that the same bistochastic matrix is being applied to each attribute. Formally, we require our social welfare index to satisfy the following axiom due to Kolm (1977).

Increasing under Common Smoothing (ICS) For every $\mathbf{N} \subset \mathbb{N} \setminus \{1\}$ and every $H', H \in \mathcal{H}_N$ such that H' is obtained from H by a common smoothing,

$$W(H') > W(H).$$ [9]

When there is only one dimension of well-being, H' and H are distributions of a single attribute, and the requirement that H' be obtained from H by a common smoothing is equivalent to saying that H' can be obtained from H by a sequence of Pigou-Dalton transfers, possibly supplemented by permutations of some of the distributions in this sequence.

Theorem 2 characterizes the subclass of $\mathcal{G}$ that satisfies ICS.

Theorem 2 *A two-parameter generalized mean social welfare index $W(H; \alpha, \beta, a)$ satisfies ICS if and only if $\alpha < 1$ and $\beta < 1$.*[10]

Proof. See Appendix B. ∎

In the definition of a generalized mean μ_γ^M, the parameter γ determines the curvature of the level surfaces (iso-achievement curves) of μ_γ^M. The restriction $\beta < 1$ implies that the aggregation function Q is strictly quasi-concave and thus has a strictly convex upper contour set. Consequently, the overall achievement score increases when one achievement vector is obtained from the second by a strictly convex combination of the achievements of the latter. Note that the first stage aggregation function is analogous to the constant elasticity of substitution (CES) function in the utility analysis. Similarly, if $\alpha < 1$, then the aggregation function Φ is also strictly quasi-concave in its arguments, which are the overall achievements of the individuals.

[8] A bistochastic matrix is a non-negative square matrix whose row and column sums are both equal to one.

[9] This axiom is also known as the Uniform Majorization Principle. See Kolm (1977) and Weymark (2006) for further discussion of this and related distribution sensitivity axioms.

[10] Note that the restriction on the parameters α and β in Seth (2009, p. 387) requires minor modification in order to satisfy SICS, which is equivalent to ICS in this article. The restriction should be strict, as in Theorem 2, and not $\alpha, \beta \leq 1$.

Association Sensitive Inequality

We now consider the sensitivity of the social welfare index W to a change in the association between dimensions while leaving the marginal distributions unaltered.[11] Association sensitivity was introduced into the literature on multidimensional social welfare by Atkinson and Bourguignon (1982) and has subsequently been considered by Tsui (1995, 1999, 2002), Bourguignon (1999), Bourguignon and Chakravarty (2003), and Decancq and Lugo (2009), among others. There are various ways in which the different dimensions of well-being may be interdependent, with the consequence that there are a number of different concepts of association sensitivity. See Joe (1997, Chapter 2) for a discussion.

Here, association sensitivity of W is obtained by requiring that the value of the index increases if an achievement matrix is subjected to an association increasing transfer. For every $\mathbf{N} \subset \mathbb{N} \backslash \{1\}$ and every $H, H' \in \mathcal{H}_N$, H' is obtained from H by an *association increasing transfer* if $H' \neq H$, H' is not a permutation of H, and there exist two individuals n_1 and n_2 such that $h'_{n_1\cdot} = (h_{n_1\cdot} \vee h_{n_2\cdot})$, $h'_{n_2\cdot} = (h_{n_1\cdot} \wedge h_{n_2\cdot})$, and $h'_{n\cdot} = h_{n\cdot}$ for all $n \in \mathbf{N} \backslash \{n_1, n_2\}$.[12] To interpret this definition, consider two individuals and an achievement matrix such that neither individual has at least as much of every attribute as the other. For each attribute, if we reallocate their achievements between these two individuals so that one of them has at least as much of every achievement as the other, then the resulting achievement matrix has been obtained from the former by an association increasing transfer. As emphasized by Bourguignon and Chakravarty (2003), whether an association increasing transfer is socially beneficial depends on whether the attributes are substitutes or complements in W. As a consequence, we have the following two different association sensitivity axioms, the choice of which depends on which of these two cases apply.

Decreasing under Increasing Association (DIA) For every $\mathbf{N} \subset \mathbb{N} \backslash \{1\}$ and every $H', H \in \mathcal{H}_N$ such that H' is obtained from H by a finite sequence of association increasing transfers,

$$W(H') < W(H).$$

Increasing under Increasing Association (IIA) For every $\mathbf{N} \in \mathbb{N} \backslash \{1\}$ and every $H', H \in \mathcal{H}_N$ such that H' is obtained from H by a finite sequence of association increasing transfers,

$$W(H') > W(H).$$

Theorem 3 characterizes the subclasses of $\mathcal{G}$ that satisfy these axioms.

Theorem 3 (i) *A two-parameter generalized mean social welfare index $W(H; \alpha, \beta, a)$ satisfies DIA if and only if $\alpha < \beta$.* (ii) *A two-parameter generalized mean social welfare index $W(H; \alpha, \beta, a)$ satisfies IIA if and only if $\alpha > \beta$.*

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[11] What we refer to as association between dimensions is often called dependence in the statistics literature and correlation in the literature on economic inequality. We do not employ the term 'correlation' here so as to emphasize that we are not restricting our attention to the correlation coefficient used in statistics.

[12] The concept of an association increasing transfer was introduced by Tsui (1999) under the name of a correlation increasing transfer. Tsui's concept in turn was based on the idea of a basic rearrangement due to Boland and Proschan (1988). These concepts are analogous to the correlation increasing switches considered by Bourguignon and Chakravarty (2003). For formal definitions of these concepts, see the articles cited above, and for a discussion of the relationship between them, see Chakravarty (2009) and Seth (2009).

Proof. See Appendix C. ■

After an association increasing transfer takes place, one of the two individuals affected by the transfer has at least as much of every attribute as the other affected individual. If the attributes are substitutes (resp. complements) from the perspective of social welfare, then such a transfer should decrease (resp. increase) the value of the social welfare index, which requires that α is less than (resp. larger than) β. For example, if two of the attributes are income and some indicator of health status, then it is natural to regard them as being substitutes because an individual with poor health can better deal with his condition if he has sufficient funds to help ameliorate this situation. On the other hand, if quality of health and housing infrastructure are two attributes of well-being, then good health is better enjoyed by an individual whose housing infrastructure is improved as well. In this situation, these two attributes are complements to each other.

Sensitivity to Both Forms of Inequality

By combining Theorems 2 and 3, we obtain the subclasses of $\mathcal{G}$ that are both distribution and association sensitive.

Theorem 4 (i) *A two-parameter generalized mean social welfare index $W(H;\alpha,\beta,a)$ satisfies ICS and DIA if and only if $\alpha<\beta<1$.* (ii) *A two-parameter generalized mean social welfare index $W(H;\alpha,\beta,a)$ satisfies ICS and IIA if and only if $\beta<\alpha<1$.*

To illustrate the significance of the parameter restrictions in Theorem 4, we consider the problem of a policy maker who needs to decide which person to allocate a marginal transfer T in his budget so as to maximize the increase in social welfare. For simplicity, in the following discussion we suppose that both α and β are non-zero. For any $\mathbf{N}\subset\mathbb{N}$ and any $H\in\mathcal{H}_N$, if the transfer T is provided to person n to improve her achievement in dimension d, then the increment in social welfare is:

$$\frac{\partial W(H;\alpha,\beta,a)}{\partial T}=\left(a_d h_{nd}^{\beta-1}C_n^{\alpha-\beta}\mathbf{C}\right)c_{nd},$$

where a_d is the weight of dimension d in the calculation of the overall achievement scores, $c_{nd}=\partial h_{nd}/\partial T$ is the increase in achievement h_{nd} due to the transfer, $C_n=\mu_\beta\left(h_{n\cdot};a\right)$ is the overall achievement score of person n, and $\mathbf{C}=\frac{1}{N}W(H;\alpha,\beta,a)^{1-\alpha}$. Note that $\mathbf{C}$ is identical across all individuals. Let $\omega_{nd}=a_d h_{nd}^{\beta-1}C_n^{\alpha-\beta}c_{nd}$ for all n and all d. To maximize the increase in social welfare, the policy maker should assist person n to increase her achievement in dimension d if

$$\omega_{nd}>\omega_{n'd'}\ \forall n'\in\mathbf{N}/\left\{n\right\}\ \text{ and }\forall d'\in\mathbf{D}/\left\{d\right\}. \tag{5}$$

First, to illustrate the role that the restriction $\alpha<1$ plays, we consider the situation in which $h_{nd}=\bar{h}_n$ for all d and $c_{nd}=\bar{c}$ for all d and all n. In this case, $\omega_{nd}=h_n^{\alpha-1}\bar{c}$. Consider the problem of determining which individual the budget increase should be spent on. Because $\alpha<1$, the policy maker should provide the transfer to the individual or individuals for which $\bar{h}_n$ is minimal.

Second, we consider the role that the restriction $\beta<1$ plays. This role is most clearly seen when $a_d=\bar{a}$ for all d and $c_{nd}=\bar{c}$ for all d and all n. Consider the problem of determining which attribute the budget increase should be spent on conditional on individual n being the person receiving the transfer. Because C_n does not depend of d and because $\beta < 1$, it follows from (5) that the transfer should be spent on the attribute or attributes for which h_{nd} is minimal.

Third, to illustrate how the substitutability and complementarity between attributes affects the allocation of the transfer, we again consider the situation in which $a_d = \bar{a}$ for all d and $c_{nd} = \bar{c}$ for all d and all n. We already know that if individual n receives a transfer, the transfer should be spent on the attribute or attributes for which h_{nd} is minimal. If the social welfare index is not association sensitive, then $\alpha = \beta$ and thus $\omega_{nd} = \bar{a}\bar{c}h_{nd}^{\beta-1}$. Hence, the transfer should be allocated to the individuals and attributes for which h_{nd} is minimal regardless of what anybody's overall achievement score is. If, however, the social welfare index is association sensitive, then $\alpha \neq \beta$ and thus $\omega_{nd} = \bar{a}\bar{c}h_{nd}^{\beta-1}C_n^{\alpha-\beta}$ and the transfer should be allocated to those individuals and attributes for which $h_{nd}^{\beta-1}C_n^{\alpha-\beta}$ are maximal. Suppose that $h_{nd} = h_{n'd'}$, where d (resp. d') is the attribute with minimal achievement for individual n (resp. n'). Then, the transfer should not go to individual n' if the attributes are substitutes ($\alpha < \beta$) and $C_{n'} > C_n$. Similarly, the transfer should not go to individual n' if the attributes are complements ($\alpha > \beta$) and $C_{n'} < C_n$. When the attributes are substitutes (resp. complements), then higher (resp. lower) association is detrimental to social welfare and, thus, the individual with the lower (resp. higher) overall achievement score should be favored whenever they have the same minimal achievements.

6 Related Social Welfare Indices

Foster *et al.* (2005) have proposed a one-parameter class of generalized mean social welfare indices, which we refer to as the FLS class. The FLS class is the subclass of our two-parameter generalized means $\mathcal{G}$ obtained by setting $\alpha = \beta \leq 1$. The FLS indices exhibit distribution sensitivity, but as can be seen from Theorem 4, they are not association sensitive. When $\alpha = \beta = 1$, the social welfare index is simply the arithmetic mean across individuals of weighted arithmetic means across attributes. This index is neither association nor distribution sensitive. Several well-known indices are simple means of weighted arithmetic means. For example, the Human Development Index (United Nations Development Programme, 2006) and the Morris (1979) physical quality of life indices have this functional form.

For the FLS class, both column-first two-stage aggregation and row-first two-stage aggregation yield an identical evaluation. This invariance property is called *path independence.*

Path Independence (PATHIN) For every $\mathbf{N} \subset \mathbb{N}$, there exist functions $\Phi : \mathbb{R}^N_{++} \to \mathbb{R}_{++}$ and $Q : \mathbb{R}^D_{++} \to \mathbb{R}_{++}$ such that for all $H \in \mathcal{H}_N$,

$$\Phi\left(Q\left(h_{1\cdot}\right), \ldots, Q\left(h_{N\cdot}\right)\right) = Q\left(\Phi\left(h_{\cdot 1}\right), \ldots, \Phi\left(h_{\cdot D}\right)\right).$$

Note that the class of two-parameter generalized mean social welfare indices cannot be simultaneously association sensitive and path independent. If the data for different attributes are available at different levels of aggregation, we do not have enough information to consider association among attributes. For example, education data may be available at the individual level, income data may be available at the household level, and health data may be available at the municipality level. In such circumstances, it may be appropriate to require the social welfare index to be path independent. Subclasses of $\mathcal{G}$ that satisfy PATHIN are characterized in Theorem 5.[13]

Theorem 5 (i) *A two-parameter generalized mean social welfare index* $W(H;\alpha,\beta,a)$ *satisfies PATHIN if and only if* $\alpha=\beta$. (ii) *A two-parameter generalized mean social welfare index* $W(H;\alpha,\beta,a)$ *satisfies PATHIN and ICS if and only if* $\alpha=\beta<1$.

Proof. For any $N\in\mathbf{N}$ and any $H\in\mathcal{H}_N$, define $W_1=\mu_\alpha^N\left(\mu_\beta^D(h_{1\cdot};a),\ldots,\mu_\beta^D(h_{N\cdot};a);\xi_N\right)$ and $W_2=\mu_\beta^D\left(\mu_\alpha^N(h_{\cdot 1};\xi_N),\ldots,\mu_\alpha^N(h_{\cdot D};\xi_N),a\right)$. It is straightforward to show that if $\alpha=\beta$, then $W_1=W_2$. By Hardy *et al.* (1934, Theorem 26), $W_1>W_2$ if $\beta<\alpha$ and $W_1<W_2$ if $\beta>\alpha$.[14] Hence, $W_1\neq W_2$ if $\alpha\neq\beta$. Part (ii) of the theorem follows directly by combining part (i) with Theorem 2. ■

The subclass of $\mathcal{G}$ for which $\alpha\in(0,1)$ and $\beta<1$ shares the same ordinal properties as the class of welfare indices proposed by Bourguignon (1999). For $a\in Int\left(S^{D-1}\right)$, $\alpha\in(0,1)$, and $\beta<1$, the Bourguignon social welfare index is defined as

$$W_B(H;\alpha,\beta,a)=\frac{1}{N}\sum\nolimits_{n=1}^{N}(\mu_\beta^D(h_{n\cdot};a))^\alpha=(W(H;\alpha,\beta,a))^\alpha, \tag{6}$$

for all $N\subset\mathbf{N}$ and all $H\in\mathcal{H}_N$. Thus, our index $W(H;\alpha,\beta,a)$ is a monotonic transformation of the corresponding Bourguignon index.

By using the inequality aversion parameter α to transform $W(H;\alpha,\beta,a)$ as in (6), it is unclear how to interpret a comparison of welfare levels for different values of α. To see why, consider any $N\in\mathbf{N}$ and suppose that there are two societies with achievement vectors H,H' $\in\mathcal{H}_N$ such that $h_{n\cdot}=h'_{n\cdot}=h$ for all n. In this situation, $W_B(H;\alpha,\beta,a)\neq W_B(H';\alpha',\beta,a)$ for any $\alpha\neq\alpha'$. However, it is not clear why differences in inequality aversion should result in different levels of social welfare when everybody has the same achievement vector.

Bourguignon has used his welfare index to construct an inequality index by setting $I_B(H;\alpha,\beta,a)=1-W_B(H;\alpha,\beta,a)/W_B(\bar{H};\alpha,\beta,a)$, where $\bar{H}=BH$ and $B=\mathbf{1}_{NN}/N$. It is shown in Seth (2009) that for some $\alpha>\alpha'>\alpha''$, $I_B(H;\alpha,\beta,a)<I_B(H;\alpha',\beta,a)>I_B(H;\alpha'',\beta,a)$. Thus, with this index, inequality is not monotonically increasing in the inequality aversion parameter for a given achievement matrix.

Recently, Decancq and Ooghe (2009) have proposed a class of welfare indices that are also constructed using a row-first two-stage aggregation procedure. In the first stage, they use the geometric mean μ_0^D to aggregate across attributes and in the second stage, they use a generalized mean μ_α^N with $\alpha<0$ to aggregate across individuals. This procedure implicitly assumes that attributes are substitutes and thus their indices can only satisfy IDA but not IIA. Note that the Decancq-Ooghe class is a subclass of $\mathcal{G}$.

[13] For a class of path independent standard of living indices, see Dutta *et al.* (2003).

[14] Although Hardy *et al.* (1934) assume that both α and β are positive, their proof can be easily extended for all α and β in $\mathbb{R}$.

7 Conclusion

In this article, we have proposed a class of two-parameter generalized mean social welfare indices and characterized it axiomatically. Under appropriate parametric restrictions, we have shown that these indices are both distribution and association sensitive. Because of their simple functional structure, our indices are easy to implement empirically. We have also shown that the indices proposed by Foster *et al.* (2005) and Decancq and Ooghe (2009), as well as the Human Development Index, are subclasses of our indices. We have also discussed how indices are related to the Bourguignon class of indices.

Our indices proposed here assume that the degree of substitution between each pair of attributes is the same. As a consequence, all attributes are either substitutes or complements to each other. A natural extension of our analysis would be to construct a more general class of indices that would treat some attributes as substitutes, while simultaneously treating other attributes as complements.

Following Tsui (1995), we have only considered association increasing transfers of the kind introduced by Boland and Proschan (1988). Alternative concepts of dependence among attributes could be used to construct indices based on them. Decancq (2009) has done this for positive orthant dependence.

Seth (2009) has used the indices proposed in this article to measure social welfare in Mexico using 2000 census data and has found that the ranking of Mexican states differs when association sensitivity is taken into account than when it is not. Applying our measures to other data sets is the subject of ongoing research.

References

Alkire, S. and Foster, J. E. (2008). Counting and multidimensional poverty measurement, Working Paper 7, Oxford Poverty & Human Development Initiative, University of Oxford.

— and Sarwar, M. B. (2009). Multidimensional measures of poverty & well-being, Working Paper, Oxford Poverty & Human Development Initiative, University of Oxford.

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory*, **2**, 244–263.

— and Bourguignon, F. (1982). The comparison of multi-dimensioned distributions of economic status. *Review of Economic Studies*, **49**, 183–201.

Ben Porath, E., Gilboa, I. and Schmeidler, D. (1997). On the measurement of inequality under uncertainty. *Journal of Economic Theory*, **75**, 194–204.

Blackorby, C., Primont, D. and Russell, R. R. (1978). *Duality, Separability, and Functional Structure: Theory and Economic Applications*. New York: North-Holland.

Boland, P. J. and Proschan, F. (1988). Multivariate arrangement increasing functions with applications in probability and statistics. *Journal of Multivariate Analysis*, **25**, 286–298.

Bourguignon, F. (1999). Comment on 'Multidimensioned approaches to welfare analysis' by E. Maasoumi. In J. Silber (ed.), *Handbook of Income Inequality Measurement*, Boston: Kluwer Academic, pp. 477–484.

— and Chakravarty, S. R. (2003). The measurement of multidimensional poverty. *Journal of Economic Inequality*, **1**, 25–49.

Chakravarty, S. R. (2009). *Inequality, Polarization, and Poverty.* New York: Springer.

Decancq, K. (2009). Copula-based measurement of dependence between dimensions of well-being, Center for Economic Studies, Katholieke Universiteit Leuven.

— and Lugo, M. A. (2008). Setting weights in multidimensional indices of well-being, Working Paper 18, Oxford Poverty & Human Development Initiative, University of Oxford.

— and — (2009). Measuring inequality of well-being with a correlation-sensitive multidimensional Gini index, ECINEQ Working Paper 2009-124.

— and Ooghe, E. (2009). Has the world moved forward? A robust multidimensional evaluation, Center for Economic Studies, Katholieke Universiteit Leuven.

Dutta, I., Pattanaik, P. K. and Xu, Y. (2003). On measuring deprivation and the standard of living in a multidimensional framework on the basis of aggregate data. *Economica*, **70**, 197–221.

Eichhorn, W. (1978). *Functional Equations in Economics.* Reading, MA: Addison-Wesley.

Foster, J. E., Lopez-Calva, L. F. and Székely, M. (2005). Measuring the distribution of human development: Methodology and an application to Mexico. *Journal of Human Development*, **6**, 5–29.

— and Sen, A. (1997). *On Economic Inequality.* Oxford: Oxford University Press.

— and Székely, M. (2008). Is economic growth good for the poor? Tracking low incomes using general means. *International Economic Review*, **49**, 1143–72.

Gajdos, T. and Weymark, J. A. (2005). Multidimensional generalized Gini indices. *Economic Theory*, **26**, 471–496.

Hardy, G. H., Littlewood, J. E. and Pólya, G. (1934). *Inequalities.* Cambridge: Cambridge University Press.

Hicks, D. A. (1997). The inequality adjusted human development index: A constructive proposal. *World Development*, **25**, 1283–1298.

Joe, H. (1997). *Multivariate Models and Dependence Concepts.* London: Chapman and Hall.

Kockläuner, G. (2006). A multidimensional poverty index. In K.-H. Waldmann and U. M. Stocker (eds.), *Operations Research Proceedings 2006*, Berlin: Springer, pp. 223–225.

— (2008). On generalized means of censored income gap ratios, unpublished manuscript, Fachbereiche Wirtschaft, Fachhochschule Kiel.

Kolm, S.-C. (1977). Multidimensional equalitarianisms. *Quarterly Journal of Economics*, **91**, 1–13.

Marshall, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.

Milgrom, P. and Roberts, J. (1990). Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, **58**, 1255–1277.

Morris, M. D. (1979). *Measuring the Condition of the World's Poor: The Physical Quality of Life Index*. New York: Pergamon Press.

Pattanaik, P. K., Reddy, S. G. and Xu, Y. (2007). On procedures for measuring deprivation and living standards of societies in a multi-attribute framework, Working Paper 08-02, Andrew Young School of Policy Studies, Georgia State University.

Seth, S. (2009). Inequality, interactions, and human development. *Journal of Human Development and Capabilities*, **10**, 375–396.

Stiglitz, J. E., Sen, A. and Fitoussi, J.-P. (2009). Report by the Commission on the Measurement of Economic Performance and Social Progress. Available at: http://www.stiglitz-sen-fitoussi.fr/en/index.htm.

Topkis, D. (1998). *Supermodularity and Complementarity*. Princeton: Princeton University Press.

Tsui, K.-Y. (1995). Multidimensional generalizations of the relative and absolute inequality indices: The Atkinson-Kolm-Sen approach. *Journal of Economic Theory*, **67**, 251–265.

— (1999). Multidimensional inequality and multidimensional generalized entropy measures: An axiomatic derivation. *Social Choice and Welfare*, **16**, 145–157.

— (2002). Multidimensional poverty indices. *Social Choice and Welfare*, **19**, 69–93.

United Nations Development Programme (2006). *Human Development Report*. New York: Oxford University Press.

Weymark, J. A. (2006). The normative approach to the measurement of multidimensional inequality. In F. Farina and E. Savaglio (eds.), *Inequality and Economic Integration*, London: Routledge, pp. 303–328.

Appendix

A *Proof of Theorem 1*

Proof. The sufficiency part of the proof is straightforward. To prove necessity, suppose that W is obtained using row-first two-stage aggregation, i.e., W takes the form (2), and that it satisfies CONT, NORM, HOM, ANON, POPRI, MON, SUBCON, and ADDSEP.

Consider any $\mathbf{N}$ and any $\hat{H} \in \mathcal{H}_N$ such that $\hat{h}_{nd} = \hat{x}_n$ for every $d \in \mathbf{D}$. By NORM, $Q_n^N(\hat{h}_{n\cdot}) = \hat{x}_n$ for every $n \in \mathbf{N}$ and, hence, $W(\hat{H}) = \Phi_N(\hat{x})$, where $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)$. Let $\bar{H} = P\hat{H}$ for some permutation matrix P. Reasoning as above, $W(\bar{H}) = \Phi_N(\bar{x})$, where $\bar{x}^T = P\hat{x}^T$ and $\hat{x}^T$ is the transpose of $\hat{x}$. ANON implies $W(\hat{H}) = W(\bar{H})$ and therefore $\Phi_N(\hat{x}) = \Phi_N(\bar{x})$. Thus, Φ_N is symmetric in its arguments. It follows from NORM that Φ_N is a reflexive function, i.e., $\Phi_N(\zeta, \ldots, \zeta) = \zeta$ for all $\zeta \in \mathbb{R}_{++}$. Consider any $H \in \mathcal{H}_N$ and let $\delta = W(H)$. Define $H^0 \in \mathcal{H}_N$ by setting $h^0_{nd} = \delta$ for all n in $\mathbf{N}$ and d in $\mathbf{D}$. By NORM, it follows that $W(H^0) = W(H)$. Now consider any $\lambda > 0$. Then by HOM we have $W(\lambda H^0) = W(\lambda H)$, and by NORM it follows that $\lambda\delta = W(\lambda H^0)$. We conclude that $\lambda W(H) = W(\lambda H)$ for any $\lambda > 0$ and any H in $\mathcal{H}_N$, and so W is homogeneous of degree one. Using the vector $\hat{x}$ defined above, it further follows that $\Phi_N(\lambda\hat{x}) = \lambda\Phi_N(\hat{x})$ and therefore Φ_N is also homogeneous of degree one.

Let $X_N \in \mathbb{R}^N_{++}$ denote the set of all vectors of overall achievement scores with the fixed population size N and let $X = \cup_{N \subset \mathbb{N}} X_N$. Define $\Phi : X \to \mathbb{R}$ so that $\Phi_N(x) = \Phi(x)$ for all N and all $x \in X_N$. The function Φ inherits continuity from W. Furthermore, Φ inherits the analogue of subgroup consistency from W. For any $r \in \mathbb{N}$, let $\tilde{H} = [\hat{H}]_r$, with the same $\hat{H}$ defined earlier. By POPRI, $W(\tilde{H}) = W(\hat{H})$ and therefore Φ satisfies replication invariance because $\Phi(\hat{x}) = \Phi(\tilde{x})$, where $\tilde{x}^T = [\hat{x}^T]_r$ for all $r \in \mathbb{N}$. We have shown that Φ satisfies all the assumptions of the Theorem in Foster and Székely (2008, p. 1149). Thus, there exists a scalar $\alpha \in \mathbb{R}$ such that Φ can be written as

$$\Phi(x) = \begin{cases} \left(\frac{1}{N}\sum_{n=1}^{N} x_n^{\alpha}\right)^{1/\alpha} & \text{if } \alpha \neq 0 \\ \left(\prod_{n=1}^{N} x_n\right)^{1/N} & \text{if } \alpha = 0 \end{cases} \tag{A.1}$$

for all $x \in X$, where N is the number of components in x.

We now prove that Q_n^N is also a generalized mean. First, for any $\mathbf{N}$, we show that $Q_n^N = Q_{n'}^N$ for all $n, n' \in \mathbf{N}$. Consider any $n, n' \in \mathbf{N}$ and any $\bar{h} \in \mathbb{R}^D_{++}$. Let $H \in \mathcal{H}_N$ be such that $h_{n\cdot} = \bar{h}$ and $h_{\hat{n}\cdot} = \mathbf{1}_D$ for all $\hat{n} \neq n$ and let $H' \in \mathcal{H}_N$ be such that $h'_{n'\cdot} = \bar{h}$ and $h_{\hat{n}\cdot} = \mathbf{1}_D$ for all $\hat{n} \neq n'$. Using NORM and (A.1), $W(H) = \mu_\alpha^N(1, \ldots, 1, Q_n^N(\bar{h}), 1, \ldots, 1; \xi_N)$ and $W(H') = \mu_\alpha^N(1, \ldots, 1, Q_{n'}^N(\bar{h}), 1, \ldots, 1; \xi_N)$. By ANON, $W(H) = W(H')$. Using the formula for a generalized mean of order α, it now follows that $Q_n^N(\bar{h}) = Q_{n'}^N(\bar{h})$. Hence, $Q_n^N = Q_{n'}^N$ for all $n, n' \in \mathbf{N}$. We denote this common function by Q^N.

Next, we prove that $Q^N = Q^{N'}$ for all $N, N' \in \mathbb{N}$. Consider any $H \in \mathcal{H}_1$. Note that $H = h$ for some $h \in \mathbb{R}^D_{++}$. By (A.1), $W(H) = Q^N(h)$. Consider any $N \in \mathbb{N}$ and let $\bar{H} = [h]_N$. By (A.1), $W(\bar{H}) = Q^N(h)$. POPRI implies that $W(\bar{H}) = W(H)$. Hence, $Q^N(h) = Q^1(h)$ for all $h \in \mathbb{R}^D_{++}$ and all $N \in \mathbb{N}$. Therefore, $Q^1 = Q^N$ for all $\mathbf{N} \subset \mathbb{N}$. We denote this common function by Q.

Because $W(\cdot) = Q(\cdot)$ when $N = 1$, Q inherits the properties of continuity, monotonicity, and homogeneity of degree one from W. ADDSEP implies that $Q(h) = U(\sum_{d=1}^{D} V_d(h_d))$ for all $h \in \mathbb{R}^D_{++}$, where $U : \mathbb{R} \to \mathbb{R}$ is continuous and increasing and $V_d : \mathbb{R}_{++} \to \mathbb{R}$ is continuous for all d. The monotonicity of Q implies that each V_d is also increasing. Hence, by Eichhorn (1978, Theorem 2.4.1), there exists a scaler $\beta \in \mathbb{R}$ and a weight vector $a \in Int(S^{D-1})$ such that Q can be written as:

$$Q(h) = \begin{cases} \left(\sum_{d=1}^{D} a_d h_d^{\beta}\right)^{1/\beta} & \text{if } \beta \neq 0 \\ \prod_{d=1}^{D} h_d^{a_d} & \text{if } \beta = 0 \end{cases} \tag{A.2}$$

for all $h \in \mathbb{R}^D_{++}$. In other words, the first-stage aggregation function Q is a generalized mean of order β. Therefore, W is a two-parameter generalized mean social welfare index. ■

B *Proof of Theorem 2*

The proof of Theorem 2 is based on Lemma B1.

Lemma B1 *For any $\mathbf{N} \subset \mathbb{N}\backslash\{1\}$, if H' is obtained from $H \in \mathcal{H}_N$ by a common smoothing, then (i) $\sum_{n=1}^{N} G(h'_{n\cdot}) > \sum_{n=1}^{N} G(h_{n\cdot})$ for strictly concave G and (ii) $\sum_{n=1}^{N} G(h'_{n\cdot}) < \sum_{n=1}^{N} G(h_{n\cdot})$ for strictly convex G.*

Proof. The proof is similar to the proof of Marshall and Olkin (1979, Theorem B.1., p. 433). Consider any $N \in \mathbb{N}\backslash\{1\}$ and suppose that H' is obtained from $H \in \mathcal{H}_N$ by a common smoothing. Thus, $H' = BH$ for some bistochastic matrix B. Denote row n of B by $b_{n\cdot}$. Because H' is not a permutation of H, there exist two individuals n_1 and n_2 such that $h'_{n_1\cdot} \neq h_{n_1\cdot}$ and $h'_{n_2\cdot} \neq h_{n_2\cdot}$. Let $G : \mathbb{R}^D_{++} \to \mathbb{R}$ be strictly concave. Strict concavity of G implies $G(h'_{n\cdot}) = G(\sum_{\hat{n}=1}^{N} b_{n\hat{n}} h_{\hat{n}\cdot}) > \sum_{\hat{n}=1}^{N} b_{n\hat{n}} G(h_{\hat{n}\cdot})$ for $n = n_1, n_2$. Because for all $n \in \mathbf{N}\backslash\{n_1, n_2\}$, either $h'_{n\cdot} = h_{n\cdot}$ or $h'_{n\cdot} = \sum_{\hat{n}=1}^{N} b_{n\hat{n}} h_{\hat{n}\cdot}$, it follows that $G(h'_{n\cdot}) \geq \sum_{\hat{n}=1}^{N} b_{n\hat{n}} G(h_{\hat{n}\cdot})$ for all $n \in \mathbf{N}\backslash\{n_1, n_2\}$. Hence, $\sum_{n=1}^{N} G(h'_{n\cdot}) > \sum_{n=1}^{N} \sum_{\hat{n}=1}^{N} b_{n\hat{n}} G(h_{\hat{n}\cdot}) = \sum_{\hat{n}=1}^{N} \sum_{n=1}^{N} b_{n\hat{n}} G(h_{\hat{n}\cdot}) = \sum_{\hat{n}=1}^{N} G(h_{\hat{n}\cdot}) = \sum_{n=1}^{N} G(h_{n\cdot})$. The second part of the lemma can be proved in a similar manner. ■

Proof of Theorem 2. (a) We first establish sufficiency. That is, we show that if $\alpha < 1$ and $\beta < 1$, then $W(\cdot; \alpha, \beta, a)$ defined in (3) satisfies ICS. We consider four cases.
Case 1. We first suppose that $\alpha \neq 0$ and $\beta \neq 0$. In this case,

$$W(H; \alpha, \beta, a) = \Psi\left(\sum_{n=1}^{N} G_1(h_{n\cdot})\right), \text{ where } G_1(h_{n\cdot}) = \frac{1}{N}\mu_\beta^D(h_{n\cdot}; a)^\alpha \text{ and } \Psi(x) = (x)^{1/\alpha}.$$

Let Q_1 denote the Hessian matrix of G_1. Then, for any non-zero vector $z = (z_1, \ldots, z_D) \in \mathbb{R}^D$, we have

$$zQ_1z' = \frac{\alpha X^{\alpha/\beta}}{N}\left[(\beta - 1)\sum_{d=1}^{D} \frac{X_d}{X}\left(\frac{z_d}{h_{nd}}\right)^2 + (\alpha - \beta)\left(\sum_{d=1}^{D} \frac{X_d}{X}\frac{z_d}{h_{nd}}\right)^2\right],$$

where $X_d = a_d h_{nd}^{\beta}$ and $X = \sum_{d=1}^{D} X_d$. By Jensen's inequality, $\sum_{d=1}^{D}(X_d/X)(z_d/h_d)^2 \geq (\sum_{d=1}^{D}(X_d/X)(z_d/h_d))^2$. Because $\alpha < 1$ and $\beta < 1$, we thus have $(\beta - 1)\sum_{d=1}^{D}(X_d/X)(z_d/h_{nd})^2 + (\alpha - \beta)(\sum_{d=1}^{D}(X_d/X)(z_d/h_{nd}))^2 < 0$. There are two subcases: (i) $0 < \alpha < 1$ and (ii) $\alpha < 0$.

In subcase (i), $zQ_1z' < 0$. Hence, $G_1(\cdot)$ is strictly concave. Therefore, if H' is obtained from H by common smoothing, then by part (i) of Lemma B1, we have $\sum_{n=1}^{N} G_1(h'_{n\cdot}) > \sum_{n=1}^{N} G_1(h_{n\cdot})$. Because $\Psi(\cdot)$ is increasing for $\alpha > 0$, it follows that $W(\cdot;\alpha,\beta,a)$ satisfies ICS.

In subcase (ii), $zQ_1z' > 0$. Hence, $G_1(\cdot)$ is strictly convex. Part (ii) of Lemma B1 then implies that $\sum_{n=1}^{N} G_1(h'_{n\cdot}) < \sum_{n=1}^{N} G_1(h_{n\cdot})$ if H' is obtained from H by common smoothing. Because $\Psi(\cdot)$ is decreasing for $\alpha < 0$, $W(\cdot;\alpha,\beta,a)$ satisfies ICS in this subcase as well.

Case 2. We now suppose that $\alpha \neq 0$ and $\beta = 0$. In this case,

$$W(H;\alpha,\beta,a) = \Psi\left(\sum_{n=1}^{N} G_2(h_{n\cdot})\right), \text{ where } G_2(h_{n\cdot}) = \frac{1}{N}\mu_0^D(h_{n\cdot};a)^{\alpha} \text{ and } \Psi(x) = (x)^{1/\alpha}.$$

Denote the Hessian matrix of G_2 by Q_2. Then for any non-zero vector $z \in \mathbb{R}^D$, we have

$$zQ_2z' = \frac{\alpha Y}{N}\left[-\sum_{d=1}^{D} a_d \frac{z_d^2}{h_d^2} + \alpha\left(\sum_{d=1}^{D} a_d \frac{z_d}{h_d}\right)^2\right].$$

Reasoning as in Case 1, it follows that $W(\cdot;\alpha,\beta,a)$ satisfies ICS.

Case 3. Next, we suppose that $\alpha = 0$ and $\beta \neq 0$. In this case,

$$W(H;\alpha,\beta,a) = \left(\prod_{n=1}^{N} G_3(h_{n\cdot})\right)^{1/N}, \text{ where } G_3(h_{n\cdot}) = \mu_\beta^D(h_{n\cdot};a).$$

Taking the logarithm on each side of this equation, it follows that $\ln[W(H;\alpha,\beta,a)] = \frac{1}{N}\sum_{n=1}^{N}\ln[G_3(h_{n\cdot})]$. Because G_3 is a generalized mean, both it and $\ln G_3$ are strictly concave for $\beta < 1$. By part (i) of Lemma B1, it follows that $W(\cdot;\alpha,\beta,a)$ satisfies ICS.

Case 4. Finally, we suppose that $\alpha = 0$ and $\beta = 0$. Then,

$$W(H;\alpha,\beta,a) = \left(\prod_{n=1}^{N}\prod_{d=1}^{D} h_{nd}^{a_d}\right)^{1/N}.$$

Equivalently, we have $\ln[W(H;\alpha,\beta,a)] = \frac{1}{N}\sum_{n=1}^{N}\sum_{d=1}^{D} a_d \log h_{nd}$. Hence, from part (i) of Lemma B1 it follows that $W(\cdot;\alpha,\beta,a)$ satisfies ICS.

(b) Next, we establish necessity by showing that ICS is violated when either (i) $\alpha \geq 1$ or (ii) $\beta \geq 1$.

(i) Suppose that $\alpha \geq 1$. For any $N \in \mathbb{N}$, consider any $h \in \mathbb{R}_{++}^N$ and let $H \in \mathcal{H}_N$ be such that $h_{\cdot d} = h \; \forall d$. For any $a \in Int(S^{D-1})$ and any $\beta \in \mathbb{R}$, the overall achievement score vector associated with H is h. Thus, $W(H;\alpha,\beta,a) = \mu_\alpha^N(h;\xi_N)$. Consider any bistochastic matrix B and let $H' = BH$. By construction, $h'_{\cdot d} = h' \; \forall d$. The overall achievement score vector associated with H' is h', where $h' = Bh$. Hence, $W(H;\alpha,\beta,a) = \mu_\alpha^N(h;\xi_N) \geq \mu_\alpha^N(h';\xi_N) = W(H';\alpha,\beta,a)$ because $\alpha \geq 1$, violating ICS.

(ii) Suppose that $\beta \geq 1$. For any $a \in Int(S^{D-1})$, let $H \in \mathcal{H}_2$ be such that $h_{1\cdot} \neq h_{2\cdot}$ but $\mu_\beta^D(h_{1\cdot};a) = \mu_\beta^D(h_{2\cdot};a) =: \bar{x}$. Thus, $W(H;\alpha,\beta,a) = \bar{x}$. Let $H' = \bar{B}H$, where $\bar{B} = \frac{1}{2}\mathbf{1}_{22}$. It follows that $\mu_\beta^D(h'_{1\cdot};a) = \mu_\beta^D(h'_{2\cdot};a) =: \bar{y}$ and $W(H';\alpha,\beta,a) = \bar{y}$. Because μ_β^D is strictly convex for $\beta > 1$, by part (ii) of Lemma B1, we have $\mu_\beta^D(h_{1\cdot};a) + \mu_\beta^D(h_{2\cdot};a) = 2\bar{x} > \mu_\beta^D(h'_{1\cdot};a) + \mu_\beta^D(h'_{2\cdot};a) = 2\bar{y}$. This implies that $W(H;\alpha,\beta,a) > W(H';\alpha,\beta,a)$. Furthermore, $W(H;\alpha,1,a) = W(H';\alpha,1,a)$ because, by construction, $\bar{x} = \bar{y}$ when $\beta = 1$. Hence, ICS is violated for any $\beta \geq 1$. ■

C *Proof of Theorem 3*

For the purpose of the proof, for every $\mathbf{N} \subset \mathbb{N}$ and every $H \in \mathcal{H}_N$, we express $W(H;\alpha,\beta,a)$ as

$$W(H;\alpha,\beta,a) = \mathcal{F}(F(G(h_{1\cdot}),\ldots,G(h_{N\cdot}))), \tag{C.1}$$

where $G : \mathbb{R}_{++}^D \to \mathbb{R}_{++}$, $F : \mathbb{R}_{++}^N \to \mathbb{R}_{++}$, and $\mathcal{F} : \mathbb{R}_{++} \to \mathbb{R}_{++}$. The functional forms of G, F, and $\mathcal{F}$ are conditional on α and β, as shown in Table C1.

Table C1: Functional Forms of G, F, and $\mathcal{F}$

	$\mathcal{F}(\cdot)$	$F(\cdot)$	$G(h_{n\cdot})$
$\alpha \neq 0,\ \beta \neq 0:$	$\left(\frac{1}{N}F(\cdot)\right)^{1/\alpha}$	$\sum_{n=1}^N G(\cdot)$	$(\mu_\beta^D(h_{n\cdot};a))^\alpha$
$\alpha \neq 0,\ \beta = 0:$	$\left(\frac{1}{N}F(\cdot)\right)^{1/\alpha}$	$\sum_{n=1}^N G(\cdot)$	$(\mu_0^D(h_{n\cdot};a))^\alpha$
$\alpha = 0,\ \beta \neq 0:$	$(F(\cdot))^{1/N}$	$\prod_{n=1}^N G(\cdot)$	$\mu_\beta^D(h_{n\cdot};a)$
$\alpha = 0,\ \beta = 0:$	$(F(\cdot))^{1/N}$	$\prod_{n=1}^N G(\cdot)$	$\mu_0^D(h_{n\cdot};a)$

To determine how W changes in response to a sequence of association increasing transfers, we first need to determine how F responds to such a sequence, which in turn depends on whether G is strictly L-subadditive, strictly L-superadditive, or a valuation.[15]

From Table C1, we see that F is either additive or multiplicative. Lemmas C1 and C2 summarize how F is sensitive to a sequence of association increasing transfers when F is additive or multiplicative, respectively.

Lemma C1 *For every $\mathbf{N} \subset \mathbb{N}\backslash\{1\}$, every $H', H \in \mathcal{H}_N$ such that H' is obtained from H by a finite sequence of association increasing transfers, and for $F(H) = \sum_{n=1}^N G(h_{n\cdot})$,* (i) *$F(H') < F(H)$ if and only if G is strictly L-subadditive,* (ii) *$F(H') > F(H)$ if and only if G is strictly L-superadditive, and* (iii) *$F(H) = F(H)$ if and only if G is a valuation.*

Proof. See Boland and Proschan (1988, Proposition 2.5 (a)). ■

Lemma C2 *For every $\mathbf{N} \subset \mathbb{N}\backslash\{1\}$, every $H, H' \in \mathcal{H}_N$ such that H' is obtained from H by a finite sequence of association increasing transfers, and for $F(H) = \prod_{n=1}^N G(h_{n\cdot})$,*

[15] A twice differentiable function $G : R_{++}^D \to R_+$, is (i) *strictly L-subadditive* if $\partial^2 G(h_{n\cdot})/\partial h_{nd_1}\partial h_{nd_2} < 0$ $\forall d_1 \neq d_2$; (ii) *strictly L-superadditive* if $\partial^2 G(h_{n\cdot})/\partial h_{nd_1}\partial h_{nd_2} > 0$ $\forall d_1 \neq d_2$; and (iii) *a valuation* if $\partial^2 G(h_{n\cdot})/\partial h_{nd_1}\partial h_{nd_2} = 0$ $\forall d_1 \neq d_2$. See Milgrom and Roberts (1990, p. 1261) or Topkis (1998, p. 43).

(i) $F(H') < F(H)$ *if and only if* $\ln G$ *is strictly L-subadditive,* (ii) $F(H') > F(H)$ *if and only if* $\ln G$ *is strictly L-superadditive, and* (iii) $F(H) = F(H')$ *if and only if* $\ln G$ *is a valuation.*

Proof. This result immediately follows from Lemma C1 by taking a logarithm on each side of $F(H) = \sum_{n=1}^{N} G(h_{n\cdot})$. ■

Table C2: Modularity Properties of G and $\ln G$

Strictly L-subadditive	**Strictly L-superadditive**	**Valuation**
$\alpha > 0$ and $\alpha < \beta$	$\alpha < 0$, $\alpha < \beta$, and $\beta \neq 0$	$\alpha = \beta \neq 0$
$\alpha = 0$ and $\beta > 0$	$\alpha > 0$, $\alpha > \beta$ and $\beta \neq 0$	$\alpha = \beta = 0$
$\alpha < 0$ and $\alpha > \beta$	$\alpha < 0$ and $\beta = 0$	
	$\alpha > 0$ and $\beta = 0$	
	$\alpha = 0$ and $\beta < 0$	

Table C2 summarizes the restrictions on α and β under which G, and hence $\ln G$, is strictly L-subadditive, strictly L-superadditive, or a valuation. With these preliminaries in hand, we now prove Theorem 3.

Proof of Theorem 3. For any N, let H' be obtained from $H \in \mathcal{H}_N$ by a sequence of association increasing transfers. We separately consider the cases in which $\alpha < \beta$, $\alpha > \beta$, and $\alpha = \beta$.

First, we show that if $\alpha < \beta$, then the social welfare index W satisfies DIA. There are four cases to consider: (i) $\alpha > 0$ and $\alpha < \beta$, (ii) $\alpha < 0$, $\alpha < \beta$, and $\alpha \neq \beta$, (iii) $\alpha < 0$ and $\beta = 0$, and (iv) $\alpha = 0$ and $\beta > 0$. In cases (i), (ii), and (iii), $F(\cdot) = \sum_{n=1}^{N} G(\cdot)$. In case (i), by Table C2, G is strictly L-subadditive and, hence, by Lemma C1, $F(H') < F(H)$. Because $\mathcal{F}(\cdot) = (\frac{1}{N}F(\cdot))^{1/\alpha}$ and $\alpha > 0$, it follows that $W(H'; \alpha, \beta, a) < W(H; \alpha, \beta, a)$. In case (ii), by Table C2, G is strictly L-superadditive and, hence, by Lemma C1, $F(H') > F(H)$. Because $\mathcal{F}(\cdot) = (\frac{1}{N}F(\cdot))^{1/\alpha}$ and $\alpha < 0$, it follows that $W(H'; \alpha, \beta, a) < W(H; \alpha, \beta, a)$. In case (iii), by Table C2, $\ln G$ is strictly L-superadditive and, hence, by Lemma C2, $F(H') > F(H)$. Because $\mathcal{F}(\cdot) = (\frac{1}{N}F(\cdot))^{1/\alpha}$ and $\alpha < 0$, it follows that $W(H'; \alpha, \beta, a) < W(H; \alpha, \beta, a)$. In case (iv), $F(\cdot) = \prod_{n=1}^{N} G(\cdot)$. By Table C2, G is strictly L-subadditive and, hence, by Lemma C1, $F(H') < F(H)$. Because $\mathcal{F}(\cdot) = (F(\cdot))^{1/N}$, $W(H'; \alpha, \beta, a) < W(H; \alpha, \beta, a)$. Therefore, W satisfies DIA if $\alpha < \beta$.

Next, we show that if $\alpha > \beta$, then the social welfare index W satisfies IIA. Again, there are four cases to consider: (i) $\alpha < 0$ and $\alpha > \beta$, (ii) $\alpha > 0$, $\alpha > \beta$, and $\beta \neq 0$, (iii) $\alpha > 0$ and $\beta = 0$, and (iv) $\alpha = 0$ and $\beta < 0$. In cases (i), (ii), and (iii), $F(\cdot) = \sum_{n=1}^{N} G(\cdot)$. In case (i), by Table C2, G is strictly L-subadditive and, hence, by Lemma C1, $F(H') < F(H)$. Because $\mathcal{F}(\cdot) = (\frac{1}{N}F(\cdot))^{1/\alpha}$ and $\alpha < 0$, it follows that $W(H'; \alpha, \beta, a) > W(H; \alpha, \beta, a)$. In case (ii), by Table C2, G is strictly L-superadditive and, hence, by Lemma C1, $F(H') > F(H)$. Because $\mathcal{F}(\cdot) = (\frac{1}{N}F(\cdot))^{1/\alpha}$ and $\alpha > 0$, it follows that $W(H'; \alpha, \beta, a) > W(H; \alpha, \beta, a)$. In case (iii), by Table C2, $\ln G$ is strictly L-superadditive and, hence, by Lemma C2, $F(H') > F(H)$. Because $\mathcal{F}(\cdot) = (\frac{1}{N}F(\cdot))^{1/\alpha}$ and $\alpha > 0$, it follows that $W(H'; \alpha, \beta, a) > W(H; \alpha, \beta, a)$.

In case (iv), $F(\cdot) = \prod_{n=1}^{N} G(\cdot)$. By Table C2, G is strictly L-superadditive and, hence, by Lemma C1, $F(H') > F(H)$. Because $\mathcal{F}(\cdot) = (F(\cdot))^{1/N}$, $W(H'; \alpha, \beta, a) > W(H; \alpha, \beta, a)$. Therefore, W satisfies IIA if $\alpha > \beta$.

It remains to be shown that if $\alpha = \beta$, then W satisfies neither DIA nor IIA. If $\alpha = \beta \neq 0$ (resp. $\alpha = \beta = 0$), then by Table C2, G (resp. $\ln G$) is a valuation. Thus, $F(H') = F(H)$ by Lemma C1 (resp. Lemma C2). It then follows that $W(H'; \alpha, \beta, a) = W(H; \alpha, \beta, a)$. Hence, W satisfies neither DIA nor IIA. ■