MEASURING THE SIZE OF THE WORLD ECONOMY
A Framework, Methodology and Results from the
International Comparison Program (ICP)

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Chapter 7: Methods of Aggregation above the Basic Heading Level within Regions

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1. Introduction

Chapter 6 discussed how the 155 Basic Heading (BH) price parities for each of the K countries in a region were constructed for ICP 2005. Once these Purchasing Power Parities (PPPs) have been constructed, aggregate measures of country prices and relative volumes between countries can be constructed using a wide variety of multilateral comparison methods that have been suggested over the years. These aggregate comparisons assume that in addition to BH price parities for each country, national statisticians have provided country expenditures (in their home currencies) for each of the 155 BH categories for the reference year 2005. Then the 155 by K matrices of Basic Heading price parities and country expenditures are used to form average price levels across all commodities and relative volume shares for each country.

There are a large number of methods that can be used to construct these aggregate Purchasing Power Parities and relative country volumes. Hill (2007a) (2007b) surveyed the main methods that have been used in previous rounds of the ICP as well as other methods that could be used.[1] Basically, only two multilateral methods have been used in previous rounds:

- The Gini Eltetö Köves Szulc (GEKS) method based on Fisher (1922) bilateral indexes and
- The Geary (1958) Khamis (1972) (GK) method, which is an additive method.

In the 2005 ICP round, aggregate PPPs and relative country volumes for countries within each region were constructed for five of the six regions using the Gini-EKS method. However, the African region wanted to use an additive method and so this region used a relatively new additive method, the Iklé Dikhanov Balk (IDB) method, for constructing

[1] For additional methods, see Rao (1990), Balk (1996), R.J. Hill (1997) (1999a) (1999b) (2001) (2004) and Diewert (1999).

PPPs and relative volumes within the region.[2] The purpose of this chapter is to describe the mechanics of these three methods (GEKS, GK and IDB) for making multilateral comparisons between countries in a region. These methods will be discussed in sections 2, 3 and 4 below. An extensive Appendix will discuss the properties of the IDB method in more detail, since this method is relatively unknown. This Appendix can be omitted by the casual reader.

A brief comment on the relative merits of the GEKS, GK and IDB methods is warranted. The GK and IDB methods are *additive methods*; i.e., the real output of each country can be expressed as a *sum* of the country's individual Basic Heading outputs where each output component is weighted by an *international price* which is constant across countries. This feature of an additive method is tremendously convenient for users and so for many purposes, it is useful to have available a set of additive international comparisons. However, additive methods are not consistent with the economic approach to index number theory. Section 5 will explain the economic approach and why additive methods are not fully consistent with this approach.

The GEKS multilateral method is fully consistent with the economic approach to making multilateral comparisons. The GEKS approach also has the property that each country in the comparison is treated in a fully symmetric manner; i.e., the method is a democratic one. This aspect of GEKS can be considered as an advantage of the method. However, from a technical point of view, there are some disadvantages to the method in that countries that are at very different stages of development and which face very different relative prices are given the same weight in the method as countries which are at a very similar stage of development and face the same structure of relative prices. Bilateral comparisons between similar in structure countries are likely to be much more accurate than comparisons between countries which are very dissimilar. Thus in Section 6, an economic approach is introduced that builds up a complete multilateral set of comparisons that rests on making bilateral comparisons between very similar in structure countries. This method is called the *spatial linking method*. It is possible that variants of this method may be used in the next ICP round.

Section 7 concludes.

2. The GEKS Method

The GEKS method is due to Gini (1924) (1931) and it was independently rediscovered by Eltetö and Köves (1964) and Szulc (1964).

[2] Iklé (1972; 203) proposed the equations for the method in a rather difficult to interpret manner and provided a proof for the existence of a solution for the case of two countries. Dikhanov (1994; 6-9) used the much more transparent equations (13) and (14) below, explained the advantages of the method over the GK method and illustrated the method with an extensive set of computations. Balk (1996; 207-208) used the Dikhanov equations and provided a proof of the existence of a solution to the sytem for an arbitrary number of countries. Van Ijzeren (1983; 42) also used Iklé's equations and provided an existence proof for the case of two countries.

In order to explain the method, it will be useful to introduce some notation at this point. Let N equal 155 and let K be the number of countries in the regional comparison for the reference year. Denote the Basic Heading PPP for commodity category n and for for country k in the region by $p_n^k > 0$ and the corresponding expenditure (in local currency units) on commodity class n by country k in the reference year by e_n^k for n = 1,...,N and k = 1,...,K. Given this information, we can define *volumes* [3] or *implicit quantity levels* q_n^k for each Basic Heading category n and for each country k as the category expenditure deflated by the corresponding commodity PPP for that country:

$$(1)\ q_n^k \equiv e_n^k/p_n^k\ ; \qquad n = 1,...,N\ ;\ k = 1,...,K.$$

It will be useful to define *country commodity expenditure shares* s_n^k for BH class n and country k as follows:

$$(2)\ s_n^k \equiv e_n^k/\textstyle\sum_{i=1}^{N} e_i^k\ ; \qquad n = 1,...,N\ ;\ k = 1,...,K.$$

Now define *country vectors of BH prices* as $p^k \equiv [p_1^k,...,p_N^k]$, *country vectors of BH volumes* as $q^k \equiv [q_1^k,...,q_N^k]$, *country expenditure vectors* as $e^k \equiv [e_1^k,...,e_N^k]$ and *country expenditure share vectors* as $s^k \equiv [s_1^k,...,s_N^k]$ for k = 1,...,K.

In order to define the GEKS parities $P^1,P^2,...,P^K$, we first need to define the *Fisher* (1922) *ideal bilateral price index* P_F between country j relative to k:[4]

$$(3)\ P_F(p^k,p^j,q^k,q^j) \equiv [p^j\cdot q^j\ p^j\cdot q^k/p^k\cdot q^j\ p^k\cdot q^k]^{1/2}\ ; \qquad j = 1,...,K\ ;\ k = 1,...,K.$$

Various justifications for the use of the Fisher ideal index in the bilateral context have been made by Diewert (1976) (1992) (2002; 569). The Fisher index can be justified from the point of view of finding the "best" symmetric average of the Laspeyres and Paasche indexes, or from the point of view of the axiomatic or test approach to index number theory, or from the viewpoint of the economic approach to index number theory; see Chapters 15, 16 and 17 in the *Consumer Price Index Manual*, ILO/IMF/OECD/UNECE/Eurostat/World Bank (2004).

The *aggregate PPP for country* j, P^j, is defined as follows:

$$(4)\ P^j \equiv \textstyle\prod_{k=1}^{K} [P_F(p^k,p^j,q^k,q^j)]^{1/K}\ ; \qquad j = 1,...,K.$$

Once the GEKS P^j's have been defined by (4), the corresponding GEKS *country real outputs or volumes* Q^j can be defined as the country expenditures $p^j\cdot q^j$ in the reference year divided by the corresponding GEKS purchasing power parity P^j:

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[3] National income accountants distinguish between a "quantity" and a "volume". A *volume* is a quantity index aggregate of a group of actual quantities. Since country expenditures in each of the Basic Heading categories are aggregates over many commodities and hence the corresponding BH PPP's (the p_n^k) are actually price indexes, it can be seen that it is appropriate to refer to the implicit quantity aggregates, the q_n^k defined by (1), as volumes.

[4] Notation: $p\cdot q \equiv \sum_{n=1}^{N} p_n q_n$ denotes the inner product between the vectors p and q.

(5) $Q^j \equiv p^j \cdot q^j / P^j$; $\quad j = 1,...,K.$

If all of the P^j defined by (4) are divided by a positive number, α say, then all of the Q^j defined by (5) can be multiplied by this same α without materially changing the GEKS multilateral method. If country 1 is chosen as the numeraire country in the region, then set α equal to P^1 defined by (4) for j = 1 and the resulting price level P^j is interpreted as the number of units of country j's currency it takes to purchase 1 unit of country 1's currency and get an equivalent amount of utility. The rescaled Q^j is interpreted as the volume of output of country j in the currency units of country 1.

It is also possible to normalize the real outputs of each country in common units (the Q^k) by dividing each Q^k by the sum $\sum_{j=1}^K Q^j$ in order to express each country's real output as a fraction or share of total regional output; i.e., define the country k's *share of regional output*, S^k, as follows:[5]

(6) $S^k \equiv Q^k / \sum_{j=1}^K Q^j$; $\quad k = 1,...,K.$

Of course, the country shares of regional real output, the S^k, remain unchanged after rescaling the PPPs by the scalar α.

This completes a brief description of the GEKS method for making multilateral comparisons.[6]

3. The Geary Khamis Method

The method was suggested by Geary (1958) and Khamis (1972) showed that the equations that define the method have a positive solution under certain conditions.

The GK system of equations involves K *country price levels* or PPPs, $P^1,...,P^K$, and N *international commodity reference prices*, $\pi_1,...,\pi_N$. The equations which determine these unknowns (up to a scalar multiple) are the following ones:

(7) $\pi_n = \sum_{k=1}^K [q_n^k / \sum_{j=1}^K q_n^j][p_n^k / P^k]$; $\quad n = 1,...,N$;
(8) $P^k = p^k \cdot q^k / \pi \cdot q^k$; $\quad k = 1,...,K$

where $\pi \equiv [\pi_1,...,\pi_N]$ is the vector of GK regional average reference prices. It can be seen that if a solution to equations (7) and (8) exists, then if all of the country parities P^k are multiplied by a positive scalar λ say and all of the reference prices π_n are divided by the

[5] There are several additional ways of expressing the GEKS PPP's and relative volumes; see Balk (1996), Diewert (1999; 34-37) and section 5 below.

[6] It should be noted that all of the multilateral methods that are described in this section can be applied to subaggregates of the 155 basic heading categories; i.e., instead of working out aggregate price and volume comparisons across all 155 commodity classifications, one could just choose to include the food categories in the list of N categories and use the multilateral method to compare aggregate food consumption across the countries in the region.

same λ, then another solution to (7) and (8) is obtained. Hence, the π_n and P^k are only determined up to a scalar multiple and an additional normalization is required such as

(9) $P^1 = 1$

in order to uniquely determine the parities. It can also be shown that only N + K − 1 of the N equations in (7) and (8) are independent. Once the parities P^k have been determined, the real output or volume for country k, Q^k, can be defined as country k's *nominal value of output in domestic currency units*, $p^k \cdot q^k$, divided by its PPP, P^k:

(10) $Q^k = p^k \cdot q^k / P^k$; k = 1,...,K
$= \pi \cdot q^k$ using (8).

Finally, if equations (10) are substituted into the regional share equations (6), then country k's share of regional output is

(11) $S^k = \pi \cdot q^k / \pi \cdot q$ k = 1,...,K

where the *region's total volume vector* q is defined as the sum of the country volume vectors:

(12) $q \equiv \sum_{j=1}^{K} q^j$.

Equations (10) show how convenient it is to have an additive multilateral comparison method: when country outputs are valued at the international reference prices, values are additive across both countries and commodities. However, additive multilateral methods are not consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two; see section 5 below. In addition, looking at equations (7), it can be seen that large countries will have a larger contribution to the determination of the international prices π_n and thus these international prices will be much more representative for the largest countries in the comparison as compared to the smaller ones.[7] This leads to the next method for making multilateral comparisons: an additive method that does not suffer from this problem of big countries having an undue influence in the comparison.

4. The Iklé Dikhanov Balk Method

Iklé (1972; 202-204) suggested this method in a very indirect way, Dikhanov (1994) (1997) suggested the much clearer system (13)-(14) below and Balk (1996; 207-208) provided the first existence proof. Dikhanov's (1994; 9-12) equations that are the counterparts to the GK equations (7) and (8) are the following ones:

(13) $\pi_n = [\sum_{k=1}^{K} s_n^k [p_n^k/P^k]^{-1} / \sum_{j=1}^{K} s_n^j]^{-1}$; n = 1,...,N
(14) $P^k = [\sum_{n=1}^{N} s_n^k [p_n^k/\pi_n]^{-1}]^{-1}$ k = 1,...,K

[7] Hill (1997) and Dikhanov (1994; 5) made this point.

where the country expenditure shares s_n^k are defined by (2) above.

As in the GK method, equations (13) and (14) involve the K country price levels or PPPs, $P^1,...,P^K$, and N international commodity reference prices, $\pi_1,...,\pi_N$. Equations (13) indicate that the *nth international price*, π_n, is a *share weighted harmonic mean of the country k prices for commodity n*, p_n^k, deflated by country k's PPP, P^k. The country k share weights for commodity n, s_n^k, do not sum (over countries k) to unity but when s_n^k is divided by $\sum_{j=1}^K s_n^j$, the resulting normalized shares do sum (over countries k) to unity. Thus equations (13) are similar to the GK equations (7), except that now a harmonic mean of the deflated commodity n prices, p_n^k/P^k, is used in place of the old arithmetic mean and in the GK equations, country k's share of commodity n in the region, $q_n^k/\sum_{j=1}^K q_n^j$, was used as a weighting factor (and hence large countries had a large influence in forming these weights) but now the weights involve country expenditure shares and so each country in the region has an equal influence in forming the weighted average. Equations (14) indicate that P^k, the *PPP for country* k, P^k, is equal to a *weighted harmonic mean of the country k commodity prices*, p_n^k, deflated by the international price for commodity n, π_n, where the summation is over commodities n instead of over countries k as in equations (13). The share weights in the harmonic means defined by (14), the s_n^k, of course sum to one when the summation is over n, so there is no need to normalize these weights as was the case for equations (13).

It can be seen that if a solution to equations (13) and (14) exists, then multiplication of all of the country parities P^k by a positive scalar λ and division all of the reference prices π_n by the same λ will lead to another solution to (13) and (14). Hence, the π_n and P^k are only determined up to a scalar multiple and an additional normalization is required such as (9), $P^1 = 1$.

Although the IDB equations (14) do not appear to be related very closely to the corresponding GK equations (8), it can be shown that these two sets of equation are actually the same system. To see this, note that the country k expenditure share for commodity n, s_n^k, has the following representation:

(15) $s_n^k = p_n^k q_n^k / p^k \cdot q^k$; $n = 1,...,N$; $k = 1,...,K$.

Now substitute equations (15) into equations (14) to obtain the following equations:

$$(16)\ P^k = 1/\sum_{n=1}^N s_n^k [p_n^k/\pi_n]^{-1} \qquad k = 1,...,K$$
$$= 1/\sum_{n=1}^N [p_n^k q_n^k/p^k\cdot q^k][\pi_n/p_n^k]$$
$$= p^k\cdot q^k/\sum_{n=1}^N \pi_n q_n^k$$
$$= p^k\cdot q^k/\pi\cdot q^k.$$

Thus equations (14) are equivalent to equations (8) and the IDB system is an additive system; i.e., equations (10)-(12) can be applied to the present method just as they were applied to the GK method for making international comparisons.

In the Appendix, several different ways of representing the IDB system of parities will be obtained and fairly weak conditions for the existence and uniqueness of the IDB parities will be obtained. Effective methods for obtaining solutions to the system of equations (13) and (14) (with a normalization) will also be presented.

As was mentioned in the introduction to this chapter, the IDB method was used by the African region in order to construct regional aggregates. Basically, this method appears to be an improvement over the GK method in that large countries no longer have a dominant influence on the determination of the international reference prices π_n and so if an additive method is required with more democratic reference prices, IDB appears to be "better" than GK. In addition, Deaton and Heston (2010) have shown empirically that the IDB method generates aggregate PPPs that are much closer to the GEKS PPPs than are GK PPPs, using ICP 2005 data. However, in the following section, it will be shown that if one takes the economic approach to index number comparisons, then any additive multilateral method will be subject to some substitution bias.

5. Additive Multilateral Methods and the Economic Approach to Making Index Number Comparisons

It is useful to begin this section by reviewing what are the essential assumptions for the *economic approach to index number theory*:

- Purchasers have preferences over alternative bundles of goods and services that they purchase.
- As a result, they buy more of things that have gone down in relative price and less of things which have gone up in relative price.

The above type of substitution behaviour is well documented and hence, it is useful to attempt to take it into account when doing international comparisons.

The economic approach to index number theory does take substitution behavior into account. This approach was developed by Diewert (1976) in the bilateral context[8] and by Diewert (1999) in the multilateral context. Basically this theory works as follows:

- Assume that all purchasers have the same preferences over commodities and that these preferences can be represented by a homogeneous utility function.
- Find a functional form that can approximate preferences to the second order[9] and has an exact index number formula associated with it. The resulting index number formula is called a *superlative index number formula*.[10]
- Use the superlative index number formula in a bilateral context so that the real output of every country in the region can be compared to the real output of a numeraire country using this formula. The resulting relative volumes are dependent on the choice of the numeraire country.

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[8] The pioneers in this approach were Konüs and Byushgens (1926).
[9] Diewert (1974;113) termed such functional forms *flexible*.
[10] Diewert (1976; 117) introduced this concept and terminology.

- Take the geometric average of all K sets of relative volumes using each country in the region as the numeraire country. This set of average relative volumes can then be converted into regional shares as in section 2 above. The resulting method is called a *superlative multilateral method*.[11]

It turns out that the GEKS method discussed in section 2 above is a superlative multilateral method; see Diewert (1999; 36). Diewert also showed that the GEKS method has quite good axiomatic properties.

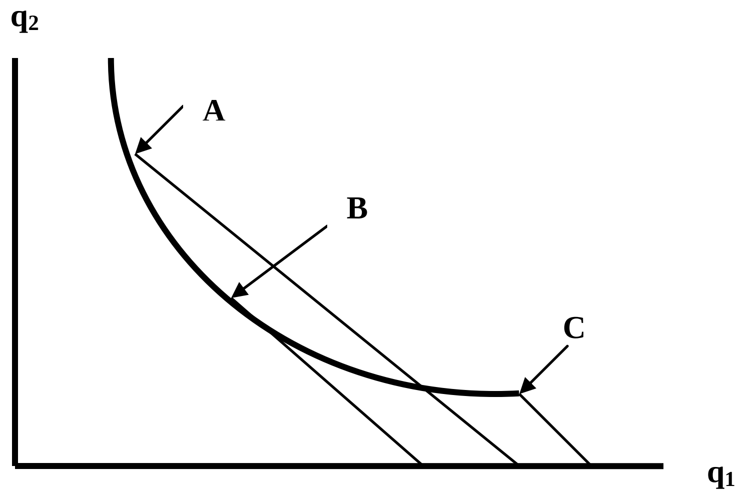
Given the importance of the GEKS multilateral method, it is worth explaining that the GEKS volume parities can be obtained by alternative methods.

The first alternative method is explained by Deaton and Heston (2010). In this method, the GEKS parities can be obtained by using a least squares minimization problem, due originally to Gini (1924), that will essentially make an K by K matrix of bilateral Fisher volume parities that are not transitive into a best fitting set of transitive parities. The second well known method for deriving the GEKS parities was implicitly explained above. Pick any country as the base country and use the Fisher bilateral quantity index to form the real output of every country relative to the chosen base country. This gives estimated volumes for all countries in the comparison relative to the chosen base country. Now repeat this process, choosing each country in turn as the base country, which leads to K sets of relative volume estimates. The final step for obtaining the GEKS relative volumes is to take the geometric mean of all of the K base country dependent sets of parities.

The problem with an additive multilateral method (from the perspective of the economic approach) if the number of countries in the region is greater than two can now be explained with the help of a diagram.[12]

Figure 1

[11] See Diewert (1999; 22).
[12] This diagram is basically due to Marris (1984; 52) and Diewert (1999; 48-50).



The solid curved line in the above Figure represents an indifference curve for purchasers of the two goods under consideration. The consumption vectors of Countries A, B and C are all on the same indifference curve and hence, the multilateral method should show the same volume for the three countries. If we use the relative prices that country B faces as "world" reference prices in an additive method, then country B has the lowest volume or real consumption, followed by country A and finally, C has the highest volume. But they all have equal volumes! It can be seen that we can devise an additive method that will make the volumes of any two countries equal but we cannot devise an additive method that will equalize the volumes for all three countries. On the other hand, the common indifference curve in Figure 1 can be approximated reasonably well by a flexible functional form that has a corresponding exact index number formula (such as the Fisher index) and thus a GEKS method that used the Fisher bilateral index as a basic building block would give the right answer to a reasonable degree of approximation. The bottom line is that an additive multilateral method is not really consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two.[13]

Although additive multilateral methods have their problems in that they are not consistent with substitution in the face of changing relative prices, the economic approach as

[13] "Figure 1.1 also illustrates the Gerschenkron effect: in the consumer theory context, countries whose price vectors are far from the 'international' or world average prices used in an additive method will have quantity shares that are biased upward. ... It can be seen that these biases are simply quantity index counterparts to the usual substitution biases encountered in the theory of the consumer price index. However, the biases will usually be much larger in the multilateral context than in the intertemporal context since relative prices and quantities will be much more variable in the former context. ... The bottom line on the discussion presented above is that the quest for an additive multilateral method with good economic properties (i.e., a lack of substitution bias) is a doomed venture: nonlinear preferences and production functions cannot be adequately approximated by linear functions. Put another way, if technology and preferences were always linear, there would be no index number problem and hundreds of papers and monographs on the subject would be superfluous!" W. Erwin Diewert (1999; 50).

explained above is not without its problems. Two important criticisms of the economic approach are:

- The assumption that all final purchasers have the same preferences over different baskets of final demand purchases is suspect and
- The assumption that preferences are homothetic (i.e., can be represented by a linearly homogeneous utility function) is also suspect.

The second criticism of the economic approach to multilateral comparisons based on superlative bilateral index number formulae has been discussed in the recent literature on international comparisons and some brief comments on this literature are in order here.

An important recent development is Neary's (2004) GAIA multilateral system, which can be described as a consumer theory consistent version of the GK system, which allows for nonhomothetic preferences on the part of final demanders. Deaton and Heston (2010) point out that a weakness of the Neary multilateral system is that it uses a single set of relative prices to value consumption or GDP in all countries, no matter how different are the actual relative prices in each country. This problem was also noticed by Feenstra, Ma and Rao (2009) and these authors generalized Neary's framework to work with two sets of cross sectional data in order to estimate preferences[14] and they also experimented with alternative sets of reference prices. Barnett, Diewert and Zellner (2009) in their discussion of Feenstra, Mao and Rao, noted that a natural generalization of their model would be to use a set of reference prices which would be representative for each country in the comparison. Using representative prices for each country would lead to K sets of relative volumes and in the end, these country specific parities could be averaged just as the GEKS method averages country specific parities. Barnett, Diewert and Zellner conjectured that this geometric average of the country estimates will probably be close to GEKS estimates based on traditional multilateral index number theory, which of course, does not use econometrics. It remains to be seen if econometric approaches to the multilateral index number problem can be reconciled with superlative multilateral methods.[15]

In the following section, another economic approach to constructing multilateral comparisons will be described: the spatial linking method.

6. The Spatial Linking or Minimum Spanning Tree Method for Making Multilateral Comparisons

Recall that the Fisher ideal quantity index can be used to construct real outputs for all K countries in the comparison, using one country as the base country. Thus as each country is used as the base country, K sets of relative outputs can be obtained. The GEKS

[14] Methods that rely on the econometric estimation of preferences across countries are probably not suitable for the ICP, since it becomes very difficult to estimate flexible preferences for 155 commodity categories.

[15] One limitation of econometric approaches is that it will be impossible to estimate flexible functional forms for preferences when the number of commodity groups is as large as 155 since approximately 12,000 parameters would have to be estimated in this case.

multilateral method treats each country's set of relative outputs as being equally valid and hence an averaging of the parities is appropriate under this hypothesis. Thus the method is "democratic" in that each bilateral index number comparison between any two countries gets the same weight in the overall method. However, it is not the case that all bilateral comparisons of volume between two countries are equally accurate: if the relative prices in countries A and B are very similar, then the Laspeyres and Paasche quantity indexes will be very close to each other and hence it is likely that the "true" volume comparison between these two countries (using the economic approach to index number theory) will be very close to the Fisher volume comparison. On the other hand, if the structure of relative prices in the two countries is very different, then it is likely that the structure of relative quantities in the two countries will also be different and hence the Laspeyres and Paasche quantity indexes will likely differ considerably and it is no longer so certain that the Fisher quantity index will be close to the "true" volume comparison. The above considerations suggest that a more accurate set of world product shares could be constructed if initially a bilateral comparison was made between the two countries which had the most *similar relative price structures*. At the next stage of the comparison, look for a third country which had the most similar relative price structure to the first two countries and link in this third country to the comparisons of volume between the first two countries and so on. At the end of this procedure, a *minimum spanning tree* would be constructed, which is a path between all countries that minimized the sum of the relative price similarity measures. This linking methodology has been developed by Robert Hill (1999a) (1999b) (2004) (2009). The conclusion is that spatial linking using Fisher ideal quantity indexes as the bilateral links is an alternative to GEKS which has some advantages over it.[16] Both methods are consistent with the economic approach to index number theory.

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two countries. Various measures of the similarity or dissimilarity of relative price structures have been proposed by Aten and Heston (2009), Diewert (2009), Hill (1997) (2009) and Sergeev (2001) (2009). The focus here will be on one of Diewert's (2009; 207) measures of relative price similarity, the *weighted log quadratic measure of relative price dissimilarity*, $\Delta_{PLQ}(p^1,p^2,q^1,q^2)$, (the smaller the measure, the more similar is the structure of relative prices between the two countries):

(17) $\Delta_{PLQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2)[\ln(p_n^2/p_n^1 P_F(p^1,p^2,q^1,q^2))]^2$

[16] Deaton (2010; 33-34) noticed the following problem with the GEKS method: suppose we have two countries where the expenditure share on commodity 1 is tiny for country A and very big for country B. Suppose also that the price of commodity 1 in country A is very large relative to the price in country B. Then looking at the Törnqvist price index between A and B, it can be seen that the overall price level for country A will be blown up by the relatively high price for good 1 in A relative to B and by the big expenditure share in B on commodity 1. Since the Törnqvist will generally approximate the corresponding Fisher index closely, it can be seen that we have ended up exaggerating the price level of country A relative to B. This problem can be mitigated by spatial linking of countries that have similar price and quantity structures.

where $P_F(p^1,p^2,q^1,q^2) \equiv [p^2\cdot q^1\ p^2\cdot q^2/p^1\cdot q^1\ p^1\cdot q^2]^{1/2}$ is the Fisher ideal price index between countries 2 and 1 and $s_n^c \equiv p_n^c q_n^c/p^c\cdot q^c$ is the country c expenditure share on commodity n for c = 1,2 and n = 1,...,N.

It can be seen that if prices are proportional for the two countries so that $p^2 = \lambda p^1$ for some positive scalar λ, then $P_F(p^1,p^2,q^1,q^2) = \lambda$ and the measure of relative price dissimilarity $\Delta_{PLQ}(p^1,p^2,q^1,q^2)$ defined by (17) will equal its minimum of 0. Thus the smaller is $\Delta_{PLQ}(p^1,p^2,q^1,q^2)$, the more similar is the structure of relative prices in the two countries.

The method of spatial linking using the relative price dissimilarity measure defined by (17) will be illustrated in section A.7 of the Appendix.[17] Basically, instead of using the GEKS country shares defined by (6) in section 2, the shares generated by the minimum spanning tree are used to link all of the countries in the comparison.

The narrowing of Paasche and Laspeyres spreads by the use of a spatial linking method is not the only advantage of this method of linking countries. There are advantages at *lower levels of aggregation* in that if similar in structure countries are compared, generally, it will be found that product overlaps are maximized:

"Many differences in quality and proportion of high tech items discussed above are likely to be more pronounced between countries with very different economic structures. If criteria can be developed to identify countries with similar economic structure and they are compared only with each other, then it may overcome many of the issues of quality and lowest common denominator item comparisons. Economically similar countries are likely to have outlet types in similar proportions carrying the same types of goods and services. So direct comparisons between such countries will do a better job of holding constant the quality of the items than comparisons across more diverse countries." Bettina Aten and Alan Heston (2009; 251).

The above quotation suggests that perhaps the similarity criterion should not be based only on the similarity of the structure of relative prices across the two countries being compared. In addition, one could look at the degree of similarity in the structure of absolute per capita quantity vectors and take a sum of the two measures of similarity as our overall measure of similarity in structure.

There are some disadvantages to the spatial linking method. The two most important disadvantages are:

- The path of bilateral links between countries generated by the method tends to be unstable; i.e., the most similar tree linking the countries tends to change when we move from one cross sectional comparison between countries to another cross sectional comparison.
- Some countries in the comparison will inevitably have lower quality data than other countries and if these poorer data quality countries end up having many bilateral links with many countries in the minimum spanning tree, then the quality of the entire comparison may be low.

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[17] Some additional examples will be presented in Chapter 8.

Hill (2009) discusses both of these problems and offers "reasonable" solutions to these difficulties. The first difficulty is not really a difficulty if the overall volume comparisons remain more or less the same even if the particular bilateral links change. In particular, it may be the case that countries break up into two or more relatively homogeneous groups. Within each group, the bilateral dissimilarity measures are all low so even if the links within each group change, the relative volume indexes within each group remain roughly unchanged. The key problem then boils down to the bilateral links between the various groupings. In order to get more stability between these groupings, it may be advisable to have more than one link between the groupings and this constraint can readily be imposed. The second difficulty can be dealt with by specifying that countries with lower quality data should not be allowed to have more than one link in the overall tree of comparisons.

Of course, a problem with the above "solutions" to the problems associated with spatial linking is that the solutions appear to have an ad hoc character and this may lead to charges by outside observers that the ICP is being manipulated. This potential problem could be mitigated by experimentation with the ICP 2005 data set so a firm a priori strategy could be put in place before the results for ICP 2011 were calculated.

7. Conclusion

This chapter discussed four multilateral methods for constructing PPPs and relative volumes for countries in a region.

Two of the methods were additive methods: the Geary Khamis (GK) method and the Iklé Dikhanov Balk (IDB) method. Additive methods are preferred by many users due to the fact that components of real GDP add up across countries and across commodities when an additive multilateral method is used. The IDB method has better axiomatic properties than the GK method; in particular, the IDB international prices are not unduly influenced by the structure of relative prices in the biggest countries in the region; i.e., the IDB method is more "democratic" than the GK method in its choice of international prices.

The other two methods discussed in this chapter were the Gini Eltetö Köves Szulc (GEKS) method and the method of spatial linking (due initially to Fisher (1922) but more recently developed by Robert Hill) using Fisher ideal indexes as basic bilateral building blocks. These two methods can be regarded as being consistent with an economic approach to a multilateral method; i.e., these methods deal adequately with substitution behavior on the part of purchasers of a country's outputs. The spatial linking method was not used in ICP 2005 but it has some attractive features that may lead to its adoption in ICP 2011.[18]

Appendix: The Properties of the Iklé Dikhanov Balk Multilateral System

A.1 Introduction and Overview

[18] The method also has some drawbacks which may prevent its use in 2011.

Unfortunately, multilateral index number theory is much more complicated than bilateral index number theory. Thus a rather long appendix is required in order to investigate the axiomatic and economic properties of the IDB multilateral system, particularly when some prices and quantities are allowed to be zero.[19] A brief overview of this appendix follows.

There are many equivalent ways of expressing the equations that define the IDB parities. Section A.2 lists these alternative systems of equations that can be used to define the method. Section A.3 provides proofs of the existence and uniqueness of solutions to the IDB equations. Section A.4 considers various special cases of the IDB equations. When there are only two countries so that K = 2, a bilateral index number formula is obtained and this case is considered along with the case where N = 2, so that there are only 2 commodities. These special cases cast some light on the structure of the general indexes. Section A.5 explores the axiomatic properties of the IDB method while section A.6 looks at the system's economic properties. Finally, section A.7 concludes by calculating a numerical example.

Throughout this appendix, it is assumed that the number of countries K and the number of commodities N is equal to or greater than two.

A.2 Alternative Representations

A.2.1 The P^k, π_n Representation

The basic data for the multilateral system are the prices and quantities for commodity n in country k at the Basic Heading level, p_n^k and q_n^k respectively, for $n = 1,...,N$ and $k = 1,...,K$ where the number of basic heading categories $N \geq 2$ and the number of countries $K \geq 2$. The N by 1 vectors of prices and quantities for country k are denoted by p^k and q^k and their inner product is $p^k \cdot q^k$ for $k = 1,...,K$. The share of country k expenditure on commodity n is denoted by $s_n^k \equiv p_n^k q_n^k / p^k \cdot q^k$ for $k = 1,...,K$ and $n = 1,...,N$.

It is assumed that for each n and k, either p_n^k, q_n^k and s_n^k are all zero or p_n^k, q_n^k and s_n^k are all positive. Thus the possibility that some countries do not consume all of the basic heading commodities is allowed for. This complicates the representations of the equations since division by zero prices, quantities or shares leads to difficulties and complicates proofs of existence.[20] For now, the following assumptions are made:

(A1) For every basic heading commodity n, there exists a country k such that p_n^k, q_n^k and s_n^k are all positive so that each commodity is demanded by some country.

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[19] Balk (1996; 207-208) has the most extensive published discussion of the properties of the IDB system but he considered only the case of positive prices and quantities for all commodities across all countries and he did not discuss the economic properties of the method.

[20] Balk's (1996; 208) existence proof assumed that all prices and quantities were strictly positive.

(A2) For every country k, there exists a commodity n such that p_n^k, q_n^k and s_n^k are all positive so that each country demands at least one basic heading commodity.

In section A.2, the above assumptions will be strengthened in order to ensure that the IDB equations have unique, positive solutions.

Recall that the IDB multilateral system was defined by the Dikhanov equations (13) and (14) (plus one normalization such as (9)). Taking into account the division by zero problem, these equations can be rewritten as follows:[21]

(A3) $\pi_n = [\sum_{j=1}^K s_n^j]/[\sum_{k=1}^K (q_n^k P^k/p^k \cdot q^k)]$; n = 1,...,N
(A4) $P^k = p^k \cdot q^k/\pi \cdot q^k$; k = 1,...,K

where π is a vector whose components are $\pi_1,...,\pi_N$.

Using assumptions (A1) and (A2), it can be seen that equations (A3) and (A4) will be well behaved even if some p_n^k and q_n^k are zero. Equations (A3) and (A4) (plus a normalization on the P^k or π_n such as $P^1 = 1$ or $\pi_1 = 1$) provide the second representation of the IDB multilateral equations.[22]

In order to find a solution to equations (A3) and (A4), one can start by assuming that $\pi = 1_N$, a vector of ones and then use equations (A4) to determine a set of P^k. These P^k can then be inserted into equations (A3) in order to determine a new π vector. Then this new π vector can be inserted into equations (A4) in order to determine a new set of P^k. And so on; the process can be continued until convergence is achieved.

A.2.2 An Alternative P^k, π_n Representation using Biproportional Matrices

It can be seen that equations (A3) and (A4) can be rewritten in the following manner:

(A5) $\sum_{k=1}^K q_n^k [p^k \cdot q^k]^{-1} \pi_n P^k = \sum_{j=1}^K s_n^j$; n = 1,...,N;
(A6) $\sum_{n=1}^N q_n^k [p^k \cdot q^k]^{-1} \pi_n P^k = \sum_{n=1}^N s_n^j = 1$; k = 1,...,K.

Define the N by K *normalized quantity matrix* A which has element a_{nk} in row n and column k where

(A7) $a_{nk} \equiv q_n^k/p^k \cdot q^k$; n = 1,...,N ; k = 1,...,K.

Define the N by K *expenditure share matrix* S which has the country k expenditure share for commodity n, s_n^k in row n and column k. Let 1_N and 1_K be vectors of ones of

[21] Equations (A3) are equivalent to Balk's (1996; 207) equations (38a) in the case where all price p_n^k are positive and equations (A4) are Balk's equations (38b).
[22] Equations (13) and (14) provide a first representation in the case where all prices and quantities are positive.

dimension N and K respectively. Then equations (A5) and (A6) can be written in matrix form as follows:[23]

(A8) $\hat{\pi}$ AP = $S1_K$;
(A9) $\pi^T A \hat{P} = 1_N{}^T S$

where $\pi \equiv [\pi_1,...,\pi_N]$ is the vector of IDB international prices, $P \equiv [P^1,...,P^K]$ is the vector of DI country PPPs, $\hat{\pi}$ denotes an N by N diagonal matrix with the elements of the vector π along the main diagonal and $\hat{P}$ denotes an K by K diagonal matrix with the elements of the vector P along the main diagonal. There are N equations in (A8) and K equations in (A9). However, examining (A8) and (A9), it is evident that if N+K−1 of these equations are satisfied, then the remaining equation is also satisfied. Equations (A8) and (A9) are a special case of the *biproportional matrix fitting model* due to Deming and Stephan (1940) in the statistics context and to Stone (1962) in the economics context (the RAS method). Bacharach (1970; 45) studied this model in great detail and gave rigorous conditions for the existence of a unique positive π, P solution set to (A8), (A9) and a normalization such as $P^1 = 1$ or $\pi_1 = 1$.[24] In section A.2 below, Bacharach's analysis will be used in order to provide simple sufficient conditions for the existence and uniqueness of a solution to equations (A8) and (A9) (plus a normalization).

In order to find a solution to (A8) and (A9), one can use the procedure suggested at the end of section A.2.1, since equations (A3) and (A4) are equivalent to (A5) and (A6).[25] Experience with the RAS method has shown that this procedure tends to converge quite rapidly.

A.2.3 The Q^k, π_n Representation

The above representations of the IDB system are in terms of a system of equations involving the N international reference prices π_n and the K country PPPs, P^k. It is useful to substitute equations (10) in the main text, $Q^k = p^k \cdot q^k / P^k$, which define the country volumes or aggregate quantities Q^k in terms of the country k price and quantity vectors p^k and q^k and the country k aggregate PPP, P^k, into equations (A3) and (A4) in order to obtain the following representation of the IDB multilateral system in terms of the Q^k and the π_n:

(A10) $\pi_n = [\sum_{j=1}^K s_n{}^j]/[\sum_{k=1}^K (q_n{}^k / Q^k)]$; n = 1,...,N

[23] Notation: when examining matrix equations, vectors such as π and P are to be regarded as column vectors and π^T and P^T denote their row vector transposes.
[24] It is obvious that if the positive vectors π and P satisfy (A8) and (A9), then $\lambda\pi$ and $\lambda^{-1}P$ also satisfy these equations where λ is any positive scalar. Dikhanov (1997; 12-13) also derived conditions for the existence and uniqueness of the solution set using a different approach.
[25] Bacharach (1970; 46) calls this method of solution the *biproportional process*. Bacharach (1970; 46-59) establishes conditions for the existence and uniqueness of a solution to the biproportional process; i.e., for the convergence of the process. The normalization (say $P^1 = 1$ or $\pi_1 = 1$) can be imposed at each iteration of the biproportional process or it can be imposed at the end of the process when convergence has been achieved.

(A11) $Q^k = \pi \cdot q^k$; $k = 1,...,K.$

Of course, a normalization such as $Q^1 = 1$ or $\pi_1 = 1$ needs to be added in order to obtain a unique positive solution to (A10) and (A11).[26] Obviously, a biproportional iteration process could be set up to find a solution to equations (A10) and (A11) along the lines suggested at the end of section A.2.1, except that now the Q^k are determined rather than the P^k.

A.2.4 The Q^k Representation

If equations (A10) are substituted into equations (A11), the following K equations are obtained, involving only the country volumes, $Q^1,...,Q^K$:

(A12) $Q^k = \sum_{n=1}^N \{[s_n^1 +...+ s_n^K]\, q_n^k / [(q_n^1/Q^1) +...+ (q_n^K/Q^K)]\}$; $k = 1,...,K.$

A normalization on the Q^k is required in order to obtain a unique solution, such as $Q^1 = 1$. It also can be seen that the K equations (A12) are not independent; i.e., if both sides of equation k in (A12) are divided by Q^k for each k and then the resulting equations are summed, the identity K equals K is obtained, using the fact that $\sum_{n=1}^N s_n^k = 1$ for each k. Thus once any K−1 of the K equations in (A12) are satisfied, the remaining equation is also satisfied.

Equations (A12) can be used in an iterative fashion in order to obtain a $Q^1,...,Q^K$ solution; i.e., make an initial guess at these volume parities and calculate the right hand side of each equation in (A12). This will generate a new set of volume parities, which can then be normalized to satisfy say $\sum_{k=1}^K Q^k$ equals 1. Then these new volume parities can again be inserted into the right hand sides of equations (A12) and so on.[27]

A.2.5 The P^k Representation

If the equations $Q^k = p^k \cdot q^k / P^k$ are substituted into equations (A12), the following K equations involving only the country PPPs, $P^1,...,P^K$, are obtained:

(A13) $(P^k)^{-1} = \sum_{n=1}^N \{[s_n^1 +...+ s_n^K][q_n^k/p^k \cdot q^k]/[(P^1 q_n^1/p^1 \cdot q^1) +...+ (P^K q_n^K/p^K \cdot q^K)]\}$;
$k = 1,...,K.$

As usual, a normalization on the P^k is needed in order to obtain a unique solution, such as $P^1 = 1$. It also can be seen that the K equations (A13) are not independent; i.e., if both sides of equation k in (A13) are multiplied by P^k for each k and then the resulting equations are summed, the identity K equals K is obtained, using the fact that $\sum_{n=1}^N s_n^k =$

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[26] It can be verified that if N+K−1 of the equations (A10) and (A11) are satisfied, then the remaining equation is also satisfied; equations (A12) may be used to establish this result.

[27] When this method was tried on the data for the numerical example in Diewert (1999; 79) (see section A.7 of this Appendix), it was found that convergence was very slow. The iterative methods described in section A.2.1 converged much more quickly.

1 for each k. Thus once any K−1 of the K equations in (A13) are satisfied, the remaining equation is also satisfied.

Equations (A13) can be used iteratively in order to find a solution in a manner similar to the method described at the end of section A.2.4.

Equations (A12) and (A13) are difficult to interpret at this level of generality but when the axiomatic properties of the method are studied, it will be seen that the IDB parities have good axiomatic properties.

A.2.6 The π_n Representation

Finally, substitute equations (A4) into equations (A3) in order to obtain the following system of N equations which characterize the IDB international prices π_n:

$$\text{(A14)} \quad \sum_{k=1}^{K} [\pi_n q_n^{k}/\pi\cdot q^{k}] = \sum_{k=1}^{K} s_n^{k} ; \qquad n = 1,...,N.$$

It can be seen that equations (A14) are homogeneous of degree 0 in the components of the π vector and so a normalization such as $\pi_1 = 1$ is required in order to obtain a unique positive solution. It also can be seen that if the N equations in (A14) are summed, the identity K equals K is obtained and so if any N−1 of the N equations in (A14) are satisfied, then so is the remaining equation.

Equations (A14) can be rewritten as follows:

$$\text{(A15)} \quad \pi_n = [\sum_{k=1}^{K} s_n^{k}]/[\sum_{k=1}^{K} q_n^{k}/\pi\cdot q^{k}] ; \qquad n = 1,...,N.$$

Equations (A15) can be used iteratively in the usual manner in order to obtain a solution to equations (A14).

Equations (A14) have an interesting interpretation. Using the international reference prices π_n, define *country k's expenditure share for commodity n using these international prices* as:

$$\text{(A16)} \quad \sigma_n^{k} \equiv \pi_n q_n^{k}/\pi\cdot q^{k} ; \qquad k = 1,...,K ; n = 1,...,N.$$

Substituting (A16) into (A14) leads to the following system of equations:

$$\text{(A17)} \quad \sum_{k=1}^{K} \sigma_n^{k} = \sum_{k=1}^{K} s_n^{k} ; \qquad n = 1,...,N.$$

Thus for each basic heading commodity group n, the international prices π_n are chosen by the IDB method to be such that the sum over countries expenditure shares for commodity n using these international reference prices, $\sum_{k=1}^{K} \sigma_n^{k}$, is equal to the corresponding sum over countries expenditure shares using domestic prices in each country, $\sum_{k=1}^{K} s_n^{\,k}$, and this equality holds for all commodity groups n.[28]

A.3 Conditions for the Existence and Uniqueness of Solutions to the IDB Equations

In order to find conditions for positive solutions to any set of the IDB equations, the biproportional matrix representation that was explained in section A.2.2 above will be used.[29]

Bacharach (1970; 43-59) provided very weak sufficient conditions for the existence of a strictly positive solution $\pi_1,...,\pi_N$, $P^1,...,P^K$ to equations (A5) and (A6), assuming that (A1) and (A2) also hold. Bacharach's conditions involve the concept of *matrix connectedness*. Let A be an N by K matrix. Then Bacharach (1970; 44) defines A to be *disconnected* if after a possible reordering of its rows and columns, it can be written in the following *block rectangular* form:

$$\text{(A18) } A = \begin{bmatrix} A_{nxk} & 0_{nx(K-k)} \\ 0_{(N-n)xk} & A_{(N-n)x(K-k)} \end{bmatrix}$$

where $1 \leq n < N$, $1 \leq k < K$, $A_{n\times k}$ and $A_{(N-n)\times(K-k)}$ are submatrices of A of dimension n by k and N−n by K−k respectively and $0_{n\times(K-k)}$ and $0_{(N-n)\times(K-k)}$ are n by K−k and N−n by K−k matrices of zeros. As Bacharach (1970; 47) noted, the concept of disconnectedness is a generalization to rectangular matrices of the concept of decomposability which applies to square matrices. Bacharach (1970; 47) defined A to be *connected* if it is not disconnected (and it can be seen that this is a generalization of the concept of indecomposibility which applies to square matrices). Bacharach (1970; 47-55) went on to show that if the matrix A defined by (A7) is connected, assumptions (A1) and (A2) hold, and a normalization like $\pi_1 = 1$ or $P^1 = 1$ is added to equations (A5) and (A6), then these equations have a unique positive solution which can be obtained by using the biproportional procedure suggested at the end of section A.2.1, which will converge.

It is useful to have somewhat simpler conditions on the matrix A defined by (A7) which will imply that it is connected. It can be seen that either of the following two simple conditions will imply that A is connected (and hence, we have sufficient conditions for the existence of unique positive solutions to any representation of the IDB equations):

(A19) There exists a commodity n which is demanded by all countries; i.e., there exists an n such that $y_n^{\,k} > 0$ for $k = 1,...,K$;

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[28] Dividing both sides of (A17) by K means that for each commodity group, the average (over countries) expenditure share using the IDB international prices is equal to the corresponding average expenditure share using the domestic prices prevailing in each country.

[29] Once the existence and uniqueness of a positive solution to any one of the representations of the IDB equations has been established, using assumptions (A1) and (A2), it is straightforward to show that a unique positive solution to the other representations is also implied.

(A20) There exists a country k which demands all commodities; i.e., there exists a k such that $y_n^k > 0$ for n = 1,...,N.

Conditions (A19) and (A20) are easy to check. These assumptions will be used in the following section.

A.4 Special Cases

In this section, some of the general N and K representations of the IDB equations will be specialized to cases where the number of commodities N or the number of countries K is equal to two.

A.4.1 The Two Country, Many Commodity Quantity Index Case

Suppose that the number of countries K is equal to 2. Set the country 1 volume equal to 1 so that Q^1 equals one and the first equation in (A12) becomes:

(A21) $\sum_{n=1}^{N} \{[s_n^1 + s_n^2]\, q_n^1 / [q_n^1 + (q_n^2/Q^2)]\} = 1.$

Equation (A21) is one equation in the one unknown Q^2 and it implicitly determines Q^2. It can be seen that Q^2 can be interpreted as a Fisher (1922) type bilateral quantity index, $Q_{IDB}(p^1,p^2,q^1,q^2)$, where p^k and q^k are the price and quantity (or more accurately, volume) vectors for country k. Thus in what follows in the remainder of this section, Q^2 will be replaced by Q.

At this point, assume that the data for country 1 satisfy assumption (A20) (so that q^1, p^1 and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (A21). With this assumption, the quantity relatives r_n are well defined as follows:

(A22) $r_n \equiv q_n^2/q_n^1 \geq 0$; n = 1,...,N.

Assumption (A2) implies that at least one quantity relative r_n is positive. Since each q_n^1 is positive and letting Q equal Q^2, (A21) can be rewritten using definitions (A22) as follows:[30]

(A23) $\sum_{n=1}^{N} \{[s_n^1 + s_n^2] / [1 + (r_n/Q)]\} = 1.$

Define the *vector of quantity relatives* r as $[r_1,...,r_N]$. Then the function on the left hand side of (A23) can be defined as $F(Q,r,s^1,s^2)$, where s^k is the expenditure share vector for country k for k = 1,2. Note that $F(Q,r,s^1,s^2)$ is a continuous, monotonically increasing function of Q for Q positive. It is assumed that the components of q^1 and hence s^1 are all positive. Now compute the limits of $F(Q,r,s^1,s^2)$ as Q tends to plus infinity:

(A24) $\lim_{Q\to+\infty} F(Q,r,s^1,s^2) = \sum_{n=1}^{N} \{[s_n^1 + s_n^2] = 2.$

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[30] (A23) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.

In order to compute the limit of $F(Q,r,s^1,s^2)$ as Q tends to 0, two cases need to be considered. For the first case, assume that both countries consume all commodities so that $q^2 >> 0_N$ (this is in addition to the earlier assumption that $q^1 >> 0_N$). In this case, it is easy to verify that:

(A25) $\lim_{Q\to 0} F(Q,r,s^1,s^2) = 0.$

For the second case, assume that one or more components of q^2 are zero and let N^* be the set of indexes n such that q_n^2 equals 0. In this case:

(A26) $\lim_{Q\to 0} F(Q,r,s^1,s^2) = \sum_{n\in N^*} s_n^1 < 1$

where the inequality in (A26) follows from the fact that it is assumed that all s_n^1 are positive and the sum of all of the s_n^1 is 1.

The fact that $F(Q,r,s^1,s^2)$ is a continuous, monotonically increasing function of Q along with (A24)-(A26) implies that a finite positive Q solution to the equation $F(Q,r,s^1,s^2) = 1$ exists and is unique. Denote this solution as

(A27) $Q = G(r,s^1,s^2)$.

Now use the Implicit Function Theorem to show that $G(r,s^1,s^2)$ is a continuously differentiable function which is increasing in the components of r. Thus:

(A28) $\partial G(r,s^1,s^2)/\partial r_n$
$= [s_n^1 + s_n^2]\,[1 + (r_n/Q)]^{-2}Q/\{\sum_{i=1}^N [s_i^1 + s_i^2][1 + (r_i/Q)]^{-2}r_i\} > 0\ ; \qquad n = 1,\ldots,N$

where Q satisfies (A27). However, the inequalities in (A28) do not imply that the IDB bilateral index number formula $Q_{IDB}(p^1,p^2,q^1,q^2)$ is increasing in the components of q^2 and decreasing in the components of q^1, since the derivatives in (A28) were calculated under the hypothesis that r_n equal to q_n^2/q_n^1 increased but the share vectors s^1 and s^2 were held *constant* as r_n was increased. In fact, it is *not* the case that $Q_{IDB}(p^1,p^2,q^1,q^2)$ is globally increasing in the components of q^2 and globally decreasing in the components of q^1.[31]

It is clear that $Q_{IDB}(p^1,p^2,q^1,q^2)$ satisfies the *identity test*; i.e., if $q^1 = q^2$ so that all quantity relatives r_n equal 1, then the only Q which satisfies (A23) is Q = 1. It is also clear that if $q^2 = \lambda q^1$ for $\lambda > 0$, then $Q_{IDB}(p^1,p^2,q^1,\lambda q^1)$ equals λ.[32]

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[31] This negative monotonicity result also applies to the Törnqvist Theil bilateral index number formula, Q_T; see Diewert (1992; 221). The logarithm of Q_T is defined as $\ln Q_T = \sum_{n=1}^N (1/2)[s_n^1 + s_n^2] \ln r_n$.

[32] It is also clear from (A23) that $Q_{IDB}(p^1,p^2,q^1,q^2)$ satisfies the following four homogeneity tests $Q_{IDB}(p^1,p^2,q^1,\lambda q^2) = \lambda Q_{IDB}(p^1,p^2,q^1,q^2)$, $Q_{IDB}(p^1,p^2,\lambda q^1,q^2) = \lambda^{-1}Q_{IDB}(p^1,p^2,q^1,q^2)$, $Q_{IDB}(\lambda p^1,p^2,q^1,q^2) = Q_{IDB}(p^1,p^2,q^1,q^2)$ and $Q_{IDB}(p^1,\lambda p^2,q^1,q^2) = Q_{IDB}(p^1,p^2,q^1,q^2)$ for all $\lambda > 0$. Equations (A21) or (A23) can be used to show that $Q_{IDB}(p^1,p^2,q^1,q^2)$ satisfies the first eleven of Diewert's (1999; 36) thirteen tests for a

Define $\alpha \geq 0$ as the minimum over n of the quantity relatives, $r_n = q_n^2/q_n^1$ and define $\beta > 0$ as the maximum of these quantity relatives. Then using the monotonicity properties of the function $F(Q,r,s^1,s^2)$ defined by the left hand side of (A23), it can be shown that

(A29) $\alpha \leq Q_{IDB}(p^1,p^2,q^1,q^2) \leq \beta$

with strict inequalities in (A29) if the r_n are not all equal. Thus the IDB bilateral quantity index satisfies the usual *mean value test* for bilateral quantity indexes.[33]

It is possible to develop various approximations to $Q_{IDB}(p^1,p^2,q^1,q^2)$ that cast some light on the structure of the index. Recall that (A23) defined Q_{IDB} in implicit form. This equation can be rewritten as a *weighted harmonic mean* equal to 2 as follows:

(A30) $\{\sum_{n=1}^N w_n [1 + (r_n/Q)]^{-1}\}^{-1} = 2$

where the *weights* w_n in (A30) are defined as follows:

(A31) $w_n \equiv (1/2)[s_n^1 + s_n^2]$; $\qquad n = 1,...,N.$

Now approximate the weighted harmonic mean on the left hand side of (A30) by the corresponding weighted arithmetic mean and we obtain the following approximate version of equation (A30):

(A31) $\sum_{n=1}^N w_n [1 + (r_n/Q)] \approx 2.$

Using the fact that the weights w_n sum up to one, (A31) implies that $Q = Q_{IDB}$ is approximately equal to the following expression:

(A32) $Q_{IDB}(r,w) \approx \sum_{n=1}^N w_n r_n = \sum_{n=1}^N (1/2)[(p_n^1 q_n^1/p^1\cdot q^1) + (p_n^2 q_n^2/p^2\cdot q^2)][q_n^2/q_n^1]$.

If the weighted arithmetic mean on the right hand side of (A32) is further approximated by the corresponding weighted geometric mean, then it can be seen that $Q_{IDB}(r,w)$ is approximately equal to the following expression:

(A33) $Q_{IDB}(r,w) \approx \prod_{n=1}^N r_n^{w_n} \equiv Q_T(r,w)$

where Q_T is the logarithm of the Törnqvist Theil quantity index defined as $\ln Q_T = \sum_{n=1}^N w_n \ln r_n$. If all of the quantity relatives r_n are equal to the same positive number, λ say, then the approximations in (A31)-(A33) will be exact and under these conditions where q^2 is equal to λq^1, then the following equalities will hold:

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bilateral quantity index, failing only the monotonicity in the components of q^1 and q^2 tests. Thus the axiomatic properties of the IDB bilateral quantity index are rather good.

[33] See Diewert (1992) for the history of these bilateral tests.

(A34) $Q_{IDB}(\lambda 1_N,w) = Q_T(\lambda 1_N,w) = \lambda$.

In the more general case, where the quantity relatives r_n are approximately equal to the same positive number so that q^2 is approximately proportional to q^1, then the Törnqvist Theil quantity index $Q_T(r,w)$ will provide a good approximation to the implicitly defined IDB quantity index, $Q_{IDB}(r,w)$.[34] However, in the international comparison context, it is frequently the case that quantity vectors are far from being proportional and in this nonproportional case, Q_{IDB} can be rather far from Q_T and other superlative indexes as will be seen in section A.7 of this Appendix.

A.4.2 The Two Country, Many Commodity Price Index Case

Again, suppose that the number of countries K is equal to 2. Set the country 1 PPP, P^1, equal to 1 and the first equation in (A13) becomes:

$$\text{(A35)}\quad \sum_{n=1}^{N} \{[s_n^1 + s_n^2](q_n^1/p^1\cdot q^1)/[(q_n^1/p^1\cdot q^1) + (P^2\, q_n^2/p^2\cdot q^2)]\} = 1.$$

Equation (A35) is one equation in the one unknown P^2 (the country 2 PPP) and it implicitly determines P^2. It can be seen that P^2 can be interpreted as a Fisher (1922) type bilateral price index, $P_{IDB}(p^1,p^2,q^1,q^2)$, where p^k and q^k are the price and quantity vectors for country k. Thus in what follows, P^2 will be replaced by P.

Again, it is assumed that the data for country 1 satisfy assumption (A20) (so that q^1, p^1 and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (A35). It is convenient to define the *country k normalized quantity vector* u^k as the country k quantity vector divided by the value of its output in domestic currency, $p^k\cdot q^k$:

$$\text{(A36)}\quad u^k \equiv q^k/p^k\cdot q^k\,; \qquad k = 1,2.$$

Since q^1 is strictly positive, so is q^1. Hence definitions (A36) can be substituted into (A35) in order to obtain the following equation, which implicitly determines $P^2 = P = P_{IDB}$:

$$\text{(A37)}\quad \sum_{n=1}^{N} \{[s_n^1 + s_n^2]/[1 + P(q_n^2/q_n^1)(p^1\cdot q^1/p^2\cdot q^2)]\} = \sum_{n=1}^{N} \{[s_n^1 + s_n^2]/[1 + P(u_n^2/u_n^1)]\} = 1.$$

Define $r_n \equiv u_n^2/u_n^1$ for n = 1,...,N and rewrite P as 1/Q. Then it can be seen that equation (A37) becomes equation (A23) in the previous section and so the analysis surrounding equations (A23)-(A29) can be repeated to give the existence of a positive solution $P(r,s^1,s^2)$ to (A37) along with some of the properties of the solution.

Equation (A37) can be used to show that the IDB bilateral price index P, which is the solution to (A37), regarded as a function of the price and quantity data pertaining to the

[34] If we regard $Q_{IDB}(r)$ and $Q_T(r)$ as functions of the vector of quantity relatives, then it can be shown directly that $Q_{IDB}(r)$ approximates $Q_T(r)$ to the second order around the point $r = 1_N$.

two countries, $P_{IDB}(p^1,p^2,q^1,q^2)$, satisfies the first eleven of the thirteen bilateral tests listed in Diewert (1999; 36)[35], failing only the monotonicity in the components of p^1 and p^2 tests; i.e., it is not necessarily the case that $P_{IDB}(p^1,p^2,q^1,q^2)$ is decreasing in the components of p^1 and increasing in the components of p^2. Thus the axiomatic properties of the IDB bilateral price index are rather good.

The bounds on the IDB bilateral quantity index given by (A29) do not have exactly analogous price counterparts. To develop counterparts to the bounds (A29), it is convenient to assume that all of the price and quantity data pertaining to both countries are positive. Under these conditions, define the following N *implicit partial price indexes* ρ_n:

(A38) $\rho_n \equiv [p^2 \cdot q^2/q_n^{\,2}]/[p^1 \cdot q^1/q_n^{\,1}] = [p^2 \cdot q^2/p^1 \cdot q^1]/[q_n^{\,2}/q_n^{\,1}]$; $\qquad n = 1,...,N.$

An implicit bilateral price index is defined as the value ratio, $p^2 \cdot q^2/p^1 \cdot q^1$, divided by a quantity index, say $Q(p^1,p^2,q^1,q^2)$, where Q is generally some type of weighted average of the individual quantity relatives, $q_n^{\,2}/q_n^{\,1}$. Thus each quantity relative, $q_n^{\,2}/q_n^{\,1}$, can be regarded as a partial quantity index and hence the corresponding implicit quantity index, which is the value ratio divided by the quantity relative, can be regarded as an implicit partial price index. Substitution of definitions (A38) into (A37) leads to the following equation which implicitly determines P equal to $P_{IDB}(p^1,p^2,q^1,q^2)$:

(A39) $\sum_{n=1}^{N} \{[s_n^{\,1} + s_n^{\,2}]/[1 + (P/\rho_n)]\} = 1.$

Define α as the minimum over n of the partial price indexes ρ_n and define β as the maximum of these partial price indexes. Then the monotonicity properties of the function defined by the left hand side of (A39) can be used in order to establish the following inequalities:

(A40) $\alpha \leq P_{IDB}(p^1,p^2,q^1,q^2) \leq \beta$

with strict inequalities in (A40) if the ρ_n are not all equal.

An approximate explicit formula for P_{IDB} can readily be developed. Recall that (A39) defined P_{IDB} in implicit form. This equation can be rewritten as a *weighted harmonic mean* equal to 2 as follows:

(A41) $\{\sum_{n=1}^{N} w_n\,[1 + (P/\rho_n)]^{-1}\}^{-1} = 2$

where the *weights* w_n in (A41) are the average expenditure shares, $(1/2)[s_n^{\,1} + s_n^{\,2}]$ for n = 1,...,N. Now approximate the weighted harmonic mean on the left hand side of (A41) by the corresponding weighted arithmetic mean and the following approximate version of equation (A30) is obtained:

[35] The role of prices and quantities must be interchanged; i.e., Diewert's (1999; 36) tests referred to quantity indexes whereas price indexes are now being considered.

(A42) $\sum_{n=1}^{N} w_n [1 + (P/\rho_n)] \approx 2$.

Using the fact that the weights w_n sum up to one, (A42) implies that $P = P_{IDB}$ is approximately equal to the following expression:

$$\text{(A43) } P_{IDB}(\rho,w) \approx [\sum_{n=1}^{N} w_n (\rho_n)^{-1}]^{-1}$$
$$= \{\sum_{n=1}^{N} (1/2)[(p_n^{1}q_n^{1}/p^1\cdot q^1) + (p_n^{2}q_n^{2}/p^2\cdot q^2)][q_n^{2}/q_n^{1}][p^1\cdot q^1/p^2\cdot q^2]\}^{-1}$$

where $\rho \equiv [\rho_1,...,\rho_N]$ and $w \equiv [w_1,...,w_N]$. Thus the IDB bilateral price index P_{IDB} is *approximately* equal to a weighted harmonic mean of the N partial price indexes ρ_n defined earlier by (A38).[36]

A.4.3 The Many Country, Two Commodity Case

Consider the case where there are K countries but only two commodities so that N = 2. Recall that equations (A4) and (A11) determine the IDB country PPPs, P^k, and the country volumes, Q^k, in terms of the country price and quantity vectors, p^k and q^k, and a vector of international reference prices $\pi \equiv [\pi_1,...,\pi_N]$. Thus once π is determined, the P^k and q^k can readily be determined. In this section, it is assumed that N = 2, so that there are only 2 commodities and K countries. In order to ensure the existence of a solution to the IDB equations, it is assumed that commodity 1 is consumed by all countries:

(A44) $q_1^{k} > 0$; $\qquad k = 1,...,K$.

The first international prices will be set equal to one:

(A45) $\pi_1 = 1$.

The equations which determine the π_n are equations (A14) but since N = 2, the second equation in (A14) can be dropped. Using the normalization (A45), the first equation in (A14) becomes the following equation:

(A46) $\sum_{k=1}^{K} q_1^{k}/[q_1^{k} + \pi_2 q_2^{k}] = \sum_{k=1}^{K} s_1^{k}$

which determines the international price for commodity 2, π_2.

Using assumptions (A44), the *country k commodity relatives* R^k (the quantities of commodity 2 relative to 1 in country k) are well defined as follows:

(A47) $R^k \equiv q_2^{k}/q_1^{k} \geq 0$; $\qquad k = 1,...,K$.

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[36] The expressions involving the reciprocals of the ρ_n require that q^2 be strictly positive (in addition to our maintained assumption that y^1 be strictly positive). Equations (A35) and (A37) require only that y^1 be strictly positive.

Assumption (A1) implies that at least one quantity relative R^k is positive. Since each q_1^k is positive, rewrite (A46) using definitions (A47) as follows:[37]

(A48) $F(\pi_2,R,s_1) \equiv \sum_{k=1}^K 1/[1 + \pi_2 R^k] = \sum_{k=1}^K s_1^k \equiv s_1$

where s_1 is defined to be the sum over countries k of the expenditure share of commodity 1 in country k, s_1^k.[38] Define the *vector of country quantity relatives* R as $[R^1,...,R^K]$. Then the function on the left hand side of (A48) can be defined as $F(\pi_2,R,s_1)$.[39] Note that $F(\pi_2,R,s_1)$ is a continuous, monotonically decreasing function of π_2 for π_2 positive, since the R^k are nonnegative with at least one R^k positive. Now compute the limits of $F(\pi_2,R,s_1)$ as π_2 tends to zero:

(A49) $\lim_{\pi_2 \to 0} F(\pi_2,R,s_1) = K > \sum_{k=1}^K s_1^k = s_1$.

In order to compute the limit of $F(\pi_2,R,s_1)$ as π_2 tends to plus infinity, consider two cases. For the first case, assume that all countries consume both commodities so that $R >> 0_K$. Using the definition in (A48), the following inequality is obtained:

(A50) $\lim_{\pi_2 \to +\infty} F(\pi_2,R,s_1) = 0 < \sum_{k=1}^K s_1^k = s_1$.

For the second case, assume that one or more components of R are zero and let K^* be the set of indexes k such that R^k equals 0. In this case, the following limit is obtained:

(A51) $\lim_{\pi_2 \to +\infty} F(\pi_2,R,s_1) = \sum_{k\in K^*} s_n^1 < \sum_{k=1}^K s_1^k = s_1$.

The fact that $F(\pi_2,R,s_1)$ is a continuous, monotonically decreasing function of π_2 along with (A49)-(A51) implies that a finite positive π_2 solution to equation (A48) exists and is unique. Denote this solution as $\pi_2 = G(R,s_1)$. It is straightforward to verify that G is decreasing in the components of R and decreasing in s_1.

Suppose that all country quantity relatives R^k are positive and define α and β to be the minimum and maximum over k respectively of these quantity relatives. Then it is also straightforward to verify that π_2 satisfies the following bounds:[40]

(A52) $[(s_1/K)^{-1} - 1]/\beta \le \pi_2 \le [(s_1/K)^{-1} - 1]/\alpha$.

[37] (A23) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.
[38] Note that s_1 satisfies the inequalities $0 < s_1 < K$.
[39] Thus the π_2 solution to (A48) depends only on the vector of country quantity relatives, R, and the sum across countries k of the expenditure shares on commodity 1, s_1^k. Alternatively, π_2 depends on the K dimensional vector R and the sum across countries commodity share vector, $s^1 +...+ s^K$, which is a two dimensional vector in the present context where N = 2.
[40] It can be verified that $0 < s_1 < K$ so that $(s_1/K)^{-1} > 1$ so that the bounds in (A52) are positive when $R >> 0_K$. In the case where $R > 0_K$, the lower bound is still valid but the upper bound becomes $+\infty$.

Thus if all of country quantity relatives $R^k = q_2^k/q_1^k$ are equal to the same positive number λ, then the bounds in (A52) collapse to the common value $[(s_1/K)^{-1} - 1]/\lambda$.

In the case where prices and quantities are positive across all countries (so that all R^k are positive), then it is possible to rewrite the basic equation (A48) in a more illuminating form as follows:

$$\text{(A53)}\ \sum_{k=1}^K s_1^k = \sum_{k=1}^K 1/[1 + \pi_2 R^k]$$
$$= \sum_{k=1}^K \{s_1^k/[s_1^k + \pi_2 s_1^k(y_2^k/y_1^k)]\}$$
$$= \sum_{k=1}^K \{s_1^k/[s_1^k + \pi_2 s_2^k(p_2^k/p_1^k)^{-1}]\}.$$

Equation (A53) shows that the π_2 which solves the equation is a function of the K country share vectors, $s^1,...,s^K$ (each of which is of dimension 2) and the vector of K country price relatives, $[p_2^1/p_1^1,..., p_2^K/p_1^K]$. It can be seen that if all of these country price relatives are equal to a common ratio, say $\lambda > 0$, then the solution to (A53) is $\pi_2 = \lambda$. In the case where all of these country price relatives are positive, let α^* and β^* to be the minimum and maximum over k respectively of these price relatives. Then it is straightforward to verify that π_2 satisfies the following bounds:

$$\text{(A54)}\ \alpha^* \le \pi_2 \le \beta^*.$$

A.4.4 The Two Country, Two Commodity Case

In this section, it is assumed that K = 2 (two countries) and that N = 2 (two commodities). In this case, it is possible to obtain an explicit formula for the country 2 volume Q^2 relative to relative to the country 1 volume Q^1 which is set equal to one; i.e., it is possible to obtain an explicit formula for the IDB bilateral quantity index, $Q^2 = Q = Q_{IDB}(p^1,p^2,q^1,q^2)$. The starting point for this case is equation (A21) which determines Q implicitly. In the case where N equals 2, this equation becomes:

$$\text{(A55)}\ \{[s_1^1 + s_1^2]q_1^1/[q_1^1 + (q_1^2/Q)]\} + \{[(1-s_1^1) + (1-s_1^2)]q_2^1/[q_2^1 + (q_2^2/Q)]\} = 1.$$

As usual, it is assumed that the data for country 1 are positive so that $q_1^1 > 0$ and $q_2^1 > 0$. Thus the two quantity relatives, $r_n \equiv q_n^2/q_n^1$ for n = 1,2, are well defined nonnegative numbers. It is assumed that at least one of the relatives r_1 and r_2 are strictly positive. Substitution of these quantity relatives into (A55) leads to the following equation for Q:

$$\text{(A56)}\ \{[s_1^1 + s_1^2]Q/[Q + r_1]\} + \{[(1-s_1^1) + (1-s_1^2)]Q/[Q + r_2]\} = 1.$$

The above equation simplifies into the following quadratic equation:[41]

$$\text{(A57)}\ Q^2 + [s_1^1 + s_1^2-1][r_2 - r_1]Q - r_1r_2 = 0.$$

[41] This equation can be utilized to show that $Q_{IDB}(p^1,p^2,q^1,q^2)$ is not necessarily monotonically increasing in the components of q^1 or monotonically decreasing in the components of q^1.

In the case where both r_1 and r_2 are positive, there is a negative and a positive root for (A57). The positive root is the desired bilateral quantity index and it is equal to the following expression:

(A58) $Q_{IDB}(p^1,p^2,q^1,q^2) = -(1/2)(s_1^1+s_1^2-1)(r_2-r_1) + (1/2)[(s_1^1+s_1^2-1)^2 (r_2-r_1)^2 + 4r_1r_2]^{1/2}$.

Now suppose that $r_1 = q_1^2/q_1^1 = 0$ so that $q_1^1 > 0$ and $q_1^2 = 0$. Then $s_1^2 = 0$ as well and using (A57):

(A59) $Q = [1 - s_1^1]r_2 = [1 - s_1^1][q_2^2/q_2^1]$.

Formula (A59) makes sense in the present context. Recall that Q is supposed to reflect the country 2 volume or average quantity relative to country 1. If, as a preliminary estimate of this relative volume, Q is set equal to the single nonzero quantity relative, r_2, then this would overestimate the average volume of country 2 relative to 1 since country 2 has a zero amount of commodity 1 while country 1 has the positive amount q_1^1. Thus r_2 is scaled down by multiplying it by one minus country 1's share of commodity 1, s_1^1. The bigger is this share, the more the preliminary volume ratio r_2 is downsized.

Now suppose that $r_2 = y_2^2/y_2^1 = 0$ so that $q_2^1 > 0$ and $q_2^2 = 0$. Then $s_1^2 = 1$ and using (A57):

(A60) $Q = s_1^1r_1 = [1-s_2^1][q_1^2/q_1^1]$.

Again, formula (A60) makes sense in the present context. If Q is set equal to the single nonzero quantity relative, r_1, then this would overestimate the average volume of country 2 relative to 1 since country 2 has a zero amount of commodity 2 while country 1 has the positive amount q_2^1. Thus scale down r_1 by multiplying it by one minus country 1's share of commodity 2, s_2^1. The bigger is this share, the more the preliminary volume ratio r_1 is downsized.

Two other special cases of (A57) are of interest. Consider the cases where the following conditions hold:

(A61) $r_1 = r_2$;
(A62) $s_1^1+s_1^2 = 1$.

If either of the above two special cases hold, then Q equals $(r_1r_2)^{1/2}$, the geometric mean of the two quantity relatives. This first result is not surprising since this result is implied by the earlier N commodity results for two countries; i.e., see (A29). The second result is more interesting. Note that if (A62) holds, so that the sum of the two country expenditure shares on commodity 1 is equal to 1, then the sum of the two country expenditure shares on commodity 2 is also equal to 1; i.e., it is also the case that $s_2^1+s_2^2 = 1$ and the IDB quantity index is equal to the geometric mean of the two quantity relatives, $(r_1r_2)^{1/2}$.[42]

The following section provides a discussion of the axiomatic or test properties of the IDB multilateral system.

A.5 The Axiomatic Properties of the Iklé Dikhanov Balk Multilateral System

Balk (1996; 207-212) developed the axiomatic properties of the IDB multilateral method using a set of nine axioms based on the earlier work of Diewert (1988).[43] Diewert (1999; 16-20) further refined his set of axioms and below, eleven of his thirteen "reasonable" axioms he proposed for a multilateral system will be listed. A slightly different system of notation will be used in the present section: $P \equiv [p^1,...,p^K]$ will signify an N by K matrix which has the domestic Basic Heading parities or price vectors $p^1,...,p^K$ as its K columns and $Q \equiv [q^1,...,q^K]$ will signify an N by K matrix which has the country quantity vectors $q^1,...,q^K$ as its K columns.

Equations (A12) plus a normalization determine the country aggregate volumes, $Q^1,...,Q^K$. These country volumes Q^k can be regarded as functions of the data matrices P and Q, so equations (A12) plus a normalization determine the functions, $Q^k(P,Q)$ for k = 1,...,K. Once these functions $Q^k(P,Q)$ have been determined, then *country k's share of regional output*, $S^k(P,Q)$, can be defined as follows:

$$\text{(A63)}\quad S^k(P,Q) \equiv Q^k(P,Q)/[Q^1(P,Q) + ... + Q^K(P,Q)]\ ; \qquad k = 1,...,K.$$

Both Balk (1996) and Diewert (1988) (1999) used the system of regional share equations $S^k(P,Q)$ as the basis for their axioms. Alternatively, instead of using equations (A12) and (A63) to define the share functions $S^k(P,Q)$, these functions can be determined as the functions $Q^k(P,Q)$ which solve equations (A12) if the following normalization is added to equations (A12):

$$\text{(A64)}\quad Q^1 + ... + Q^K = 1.$$

Eleven of Diewert's (1999; 16-20) 13 tests or axioms for a multilateral share system, $S^1(P,Q),...,S^K(P,Q)$, will now be listed.[44] The domain of definition for these functions is the set of price and quantity vectors in the matrices P and Q which satisfy the

[42] Under these conditions, it is also the case that all prices and quantities are positive in the two countries since it was assumed that y^1 is strictly positive and y^2 is nonnegative and nonzero; i.e., $q^1>>0_2$ and $q^2 > 0_2$.

[43] Balk's axioms were somewhat different from those proposed by Diewert since Balk also introduced an extra set of country weights into Diewert's axioms. Balk's example will not be followed here since it is difficult to determine precisely what these country weights should be.

[44] Diewert's (1999; 18) bilateral consistency in aggregation test (which the IDB system does not satisfy) is omitted, since this test depends on choosing a "best" bilateral quantity index and there may be no consensus on what this "best" functional form is. His final axiom involving the consistency of the multilateral system with the economic approach to index number theory will be discussed in section A.6 below.

assumptions listed in the first paragraph of section A.2.1 plus it is assumed that at least one of assumptions (A1) or (A2) hold.

T1: *Share Test*: There exist K continuous, positive functions, $S^k(P,Q)$, k = 1,...,K, such that $\sum_{k=1}^K S^k(P,Q) = 1$ for all P,Q in the appropriate domain of definition.

T2: *Proportional Quantities Test*: Suppose that $q^k = \beta_k q$ for some $q >> 0_N$ and $\beta_k > 0$ for k = 1,...,K with $\sum_{k=1}^K \beta_k = 1$. Then $S^k(P,Q) = \beta_k$ for k = 1,...,K.

T3: *Proportional Prices Test*: Suppose that $p^k = \alpha_k p$ for $p >> 0_N$ and $\alpha_k > 0$ for k = 1,...,K. Then $S^k(P,Q) = p \cdot q^k/[p \cdot \sum_{i=1}^K q^i]$ for k = 1,...,K.

T4: *Commensurability Test*: Let $\delta_n > 0$ for n = 1,...,N and let Δ denote the N by N diagonal matrix with the δ_n on the main diagonal. Then $S^k(\Delta P, \Delta^{-1}Q) = S^k(P,Q)$ for k = 1,...,K.

This test implies that the country shares $S^k(P,Q)$ are invariant to changes in the units of measurement.

T5: *Commodity Reversal Test*: Let Π denote an N by N permutation matrix. Then $S^k(\Pi P, \Pi Q) = S^k(P,Q)$ for k = 1,...,K.

This test implies that each country's share of world output remains unchanged if the ordering of the N commodities is changed.

T6: *Multilateral Country Reversal Test*: Let S(P,Q) denote a K dimensional column vector that has the country shares $S^1(P,Q)$,.., $S^K(P,Q)$ as components and let Π^* be a K by K permutation matrix. Then $S(P\Pi^*, Q\Pi^*) = S(P,Q)\Pi^*$.

This test implies that countries are treated in a symmetric manner; i.e., the country shares of world output are not affected by a reordering of the countries. The next two tests are homogeneity tests.

T7: *Monetary Units Test*: Let $\alpha_k > 0$ for k = 1,...,K. Then $S^k(\alpha_1 p^1,...,\alpha_K p^K,Q) = S^k(p^1,...,p^K,Q) = S^k(P,Q)$ for k = 1,...,K.

This test implies that the absolute scale of domestic prices in each country does not affect each country's share of world output; i.e., only relative prices within each country affect the multilateral volume parities.

T8: *Homogeneity in Quantities Test*: For i = 1,...,K, let $\lambda_i > 0$ and let j denote another country not equal to country i. Then $S^i(P,q^1,...,\lambda_i q^i,...,q^K)/S^j(P,q^1,...,\lambda_i q^i,...,q^K) = \lambda_i S^i(P, q^1,..., q^i,...,q^K)/S^j(P,q^1,..., q^i,...,q^K) = \lambda_i S^i(P,Q)/S^j(P,Q)$.

This test says that the volume share of country i relative to country j is linearly homogeneous in the components of the country i quantity vector q^i.

T9: *Monotonicity Test in Quantities Test*: For each k, $S^k(P,q^1,...,q^{k-1},q^k,q^{k+1},...,q^K) = S^k(P,Q)$ is increasing in the components of q^k.

This test says that country k's share of world output increases as any component of the country k quantity vector q^k increases.

T10: *Country Partitioning Test*: Let A be a strict subset of the indexes (1,2,...,K) with at least two members. Suppose that for each $i\in A$, $p^i = \alpha_i p^a$ for $\alpha_i > 0$, $p^a >> 0_N$ and $q^i = \beta_i q^a$ for $\beta_i > 0$, $q^a >> 0_N$ with $\sum_{i\in A} \beta_i = 1$. Denote the subset of $\{1,2,...,K\}$ that does not belong to A by B and denote the matrices of country price and quantity vectors that belong to B by P^b and Q^b respectively. Then: (i) for $i\in A$, $j\in A$, $S^i(P,Q)/S^j(P,Q) = \beta_i/\beta_j$ and (ii) for $i\in B$, $S^i(P,Q) = S^{i*}(p^a,P^b,q^a,Q^b)$ where $S^{k*}(p^a,P^b,q^a,Q^b)$ is the system of share functions that is obtained by adding the group A aggregate price and quantity vectors, p^a and q^a respectively, to the group B price and quantity data, P^b,Q^b.

Thus if the aggregate quantity vector for the countries in group A were distributed proportionally among its members (using the weights β_i) and each group A country faced prices that were proportional to p^a, then part (i) of T10 requires that the group A share functions reflect this proportional allocation. Part (ii) of T10 requires that the group B share functions are equal to the same values no matter whether we use the original share system or a new share system where all of the group A countries have been aggregated up into the single country which has the price vector p^a and the group A aggregate quantity vector q^a. Conversely, this test can be viewed as a *consistency in aggregation test* if a single group A country is partitioned into a group of smaller countries.

T11: *Additivity Test*: For each set of price and quantity data, P,Q, belonging to the appropriate domain of definition, there exists a set of positive world reference prices $\pi >> 0_N$ such that $S^k(P,Q) = \pi\cdot q^k/[\pi\cdot\sum_{i=1}^K q^i]$ for k = 1,...,K.

The IDB multilateral system obviously satisfies the additivity test T11.

Proposition 1: Assume that the country price and quantity data P,Q satisfy assumptions (A1), (A2) and at least one of the assumptions (A19) and (A20). Then the IDB multilateral system fails only Tests 9 and 10 in the above list of 11 tests.

Proof: The existence and uniqueness of a solution to any one of the representations of the IDB equations have been discussed in section A.3 above. The continuity (and once continuous differentiability) of the IDB share functions $S^k(P,Q)$ in the data follow using the Implicit Function Theorem on the system of equations (A8) and (A9) (plus a normalization) by adapting the arguments in Bacharach (1970; 67-68). This establishes T1.

The proofs of tests T2 and T4-T8 follow by straightforward substitution into equations (A12).

The proof of T3 follows by setting $\pi = p$ and then showing that this choice of π satisfies equations (A14). Once π has been determined as p, then the Q^k are determined as $\pi \cdot q^k$ for $k = 1,...,K$ and finally the share functions are determined using (A63).

The results in section A.4.4 can be used to show that the monotonicity test T9 fails.

Finally, it can be seen that the "democratic" nature of the IDB system (each country's shares are treated equally in forming the reference prices π) leads to a failure of test T10.[45]

The main text showed that the IDB method satisfied the additivity test T11. Q.E.D.

It is useful to contrast the IDB method with the other additive method that has been used in ICP and that is the Geary Khamis system. Both methods satisfy tests T1-T7 and T11 and both methods fail the monotonicity in quantities test T9. Thus the tests that discriminate between the two methods are T8 and T10: the IDB multilateral system passes the homogeneity test T8 and fails the country partitioning test T10 and vice versa for the GK system.[46] There has been more discussion about test T10 than test T8. Proponents of the GK system like the fact that it has good aggregation (across countries) properties and the fact that big countries have more influence on the determination of the world reference price vector π is regarded as a reasonable price to pay for these good aggregation properties. On the other hand, proponents of the IDB method like the fact that the world reference prices are more democratically determined and they place less weight on having good aggregation properties. Also, from evidence presented by Deaton and Heston (2010) using the ICP 2005 data base, it appears that the IDB parities are closer to the GEKS parities than the GK parities. Thus the IDB method has the advantage that it is an additive method that does not depart too far from the parities that are generated by the GEKS method.

A.6 The Economic Properties of the Iklé Dikhanov Balk Multilateral System

An economic approach to bilateral index number theory was initiated by Diewert (1976) and generalized to multilateral indexes in Diewert (1999; 20-23). The properties of the IDB system in this economic framework will be examined in the present section.

The basic assumption in the economic approach to multilateral indexes is that the country k quantity vector q^k is a solution to the following country k utility maximization problem:

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[45] Diewert (1999; 27) showed that the GK system satisfied all of the 11 tests except the homogeneity test T8 and the monotonicity test T9. The GK system is a "plutocratic" method where the bigger countries have a greater influence in the determination of the international price vector π.

[46] Balk (1996; 212) also compares the performance of the two methods (along with other multilateral methods) using his axiomatic system.

(A65) $\max_q \{f(q) : p^k \cdot q = p^k \cdot q^k\} = u_k = Q^k$

where $u^k \equiv f(q^k)$ is the utility level for country k which can also be interpreted as the country's volume Q^k, $p^k >> 0_N$ is the vector of positive prices for outputs that prevail in country k, for k = 1,...,K[47] and f is a linearly homogeneous, increasing and concave aggregator function that is assumed to be the same across countries. This aggregator function has a *dual unit cost or expenditure function* c(p) which is defined as the minimum cost or expenditure required to achieve unit volume level if purchasers face the positive commodity price vector p.[48] Since purchasers in country k are assumed to face the prices $p^k >> 0_N$, the following equalities hold:

(A66) $c(p^k) \equiv \min_q \{p^k \cdot q : f(q) \geq 1\} \equiv P^k$; $\qquad$ k = 1,...,K

where P^k is the (unobserved) minimum expenditure that is required for country k purchasers to achieve unit utility or volume level when the purchasers face prices p^k. P^k can also be interpreted as country k's aggregate PPP. Under assumptions (A65), it can be shown[49] that the country k price and quantity vectors, p^k and q^k, satisfy the following equation:

(A67) $p^k \cdot q = c(p^k)f(q^k) = P^k u_k = P^k Q^k$ $\qquad$ k = 1,...,K.

In order to make further progress, it is assumed that either the utility function f(q) is once continuously differentiable with respect to the components of q *or* the unit cost function c(p) is once continuously differentiable with respect to the components of p (or both).

In the case where f is assumed to be differentiable, the first order necessary conditions for the utility maximization problems in (A65), along with the linear homogeneity of f, imply the following relationships between the country k price and quantity vectors, p^k and q^k respectively, and the country unit expenditures, e_k defined in (A66):[50]

(A68) $p^k = \nabla f(q^k) P^k$; $\qquad$ k = 1,...,K

where $\nabla f(q^k)$ denotes the vector of first order partial derivatives of f with respect to the components of q evaluated at the country k quantity vector, q^k.

In the case where c(p) is assumed to be differentiable, then Shephard's Lemma implies the following equations:

(A69) $q^k = \nabla c(p^k) u_k = \nabla c(p^k) Q^k$ $\qquad$ k = 1,...,K

[47] In this section, it will be assumed that all country prices and quantities are positive so that $p^k >> 0_N$ and $q^k >> 0_N$ for k = 1,...,K.
[48] The unit cost function c(p) is an increasing, linearly homogeneous and concave function in p for $p >> 0_N$.
[49] See Diewert (1974) for material on duality theory and unit cost functions.
[50] See Diewert (1999; 21) for more details on the derivation of these equations.

where $u_k = f(q^k) = Q^k$ denotes the utility level for country k and $\nabla c(p^k)$ denotes the vector of first order partial derivatives of the unit cost function c with respect to the components of p evaluated at the country k price vector p^k.

If f(q) or c(p) are differentiable, then since both of these functions are assumed to be linearly homogeneous, Euler's Theorem on homogeneous functions implies the following relationships:

(A70) $f(q^k) = \nabla f(q^k)\cdot q^k = \sum_{n=1}^N [\partial f(q^k)/\partial q_n]q_n^k$; $\quad k = 1,\ldots,K$;
(A71) $c(p^k) = \nabla c(p^k)\cdot p^k = \sum_{n=1}^N [\partial c(p^k)/\partial p_n]p_n^k$; $\quad k = 1,\ldots,K$.

Recall that the *expenditure share* on commodity n for country k was defined as $s_n^k \equiv p_n^k q_n^k/p^k\cdot q^k$. In the case where f(q) is differentiable, substitution of (A68) and (A70) into these shares leads to the following expressions:

(A72) $s_n^k = q_n^k f_n(q^k)/f(q^k)$; $\quad n = 1,\ldots,N$; $k = 1,\ldots,K$

where $f_n(q^k) \equiv \partial f(q^k)/\partial q_n$. In the case where c(p) is differentiable, substitution of (A69) and (A71) into the expenditure shares s_n^k leads to the following expressions:

(A73) $s_n^k = p_n^k c_n(p^k)/c(p^k)$; $\quad n = 1,\ldots,N$; $k = 1,\ldots,K$

where $c_n(p^k) \equiv \partial c(p^k)/\partial p_n$. With the above preliminaries laid out, we are now ready to attempt to determine what classes of preferences (i.e., differentiable functional forms for f or c) are consistent with the IDB system of equations (A12).

Start out by considering the case of a differentiable utility function, f(q), which is positive, increasing, linearly homogeneous and concave for $q >> 0_N$.[51] Let $q^k >> 0_N$, $Q^k = f(q^k)$ for k = 1,...,K and substitute these equations and (A72) into equations (A12). Then f must satisfy the following system of K functional equations:

(A74) $\sum_{n=1}^N\{[q_n^1 f_n(q^1)/f(q^1)]+\ldots+[q_n^K f_n(q^K)/f(q^K)]\}[q_n^k/f(q^k)]/\{[q_n^1/f(q^1)]+\ldots+[q_n^K/f(q^K)]\}$
$= 1$; $\quad k = 1,\ldots,K$.

Note that all of the terms in the above system of K equations are the same in each equation except the terms $q_n^k/f(q^k)$ in the middle of equation k. Suppose that f(q) is a *linear function* of q so that:

(A75) $f(q) = f(q_1,\ldots,q_N) = a_1q_1 + \ldots + a_Nq_N$; $a_1 > 0 , \ldots , a_N > 0$.

It is straightforward to verify that the linear function f(q) defined by (A75) satisfies the maintained hypotheses on f and it also satisfies the system of functional equations (A75). Thus the IDB multilateral system is consistent with linear preferences.

[51] The functions f or c are defined to be *regular* if they satisfies these regularity conditions.

Now consider the case of a differentiable unit cost function c(p), which is positive, increasing, linearly homogeneous and concave for $p >> 0_N$. Let $p^k >> 0_N$, $P^k = c(p^k)$ for k = 1,...,K and substitute these equations and (A69) into equations (A12). Then c must satisfy the following system of K functional equations:

(A76) $\sum_{n=1}^{N} \{[p_n^1 c_n(p^1)/c(p^1)] + ... + [p_n^K c_n(p^K)/c(p^K)]\} c_n(p^k)/\{c_n(p^1) + ... + c_n(p^K)\} = 1$; $k = 1,...,K$.

Note that all of the terms in the above system of K equations are the same in each equation except the partial derivative terms $c_n(p^k)$ in the middle of equation k. Now suppose that c(p) is a *linear function* of p so that:

(A77) $c(p) = c(p_1,...,p_N) = b_1y_1 + ... + b_Ny_N$; $b_1 > 0 , ... , b_N > 0$.

It is straightforward to verify that the linear function c(p) defined by (A77) satisfies the maintained hypotheses on c and it also satisfies the system of functional equations (A76). Thus the IDB multilateral system is consistent with Leontief (no substitution) preferences.

The above computations show that the IDB multilateral system is consistent with preferences that exhibit perfect substitutability between commodities (the linear utility function case) and with preferences that exhibit no substitution behavior as prices change (the case of Leontief preferences where the unit cost function is linear). It turns out that if the number of countries is three or more, then these are the only (differentiable) preferences that are consistent with the IDB system as is shown by the following result:

Proposition 2: If the number of countries is greater than two, then the linear utility function defined by (A75) is the only regular differentiable utility function that is consistent with the IDB equations (A74) and the preferences that are dual to the linear unit cost function defined by (A77) are the only differentiable dual preferences that are consistent with the IDB equations (A76).

Proof: Let $K \geq 3$ and let $q^k >> 0_N$ for k = 1,...,K. Then the first two equations in (A74) can be rearranged into the following equation:

(A78) $f(q^2) - f(q^1)$
$= \sum_{n=1}^{N}\{[q_n^1 f_n(q^1)/f(q^1)]+...+[q_n^K f_n(q^K)/f(q^K)]\}[q_n^2 - q_n^1]/\{[q_n^1/f(q^1)]+...+[q_n^K/f(q^K)]\}$.

Fix n and let the components of q^1 and q^2 satisfy the following assumptions:

(A79) $q_n^2 \neq q_n^1$; $q_i^2 = q_i^1$ for $i \neq n$.

Now look at the equation (A78) when assumptions (A79) hold. The left hand side is independent of the components of q^3 and hence the right hand side of (A78) must also be independent of q^3. Using the linear homogeneity of f, this is sufficient to show that $f_n(q^3)$ must be a constant for any $q^3 >> 0_N$; i.e., for all $q >> 0_N$, $f_n(q)$ is equal to a constant a_n, which must be positive under our regularity conditions on f. This proof works for n = 1,...,N, which completes the proof of the first part of the proposition.

Let $K \geq 3$ and let $p^k >> 0_N$ for k = 1,...,K. Then equations (A76) can be rewritten as follows:

(A80) $\sum_{n=1}^{N} \rho_n(p^1,...,p^K)c_n(p^k) = 1$; k = 1,...,K

where the coefficients $\rho_n(p^1,...,p^K)$ in (A80) are defined for n = 1,...,N as follows:

(A81) $\rho_n(p^1,...,p^K) \equiv \{[p_n^1 c_n(y^1)/c(p^1)] +...+ [p_n^K c_n(p^K)/c(p^K)]\}/\{c_n(p^1) +...+ c_n(p^K)\}$.

The first two equations in (A80) can be subtracted from each other to give the following equation:

(A82) $\sum_{n=1}^{N} \rho_n(p^1,...,p^K)[c_n(p^2) - c_n(p^1)] = 0$.

Define the vector $\rho(p^1,...,p^K) \equiv [\rho_1(p^1,...,p^K), ..., \rho_N(p^1,...,p^K)]$. Since $K \geq 3$, looking at definitions (A81), it can be seen that the components of p^3 can be varied (holding the remaining price vectors constant) so that we can find N linearly independent $\rho(p^1,...,p^K)$ vectors. Substitution of these linearly independent vectors into equation (A82) implies that

(A83) $\nabla c(p^2) = \nabla c(p^1)$.

Since equations (A83) hold for all positive p^1 and p^2, the partial derivatives of c(p) are constant, which completes the proof of the proposition. Q.E.D.

Thus the IDB multilateral system suffers from the same defect as the GK system[52]; both of these additive systems are not consistent with an economic approach that allows consumer preferences to be represented by flexible functional forms, whereas the GEKS system is consistent with preferences that are representable by flexible functional forms.[53]

A.7 An Artificial Data Set Numerical Example

In this section of the Appendix, the IDB price and volume parities will be computed and compared with other multilateral systems in a simple numerical example that was used in Diewert (1999; 79-84), which was a three country, two commodity example. The price and quantity vectors for the three countries were as follows:

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[52] Diewert (1999; 27) showed that when $K \geq 3$, the GK system is only consistent with a linear or Leontief aggregator function.

[53] See Diewert (1999; 46) for descriptions of multilateral methods that have good economic properties; i.e., methods that are consistent with maximizing behavior on the part of consumers with preferences represented by flexible functional forms. See Diewert (1976) for the concept of a flexible functional form and the economic approach to index number theory. In addition to the GEKS system, the Own Share and van Ijzeren's (1983) weighted and unweighted balanced methods have good economic properties.

(A84) $p^1 \equiv [1,1]$; $p^2 \equiv [10, 1/10]$; $p^3 \equiv [1/10,10]$; $q^1 \equiv [1,2]$; $q^2 \equiv [1,100]$; $q^3 \equiv [1000,10]$.

Note that the geometric average of the prices in each country is 1, so that average price levels are roughly comparable across countries, except that the price of commodity 1 is very high and the price of commodity 2 is very low in country 2 and vice versa for country 3. As a result of these price differences, consumption of commodity 1 is relatively low and consumption of commodity 2 is relatively high in country 2 and vice versa in country 3. Country 1 can be regarded as a tiny country, with total expenditure (in national currency units) equal to 3, country 2 is a medium country with total expenditure equal to 20 and country 3 is a large country with expenditure equal to 200.

The Fisher (1922) quantity index Q_F can be used to calculate the volume Q^k of each country k relative to country 1; i.e., calculate Q^k/Q^1 as $Q_F(p^1,p^k,q^1,q^k) \equiv [p^1 \cdot q^k p^k \cdot q^k/p^1 \cdot q^1 p^k \cdot q^1]^{1/2}$ for k = 2,3. Set $Q^1 = 1$ and then Q^2 and Q^3 are determined and these volumes using country 1 as the base or star country are reported in the Fisher 1 column of Table 1. In a similar manner, use country 2 as the base and use the Fisher formula to calculate Q^1, $Q^2 = 1$ and Q^3. Then divide these numbers by Q^1 and the numbers listed in the Fisher 2 column of Table 1 are obtained. Finally, use country 3 as the base and use the Fisher formula to calculate Q^1, Q^2 and $Q^3 =1$. Then divide these numbers by Q^1 and obtain the numbers listed in the Fisher 3 column of Table 1. Ideally, these Fisher star parities would all coincide but since they do not, take the geometric mean of them and obtain the GEKS parities which are listed in the fourth column of Table 1. Thus for this example, the GEKS economic approach to forming multilateral quantity indexes leads to the volumes of countries 2 and 3 to be equal to 7.26 and 64.81 times the volume of country 1.[54]

Table 1: Fisher Star, GEKS, GK and IDB Relative Volumes for Three Countries

	Fisher 1	**Fisher 2**	**Fisher 3**	**GEKS**	**GK**	**IDB**
$\mathbf{Q^1}$	1.00	1.00	1.00	1.00	1.00	1.00
$\mathbf{Q^2}$	8.12	8.12	5.79	7.26	47.42	33.67
$\mathbf{Q^3}$	57.88	81.25	57.88	64.81	57.35	336.67

Turning to the spatial linking method, it can be seen that country 1 has the most similar structure of prices with both countries 2 and 3; i.e., countries 2 and 3 have the most dissimilar structure of relative prices. Thus in this case, the spatial linking method leads to the Fisher star parities for country 1; i.e., the spatial linking relative outputs are given by the Fisher 1 column in Table 1. Note that these parities are reasonably close to the GEKS parities.

[54] Since the Fisher star parities are not all equal, it needs to be recognized that the GEKS parities are only an approximation to the "truth". Thus it could be expected that an economic approach would lead to a Q^2/Q^1 parity in the 5 to 9 range and to a Q^3/Q^1 parity in the 50 to 90 range. Note that the IDB parities are well outside these ranges and the GK parity for Q^2/QY^1 is well outside this suggested range.

The GK parities for P^k and π_n can be obtained by iterating between equations (7) and (8) until convergence has been achieved.[55] Once these parities have been determined, the Q^k can be determined using equations (10). These country volumes were then normalized so that $Q^1 = 1$. The resulting parities are listed in the GK column in Table 1. It can be seen that the GK parity for Q^3/Q^1, 57.35, is reasonable but the parity for Q^2/Q^1, 47.42, is much too large to be reasonable from an economic perspective. The cause of this unreasonable estimate for Q^2 is the fact that the GK international price vector, $[\pi_1,\pi_2]$, is equal to [1, 9.00] so that these relative prices are closest to the structure of relative prices in country 3, the large country. Thus the relatively large consumption of commodity 2 in country 2 gets an unduly high price weight using the GK vector of international reference prices, leading to an exaggerated estimate for its volume, Q^2. This illustrates a frequent criticism of the GK method: the structure of international prices that it gives rise to is "biased" towards the price structure of the biggest countries.

The IDB parities for the above numerical example are now calculated in order to see if the method can avoid the unreasonable results generated by the GK method. The parities for P^k and π_n can be obtained by iterating between equations (13) and (14) until convergence has been achieved.[56] Once these parities have been determined, the Q^k can be determined using equations (10). These country volumes were then normalized so that $Q^1 = 1$. The resulting parities are listed in the IBD column in Table 1. It can be seen that the GK parity for Q^2/Q^1 is 33.67 which is well outside the suggested reasonable range (from the viewpoint of the economic approach) of 5 to 9 and the GK parity for Q^3/Q^1 is 336.7 which is well outside the suggested reasonable range of 50 to 90. What is the cause of these problematic parities?

The problematic IDB volume estimates are not caused by an unrepresentative vector of international prices since the IBD international price vector, $[\pi_1,\pi_2]$, is equal to [1, 1], which in turn is equal to the vector of (equally weighted) geometric mean commodity prices across countries. The problem is due to the fact that *any* additive method cannot take into account the problem of declining marginal utility as consumption increases if there are 3 or more countries in the comparison. Thus the IBD vector of international prices $\pi = [1,1]$ is exactly equal to the country 1 price vector $p^1 = [1,1]$ and so the use of these international prices leads to an accurate volume measure for country 1. But the structure of the IBD international prices is far different from the prices facing consumers in country 2, where the price vector is $p^2 \equiv [10, 1/10]$. The very low relative price for commodity 2 leads consumers to demand a relatively large amount of this commodity (100 units) and the relatively high price for commodity 1 leads to a relatively low demand for this commodity (1 unit). Thus at international prices, the output of country 2 is $\pi \cdot q^2$ which is equal to 101 as compared to its nominal output $p^2 \cdot q^2$ which is equal to 20. Thus the use of international prices overvalues the output of country 2 relative to country 1 because the international price of commodity 2 is equal to 1 which is very much larger than the actual price of commodity 2 in country 2 (which is 1/10). Note that Q^2/Q^1 is

[55] Only 5 iterations were required for convergence.

[56] Since all of the prices and quantities are positive in this example, equations (13) and (14) in the main text can be used instead of the more robust (to zero entries) equations (A3) and (A4). Eighteen iterations were required for convergence.

equal to $\pi \cdot q^2/\pi \cdot q^1$ = 101/3 = 33.67, an estimate which fails to take into account the declining marginal utility of the relatively large consumption of commodity 2 in country 1. A similar problem occurs when the outputs of countries 1 and 3 are compared using international prices except in this case, the use of international prices tremendously overvalues country 3's consumption of commodity 1. The problem of finding international reference prices that are "fair" for two country comparisons can be solved[57] but the problem cannot be solved in general if there are three or more countries in the comparison as was seen in section 5 of the main text.

The tentative conclusion at this point is that additive methods for making international price and quantity comparisons where there are tremendous differences in the structure of prices and quantities across countries are likely to give rather different answers than methods that are based on economic approaches. Hence although additive methods are very convenient, they are likely to lead to biased comparisons from the viewpoint of the economic approach to index number theory.

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[57] See Diewert (1996; 246) for examples of superlative indexes that are additive if there are only two countries or observations.

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