Session Number: Parallel Session 8A
Time: Friday, August 27, PM

Paper Prepared for the 31st General Conference of
The International Association for Research in Income and Wealth

St. Gallen, Switzerland, August 22-28, 2010

How to Measure Living Standards and Productivity

Nicholas Oulton

For additional information please contact:

Name: Nicholas Oulton
Affiliation: London School of Economics

Email Address: N.Oulton@lse.ac.uk

How to measure living standards and productivity

Nicholas Oulton
Centre for Economic Performance, London School of Economics

Original version: 9 September 2009
Revised version: 28 July 2010

ABSTRACT

I set out a general algorithm for calculating true cost-of-living indices when demand is not homothetic and when the number of products may be large. The non-homothetic case is the important one empirically as Engel's Law demonstrates. The algorithm can be applied in both time series and cross section, eg to estimate true PPPs. The algorithm can also be used to estimate true producer price indices and consequently TFP in the presence of input-biased economies of scale. The data required are no more than are needed for the construction of conventional index numbers.

JEL codes: C43, D11, D12, E31, D24, I31, O47

Key words: consumer price index, Konüs, cost of living, measurement of welfare change, Quadratic Almost Ideal Demand System, producer price index, homothetic, productivity

1. Introduction[1]

This paper sets out an algorithm for measuring the true cost of living in the important case where demand is non-homothetic. The algorithm can be applied both to time series and to cross sections, eg cross-country studies of living standards. Essentially the same algorithm can be applied to the parallel problem of measuring the price of producers' inputs, which in turn is a step on the road to measuring technical progress. The algorithm is practical since it requires no more data than is needed to calculate conventional index numbers. And in principle it can be implemented at the same level of product detail at which conventional index numbers are constructed by national statistical agencies.

Economic theory tells us how to measure the true cost of living: estimate the expenditure function econometrically and then calculate the Konüs price index. The Konüs price index for period t relative to some other period r is defined as the ratio of the (minimum) cost of achieving a given utility level at the prices of period t to the cost of achieving the same utility level at the prices of period r (Konüs, 1939). If we know the expenditure function then we can calculate the Konüs price index, for any chosen utility level. Similarly, economic theory tells us how to measure the true index of the cost of a producer's inputs: estimate the producer's cost function and calculate the analogue of the Konüs price index. If we know the cost function then we also know the degree of economies of scale, the size of any input biases in economies of scale, the growth rate of technical progress, and the size of any input biases in technical progress.

However, though much work has been done on estimating systems of consumer demand or producers' cost functions, the results of these studies are not typically employed by other economists in empirical work. For example, when macro economists study inflation empirically, they do not usually employ their micro colleagues' estimates of expenditure functions. Rather they use consumer price indices constructed by national statistical agencies. The reason is clear. The economic approach cannot be applied at a level useful for other empirical economists because of data limitations.

[1] I owe thanks to Erwin Diewert for detailed comments and helpful suggestions; he is not responsible for my conclusions or any errors. This paper benefited from the comments of participants at the 2010 Royal Economic Society conference (Surrey University, Guildford) and the 6th North American Productivity Workshop (Rice University, Houston), particularly Bert Balk. I am grateful also to the U.K. Economic and Social Research Council which has financed this research under ESRC grant number RES-000-22-3438.

1.1 The data problem

The economic approach cannot be employed because the number of parameters to be estimated is large and the number of observations is comparatively small. In other words the problem is a purely practical one which might in theory be solved just by waiting long enough (possibly for hundreds of years). This causes a dilemma for the empirical economist who is unwilling to wait. Either the economic approach must be abandoned and index numbers employed instead. Or the data must be aggregated and the economic approach applied at a higher level. The first way, I shall argue later, is perfectly all right if demand (for consumer goods or producer inputs) is homothetic. But if it is not, then index numbers will not measure what they are supposed to measure. The second approach is more relevant to testing economic theory rather than to using it. In practice, empirical economists tend to use the index numbers (for output, inputs and prices) supplied to them by statistical agencies, without asking too many questions about the assumptions on which they are based.[2]

The data problem can be illustrated by taking the Quadratic Almost Ideal Demand System (QAIDS) for N products of Banks, Blundell and Lewbel (1997) as an example. In the expenditure function of this system there are $\frac{1}{2}(N-1)(N+2)$ independent parameters relating to the consumer's response to prices and $2(N-1)$ independent parameters relating to the consumer's response to income, for a total (excluding a scale parameter) of $\frac{1}{2}(N-1)(N+6)$ independent parameters. The QAIDS is a system of $N-1$ independent equations for the expenditure shares. Roughly speaking, each of these equations contains on average $\frac{1}{2}(N+2)$ independent coefficients relating to prices and two coefficients relating to income. To have any chance of estimating these coefficients econometrically we must have more observations than coefficients; ie if we have T aggregate time series observations, then we require $T > \frac{1}{2}(N+6)$.

This is where the empirical study of demand and the practice of index number construction part company. National statistical agencies construct their indices of the cost of living from hundreds of components. For example, the U.S. Bureau of Labor Statistics

[2] See for example the remarks of Tobin (1987) on the contributions of Irving Fisher to index number theory: "These index number issues do not seem as important to present-day economists as they did to Fisher. Knowing that they are intrinsically unsolvable, we finesse them and use uncritically the indexes that government statisticians provide". Of course, I do not agree that these "index number issues" are "intrinsically unsolvable", otherwise I would not have written this paper.

constructs its Consumer Price Index from 305 "entry-level items" (U.S. Bureau of Labor Statistics, 2007). The U.K.'s Consumer Prices Index and Retail Prices Index have some 650 "items" (Office for National Statistics, 1998 and 2006). To estimate the parameters of the QAIDS for 650 products would require over three centuries of annual data, a requirement that is not and is never likely to be met. So when econometricians use time series data to test the theory of demand, they are forced to aggregate the products into a small number of groups; eg Christensen *et al*. (1975) tested the theory of demand using three product groups over 1929-72. But additional, strong assumptions on separability are needed to justify this aggregation (Deaton and Muellbauer, 1980b, chapter 5; Blackorby *et al*. (2008)); to test these assumptions would run into the same problem of insufficient data as just outlined and in practice this is never done. So the "prices" and "quantities" which are the basic data for testing the theory of demand in this kind of study are themselves index numbers.[3] But then the theoretical justification for these index numbers is unclear. Cross section studies of household demand fare better since in any given year it may be reasonable to assume prices are the same for all households (except for regional effects). With typically several thousand observations in any cross section, lack of observations is not such a problem. But then only the effects of income (and of household composition) on demand can be measured, as in eg Blow, Leicester and Oldfield (2004).[4]

The upshot is that all the empirical work that economists have done on household demand has had no effect on the measurements actually made by national statistical agencies (although the underlying theory may have been influential). Similar remarks apply to the measurement of other indices such as the producer price index.

1.2 Non-homotheticity

Actually, none of this matters much *provided that demand (for consumer goods or inputs) is homothetic*. If this condition holds and if we are prepared to accept that economic theory is

[3] Latent separability (Blundell and Robin, 2000) imposes fewer restrictions than weak separability. But it is still necessary to estimate a complete demand system in order to determine which goods belong in which groups.

[4] Cross section studies also often employ highly aggregated data: five product groups in the case of Banks *et al.* (1997), eight in the case of Blundell *et al*. (2007), both studies of British household budgets, and 11 in the case of Neary (2004), a cross-country study of 1980 PPPs. The panel study on Canadian households of Lewbel and Pendakur (2009) employed nine groups.

true,[5] then we have no need to estimate cost or expenditure functions. We can instead estimate a discrete approximation to a Divisia index (which I show is the ideal measure in this case), using the superlative index numbers of Diewert (1976) with their flexibility improved by chaining.

Unfortunately, an overwhelming body of empirical evidence establishes that consumer demand is *not* homothetic. The most obvious manifestation of this is Engel's Law: the proportion of total household expenditure devoted to food falls as expenditure rises. Since its original publication in 1857, Engel's Law has been repeatedly confirmed. Houthakker (1957) showed that the Law held in some 40 household surveys from about 30 countries.[6] Engel's Law also holds in the much more econometrically sophisticated study of Banks *et al*. (1997) on UK household budgets. The prevalence of non-homotheticity is also confirmed by the more disaggregated studies of Blow *et al.* (2004), also on U.K. household budgets, which considered 18 product groups, Oulton (2008) who considered 70 product groups and Oulton (2010), 100 product groups.[7]

If demand is not homothetic, then superlative index numbers are not guaranteed to be good approximations to Konüs price indices, even locally. In fact the true price index may lie outside the Paasche-Laspeyres spread. And the true price index is no longer unique but depends on the reference level chosen for utility (or, for the producer price index, on the

---

[5] Throughout this paper I adopt the economic approach to index numbers; see Diewert (1981) and (2008) for surveys of this and of the alternative axiomatic and stochastic approaches, also Balk (1995) on the axiomatic approach.

[6] Engel's (1857) results for expenditure by households of various income levels in Saxony are described more accessibly in Marshall (1920), chapter IV; see Chai and Moneta (2010) for a modern account of Engel's work. In each of the surveys that he collected Houthakker (1957) estimated the elasticity of expenditure on food and three other groups (clothing, housing and miscellaneous) with respect to total expenditure and to household size. For each product group, he regressed the log of expenditure on that group on the log of total expenditure and the log of family size. He used weighted least squares on grouped data; individual data was not available to him. The results for food were clear-cut: demand was inelastic with respect to expenditure in every survey. The results for clothing and miscellaneous were equally clear-cut: demand was expenditure-elastic. The result for housing was more mixed.

[7] An exception to this consensus is Dowrick and Quiggin (1997). They studied the 1980 and 1990 PPPs for 17 OECD countries, using 38 components of GDP, and argued that the data could be rationalised by a homothetic utility function. But their anomalous finding may be due partly to the fact that the per capita incomes of these countries were fairly similar, partly to the fact that some of the 38 components were not household spending, and partly to the low power of their nonparametric test (Neary, 2004). By contrast Crawford and Neary (2008) found that the cross-country data in Neary (2004) — 11 commodity groups in 60 countries from the World Bank's 1980 ICP — are rationalizable by a single non-homothetic utility function, but not by any homothetic utility function.

reference output level). The fact that the Konüs price index generally varies with the reference utility level is sometimes taken as puzzlingly paradoxical. But it can be given a simple intuitive justification. Consider a household with a very low standard of living spending 60% of its budget on food (as was the case with the working class households studied by Engel in 1857). Suppose the price of food rises by 20%, with other prices constant. Then money income will probably have to rise by close to (0.60 x 20% =) 12%, to leave utility unchanged, since there are limited possibilities for substituting clothing and shelter for food. Compare this household to a modern day British one, spending 15% of its budget on food prepared and served at home (Blow *et al*., 2004). Now the maximum rise in income required to hold utility constant is only (0.15 x 20 =) 3% and probably a good bit less as substitution opportunities are greater.

This leaves the welfare interpretation of conventional consumer price indices and their cross-country cousins, the Purchasing Power Parities (PPPs) constructed by the OECD and the World Bank, somewhat up in the air. If the true price index depends on the reference level of utility, how are we to interpret real world price indices? The answer in the time series context is that a chained, superlative index is likely to be approximately equal to a true price index with reference utility level at the midpoint of the sample period (Diewert, 1976 and 1981; Feenstra and Reinsdorf, 2000; Balk 2004).[8] For a cross-country comparison, the viewpoint will be that of a "middle" country. While there is nothing wrong with this viewpoint, there is no special reason why the midpoint should be so privileged. There is also the disadvantage that when the sample period is extended (or the number of countries in the comparison increased), the viewpoint changes.

A parallel issue arises on the production side and takes the form of input biases in economies of scale: if output is doubled, holding prices and technology constant, does that

---

8 Suppose a utility function exists which rationalises the data but may be non-homothetic. Diewert (1981) showed that there exists a utility level which is intermediate between the levels at the endpoints of the interval under study such that a Konüs price index over this interval, with utility fixed at the intermediate level, is bounded below by the Paasche and above by the Laspeyres. Balk (2004) showed that when the growth of prices is piecewise log linear a chained Fisher price index approximates a Konüs price index over an interval when the reference utility level is fixed at that of some intermediate point in the interval. More precise results are available for specific functional forms. Diewert (1976) showed that a Törnqvist price index is exact for a non-homothetic translog cost function when the reference utility level is the geometric mean of the utility levels at the endpoints; see also Diewert (2009) for extensions. For the AIDS, Feenstra and Reinsdorf (2000) showed that, if prices are growing at constant rates, the Divisia index between two time periods equals the Konüs price index when the reference utility level is a weighted average of utility levels along the path.

leave all cost shares unchanged? The possibility that this is *not* the case has certainly been entertained as a matter of theory, though I am not aware of any substantial body of empirical work devoted to this issue. But such a situation may be quite common. Consider a firm which has fixed and variable costs, where the fixed costs are white collar workers and the variable costs are blue collar workers. Then an expansion of output will lower the share of white collar workers in total costs. In this case the cost function is non-homothetic and also non-homogeneous in output. So it would certainly seem desirable to take non-homotheticity into account when trying to measure TFP.

1.3 The algorithm

The proposed algorithm can be summarised as follows. The growth rate of a Konüs consumer price index resembles that of a Divisia index (or the latter's empirical counterpart, a chain index) in that it is an expenditure-share-weighted average of the growth rates of the component prices. But for the Konüs index the shares are not the actual, observed ones, but rather what I call the compensated shares: the shares that would be observed if prices were the actual, observed ones but utility were held constant at some given reference level. I derive a relationship between the compensated and the actual shares: the compensated shares are equal to the actual ones, adjusted for the difference in real income (utility) between the actual situation and the reference level. The adjustment requires us to know, for each product, the consumer's response to real income changes but not the response to price changes. This is why the algorithm can be implemented at a very disaggregated level, since the number of parameters needed to describe the consumer's response to income changes is quite small: in the case of the QAIDS only two parameters for each product need to be known. These income response parameters can be estimated econometrically, provided we do not try at the same time to estimate the responses to individual price changes. This can be done by estimating a flexible demand system such as the QAIDS but with the price variables replaced by a much smaller number of principal components. In this way the data limitation problem can be overcome.

It is important to note that the algorithm proposed here is not designed as a test of whether the theory of consumer (or producer) demand is true. Rather it seeks to *use* demand theory to construct better measures of living standards and productivity. In fact, the algorithm *assumes* that demand theory is true and hence that the consumer's or producer's responses can be approximated by a flexible system like the QAIDS.

1.4 Plan of the paper

I start in section 2 with the homothetic case. I show that a Divisia index provides an ideal measure and that this can be well approximated by a chained, superlative index number. In section 3 I go on to consider the non-homothetic case and present a general algorithm for estimating a true (Konüs) price index for a representative consumer. The algorithm requires just the same data (and no more) as would be required to estimate a conventional index number. This algorithm is illustrated more specifically for the QAIDS. I argue that it can be applied both to time series and to cross section (eg cross country studies). In section 4 the analysis is extended by dropping the assumption of a representative consumer. I show how the QAIDS can be adapted to allow for inequality in the distribution of income. It turns out that this just requires adding two additional variables, both statistics of the income distribution, to the share equations of the QAIDS. The algorithm derived for the simpler case of a representative consumer can then be applied much as before. This section also discusses including household characteristics as additional determinants of demand. Section 5 shows how the general method applies, after some adaptation, to the estimation of a true input price index for producers, in the case where economies of scale may exist and may be input-biased. The algorithm enables input biases in economies of scale and in technical progress to be estimated simultaneously. Finally, section 6 concludes.

2. Price indices: the homothetic case

In this section I argue that chained, superlative index numbers have solved the problem of measuring the true cost of living for a single, representative consumer in the case where demand is homothetic.

Let the consumer's expenditure function be

$$x = E(\mathbf{p}, u), \quad \partial x / \partial u > 0$$

This shows the minimum expenditure x needed to reach utility level u when $\mathbf{p} = (p_1\, p_2 \ldots p_N)$ is the Nx1 price vector faced by the consumer**;** $x = \sum_i p_i q_i$ where the q_i are the quantities purchased. Expenditure at time t is therefore a function of prices at time t and the utility level. The expenditure function is assumed to possess derivatives of all orders. Suppose that, hypothetically, utility were held at its level at time *b* while the consumer faced the prices of time *t*. Let $x(t,b)$ denote the minimum expenditure at the prices of time *t* required to achieve the utility level of time *b*. Then

$$x(t,b) = E(\mathbf{p}(t), u(b)) \tag{1}$$

For brevity write the right hand side as

$$E(t,b) = E(\mathbf{p}(t), u(b))$$

where the first argument of $E(t,b)$ is the time period for prices and the second is the time period for utility. The Konüs price index at time *t*, with time *b* as the base period for utility, is defined as the ratio of the minimum expenditure required with the prices of time *t* to attain the utility level of time *b*, to the minimum expenditure required to attain this same utility level, when the consumer faces the prices of time *b*:[9]

$$P^K(t,b) = E(t,b) / E(b,b) \tag{2}$$

(Clearly, $P^K(b,b) = 1$). In general, the Konüs price index depends on both the prices and the specified utility level. However as is well known, the index is independent of the utility level and depends only on the prices if and only if demand is homothetic, ie if all income elasticities are equal to one (Konüs, 1939; Samuelson and Swamy, 1974; Deaton and Muellbauer, chapter 7, 1980b).

Let s_i denote the share of product *i* in total expenditure. Applying Shephard's Lemma to the expenditure function, we obtain the *share functions*:

$$s_i = \frac{\partial \ln E(\cdot,\cdot)}{\partial \ln p_i}, \quad i = 1,\ldots,N \tag{3}$$

The expenditure shares clearly depend on both prices and utility. Let the share of product *i* in total expenditure at time *t*, if utility were fixed at the level of period *v*, be $s_i(t,v)$. Evaluating this function with the prices of time *t* and the utility level of time *b* we have

$$s_i(t,b) = \frac{\partial \ln E(t,b)}{\partial \ln p_i(t)}, \quad i = 1,\ldots,N$$

These can be called the hypothetical or *compensated* (Hicksian) shares, the shares that would be observed if utility were held constant at some reference level (here, the level prevailing in

9 It is convenient if the reference period for the Konüs price index (the period when the index equals 1) is the same as the base period. But nothing important would be changed if we chose the reference period to be *r* and defined the Konüs price index with base period *b* and reference period *r* as $P^K(t,b,r) = E(t,b)/E(r,b) = [E(t,b)/E(b,b)]/[E(r,b)/E(b,b)] = P^K(t,b)/P^K(r,b)$.

period *b*), while prices followed their observed path. The actual, observed shares in period *t* are

$$s_i(t,t) = \frac{\partial \ln E(t,t)}{\partial \ln p_i(t)}, \quad i = 1,\ldots,N$$

Note that the compensated shares in the base period *b*, $s_i(b,b)$, are the same as the actual shares in that period.

By totally differentiating the Konüs price index of equation (2) with respect to time, we obtain

$$\frac{d \ln P^K(t,b)}{dt} = \sum_{i=1}^{i=N} \frac{\partial \ln E(t,b)}{\partial \ln p_i(t)} \frac{d \ln p_i(t)}{dt} = \sum_{i=1}^{i=N} s_i(t,b) \frac{d \ln p_i(t)}{dt} \quad (4)$$

So the level of the Konüs price index in some period *T*, relative to its level in the base period *b*, is found by integration:

$$\ln P^K(T,b) = \int_b^T \left[\sum_{i=1}^{i=N} s_i(t,b) \left(\frac{d \ln p_i(t)}{dt} \right) \right] dt, \quad P^K(b,b) = 1 \quad (5)$$

The Konüs price index resembles a Divisia index (P^D) which can be written as:

$$\ln P^D(T,b) = \int_b^T \left[\sum_{i=1}^{i=N} s_i(t,t) \left(\frac{d \ln p_i(t)}{dt} \right) \right] dt, \quad P^D(b,b) = 1 \quad (6)$$

The only difference between them is that the Konüs index employs the compensated, not the actual, shares as weights (Balk, 2005; Oulton, 2008).[10] However, in the homothetic case the compensated and the actual shares are always the same: $s_i(t,b) = s_i(t,t), \forall i,b$, since shares depend only on prices, not on utility (or real income); that is, the Konüs and Divisia indices are identical. So in this case the task of index number theory is to find the best discrete approximation to the continuous Divisia index of equation (6).

In fact in the homothetic case the problem of estimating true cost-of-living indices and indices of the standard of living, together with their counterparts on the production side, has been solved, at least within the limit of what is empirically possible. The solution was in fact provided by Diewert’s superlative index numbers, index numbers which are exact for some flexible functional form (Diewert, 1976). In the homothetic case, the true index is bounded by the Laspeyres and Paasche indices (Konüs, 1939). But superlative index numbers are only

[10] Since it is a line integral, the Divisia index is in general path-dependent unless demand is homothetic, as its inventor Divisia (1925-26) was well aware; see Hulten (1973) for detailed discussion and Apostol (1957), chapter 10, for the underlying mathematics. But the Konüs price index is not path-dependent since by definition utility is being held constant along the path (Oulton, 2008).

guaranteed to be good approximations locally, so they need to be chained together in order to approximate better the continuously changing weights in the Divisia index (6).[11,12]

Unfortunately, the assumption of homotheticity is a very dubious one for consumer demand. As argued earlier, there is overwhelming evidence from household surveys that income elasticities are not all equal to one. Economists have been somewhat readier to accept the assumption of constant returns to scale in the case of producers, but even so this assumption should ideally be tested. The next section therefore turns to the non-homothetic case.

3. Estimating a true cost-of-living index over time: the non-homothetic case

3.1 The Taylor series approach

In this section I consider the problem of how to estimate a true cost-of-living index over time when demand is non-homothetic and there are insufficient time series observations available to estimate the consumer's expenditure function.[13] This might be called the "large *N*, small *T*" problem: there are a large number of products but only a small number of time periods. This is the typical situation faced by national statistics agencies when for example estimating the consumer price index. Throughout this section I assume a single, representative consumer. In the next section this assumption will be relaxed.

Equation (5) shows that in order to calculate the Konüs price index in practice, we need to know the compensated shares, which differ in general from the actual ones in the non-homothetic case. We seek a way of at least approximating the compensated shares, which cannot of course be directly observed (except for the $s_i(b,b)$ which are both the actual and the compensated shares in period *b*). We can do this by expressing the actual shares $s_i(t,t)$ in

[11] Diewert (1976) was well aware of the need for chaining: see his footnote 16. For more on superlative indices, including discussion of the critique of them by Hill (2006), see section A.1 of the Appendix.

[12] Using an axiomatic approach, van Veelen (2002) has proved an impossibility theorem which purports to rule out an economically acceptable solution to the problem of measuring the standard of living, both internationally and intertemporally. However, his 4th and final axiom, "Independence of irrelevant countries" (or irrelevant time periods), would rule out the use of chain indices. On the economic approach the latter are essential to derive good approximations to Divisia indices.

[13] The argument of this section is a generalisation of the one set out in Oulton (2008).

terms of a Taylor series expansion of the compensated shares $s_i(t,b)$ in equation (3) around the point $\ln x = \ln E(t,b)$, ie holding prices constant at their levels at time *t* and varying expenditure (utility). When this is done we can establish the following Proposition:

Proposition 1 The differences between the compensated and the actual shares depend on (a) the difference in real expenditure between the base period and the current period and (b) the consumer's response to real expenditure changes. The differences do *not* depend on the consumer's response to price changes. More precisely,

$$s_i(t,b) = s_i(t,t) - \eta_{i1}(t,b)\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right] - \frac{\eta_{i2}(t,b)}{2!}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right]\right\}^2$$

$$-\frac{\eta_{i3}(t,b)}{3!}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right]\right\}^3 - \dots, \qquad (7)$$

$$i = 1,\dots,N;\ t \in [0,T]$$

where

$$\eta_{ik}(t,b) = \left(\frac{\partial^k s_i(\cdot,\cdot)}{\partial \ln E(\cdot,\cdot)^k}\right)_{\substack{\mathbf{p}=\mathbf{p}(t),\\ x=E(t,b)}}, \quad k = 1,2,\dots;\ i = 1,\dots,N \qquad (8)$$

Proof See section A.2 of the Appendix.

The partial derivative $\eta_{i1}(t,b)$ is the semi-elasticity of the budget share of the *i*th product with respect to expenditure, with prices held constant; it is evaluated at base year utility and at the prices of time *t*. It measures the consumer's response to expenditure changes, as asserted in Proposition 1. These semi-elasticities and the higher order derivatives in (7) measure basic aspects of consumer behaviour. The terms in square brackets measure the proportionate difference between real expenditure at time *t* and at time *b*. Note that if the expenditure function is a *K*th order polynomial in log expenditure, then the Taylor series effectively terminates after *K* terms, since $\eta_{i,K+1} = \eta_{i,K+2} = \dots = 0$. So equation (7) with terms higher than powers of *K* in log expenditure omitted is then exact and not an approximation.

The system of equations (7) might not appear to take us very much further if our goal is to estimate the Konüs price index, since the latter appears on the right hand side. But in fact this system, together with (4), is the basis for a practical method of estimating the Konüs price index. Suppose that the $\eta_{i1}(t,b)$ and the higher order derivatives $\eta_{i2}(t,b)$, $\eta_{i3}(t,b)$, etc, that are required for a good approximation were somehow known or could be estimated (see the next section on ways to do this). Then we could estimate the Konüs price index using equation (4) and (7). This is because these equations constitute a set of equations for $P^K(t,b)$, in which the compensated shares and the Konüs price index are the only unknowns; the actual shares $s_i(t,t)$, the nominal expenditures $x(t,t)$ and $x(b,b)$, and (by assumption) the semi-elasticities ($\eta_{i1}(t,b)$, $\eta_{i2}(t,b)$, etc) are all known.

The general procedure for solving these equations is straightforward in principle. First, we need to take discrete approximations. Equations (7) must be understood to hold in discrete not continuous time, ie for $t = 0,1,...,T$. We must also decide how many terms in the Taylor series are required. If the utility function is quadratic in log expenditure, then only the first two terms of the Taylor series are needed: see the next section. Equation (4) must be replaced by a discrete approximation, eg a chained Törnqvist (P^T) or chained Fisher formula (P^F).

Let us define the following chained, *compensated* index numbers. Each index number is for time t relative to time $t-1$, with utility held constant at the level of period *b*.

Compensated Törnqvist:

$$\ln P^T(t,t-1,b) = \sum_{i=1}^{i=N}\left(\frac{s_i(t,b)+s_i(t-1,b)}{2}\right)\ln\left(\frac{p_i(t)}{p_i(t-1)}\right) \tag{9}$$

Compensated Laspeyres:

$$P^L(t,t-1,b) = \sum_{i=1}^{i=N} s_i(t-1,b)\frac{p_i(t)}{p_i(t-1)} \tag{10}$$

Compensated Paasche:[14]

$$P^P(t,t-1,b) = \left[\sum_{i=1}^{i=N} s_i(t,b)\frac{p_i(t-1)}{p_i(t)}\right]^{-1} \tag{11}$$

Compensated Fisher:

$$P^F(t,t-1,b) = [P^L(t,t-1,b)\cdot P^P(t,t-1,b)]^{1/2} \tag{12}$$

Each of these index numbers is defined in the same way as its empirical counterpart, except that compensated, not actual, shares are used. The natural choices for discrete approximations to the continuous Konüs price index are either the compensated Törnqvist, equation (9), or the compensated Fisher, equation (12). We now have

[14] The formula for the Paasche is not the usual one but is mathematically equivalent to the usual one.

Proposition 2 The true index is bounded by the compensated Laspeyres and the compensated Paasche:

$$P^L(t,t-1,b) \geq P^K(t,b) / P^K(t-1,b) \geq P^P(t,t-1,b) \qquad (13)$$

Proof Since utility is being held constant at its level in period *b*, this follows from the well-known Konüs (1939) inequalities: see section A.2 of the Appendix for the details.

The Paasche-Laspeyres spread, calculated using the compensated shares, can be used as a check on the accuracy of whatever index number formula is adopted.[15]

Equations (7) now constitute a system of $(N-1)(T+1)$ independent equations since the N shares sum to one in each period.[16] Together with (4), this system can be solved iteratively:[17]

1. Start with an initial guess at $P^K(t,b)$: this could be derived as a chained Törnqvist or chained Fisher index which uses actual not compensated shares.

2. Substitute this estimate of $P^K(t,b)$ into (7) to get estimates of the compensated shares for each of $N-1$ products and for each of $T+1$ time periods; the share of the *N*th product can be derived as a residual.

3. Use these estimates of the compensated shares to obtain a new estimate of $P^K(t,b)$ from either of the two discrete approximations to (4), the Törnqvist (equation (9)) or the Fisher (equation (12)).[18]

4. Check whether the estimate of $P^K(t,b)$ has converged. If not, return to step 2.

The intuition behind this result is as follows. In the homothetic case it turns out that we do not need to know the individual parameters of the expenditure function: the observed shares encapsulate all the required information. In the non-homothetic case, we need to know the compensated shares. These can be thought of as like the actual shares, but contaminated by the effects of changes in real income (expenditure). What is needed is to purge the actual shares of income effects.

The algorithm is not guaranteed to converge; the convergence issue is discussed in section A.2 of the Appendix. A practical approach to convergence is suck it and see. If the

[15] Of all superlative index numbers, only the Fisher is guaranteed to lie within the Laspeyres-Paasche spread (Hill, 2006).

[16] The actual shares of course sum to one and since they derive from the expenditure function so do the compensated shares: see equation (3).

[17] If the Engel curves are log-linear, ie all the η_{ik} are zero except the η_{i1}, then the whole system is linear and an explicit solution for the compensated shares is available: see section A.2 of the Appendix.

[18] In step 3 of the algorithm it is assumed that the observations are arranged in the natural time order. See below for a refinement.

algorithm diverges there are refinements which improve the chances of convergence: see the discussion of dampening in section 3.9 of Judd (1998).

So given knowledge of the η_{ik} up to the required order, we can estimate the Konüs price index. Estimating the η_{ik} themselves may still seem a difficult task but notice that only the response of demand to changes in real income needs to be known, not the response to price changes. This is a very significant reduction in the complexity of the task empirically.

It is possible that estimates of the η_{ik} are available "off the shelf" in which case the problem is solved. The response to expenditure changes can be estimated from cross-section data since prices can usually be assumed to be the same for all households in a given region (see eg Blow *et al*. (2004)). But cross-section estimates may not be available[19] or, even if they are, the product classification may be different. In the absence of ready-made estimates, is it possible to estimate these parameters from the aggregate data available to national statistical agencies — the same data that they use to construct conventional index numbers? The answer is yes. To make further progress I turn now to consider systems of demand which are consistent with economic theory and also seem capable of fitting the data reasonably well.

3.2 Demand systems

The PIGLOG demand system, introduced by Muellbauer (1976) (see also Deaton and Muellbauer (1980a and 1980b, chapter 3)) has found wide application empirically; an example of the PIGLOG is the Almost Ideal Demand System (AIDS). The PIGLOG expenditure function is:

$$\ln x = \ln A(\mathbf{p}) + B(\mathbf{p}) \ln u \tag{14}$$

Here $A(\mathbf{p}) \geq 0$ and $B(\mathbf{p}) > 0$ (non-satiation). Also, $A(\mathbf{p})$ is assumed homogeneous of degree one and $B(\mathbf{p})$ homogeneous of degree zero in prices.

This expenditure function gives rise to Engel curves which are linear in the log of expenditure. However, a linear relationship does not fare well empirically (Banks *et al*., 1997; Blow *et al*., 2004; Oulton, 2008) and it is found necessary to add a squared term in the log of expenditure to the share equations. It turns out that this can be justified theoretically along the following lines. Lewbel (1991) defined the rank of a demand system to be the

[19] The latest round of the World Bank's International Comparison Program has generated prices and expenditures for 106 products classified to "Actual individual consumption", for each of 146 countries. But there are no corresponding micro data for these countries.

dimensions of the space spanned by its Engel curves. Exactly aggregable demand systems are those which are linear in functions of *x*. Gorman (1981) proved that the maximum possible rank of any exactly aggregable demand system is 3. The empirical evidence on Engel curves indicates that observed demands are at least rank 3. Theorem 1 of Banks *et al.* (1997) states that all exactly aggregable, rank 3, demand systems which just add a differentiable function of deflated expenditure to the utility function corresponding to equation (14) are derived from a utility function of the form

$$\ln u = \left\{ \left[\frac{\ln x - \ln A(\mathbf{p})}{B(\mathbf{p})} \right]^{-1} + \lambda(\mathbf{p}) \right\}^{-1} = \frac{\ln[x / A(\mathbf{p})]}{B(\mathbf{p}) + \ln[x / A(\mathbf{p})]\lambda(\mathbf{p})} \qquad (15)$$

where $\lambda(\mathbf{p})$ is a differentiable, homogeneous function of degree zero in prices $\mathbf{p}$ and $\lambda(\mathbf{p}) \geq 0$. The corresponding expenditure function is:

$$\ln x = \ln A(\mathbf{p}) + \frac{B(\mathbf{p}) \ln u}{1 - \lambda(\mathbf{p}) \ln u} \qquad (16)$$

(This reduces to the PIGLOG system (14) when $\lambda(\mathbf{p}) = 0$).

Applying Shephard's Lemma, and after substituting for *u* from (15), the expenditure shares in this demand system are:

$$s_i = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \frac{\ln[x / A(\mathbf{p}]}{B(\mathbf{p})} \frac{\partial B(\mathbf{p})}{\partial \ln p_i} + \frac{[\ln[x / A(\mathbf{p})]]^2}{B(\mathbf{p})} \frac{\partial \lambda(\mathbf{p})}{\partial \ln p_i} \qquad (17)$$

I now follow Banks *et al.* (1997) and adopt the Quadratic AIDS (QAIDS) specification as an example of a generalised PIGLOG system. They specify that

$$B(\mathbf{p}) = \prod_{k=1}^{k=N} p_k^{\beta_k}, \sum_{k=1}^{k=N} \beta_k = 0 \qquad (18)$$

and

$$\lambda(\mathbf{p}) = \sum_{k=1}^{k=N} \lambda_k \ln p_k, \sum_{k=1}^{k=N} \lambda_k = 0 \qquad (19)$$

Under this specification,

$$\frac{\partial \ln B(\mathbf{p})}{\partial \ln p_i} = \beta_i$$

$$\frac{\partial \lambda(\mathbf{p})}{\partial \ln p_i} = \lambda_i$$

so the system of share equations (17) becomes

$$s_i = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln\left[\frac{x}{A(\mathbf{p})}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{\ln\left[\frac{x}{A(\mathbf{p})}\right]\right\}^2 \tag{20}$$

What is the relationship between compensated and actual shares in this demand system? In equation (7) above we found a Taylor series expansion for the compensated shares which involved the semi-elasticity of the shares with respect to real income, $\partial s_i / \partial \ln E$, and higher order derivatives, $\partial^2 s_i / \partial \ln E^2$, etc. Now from (17) we get that

$$\frac{\partial s_i}{\partial \ln x} = \beta_i + \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \ln\left[\frac{x}{A(\mathbf{p})}\right]$$

$$\frac{\partial^2 s_i}{\partial [\ln x]^2} = \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \tag{21}$$

and higher order derivatives are zero.

These derivatives have to be evaluated when $x = E(t,b)$. The simplest way to do this is to adopt the normalisation that $\ln u(b) = 0$. This is always possible by appropriate choice of utility units. It now follows also from (16) that

$$\ln x(t,b) = \ln A_b(\mathbf{p}(t)) + \frac{B(\mathbf{p}(t)) \ln u(b)}{1 - \lambda(\mathbf{p}(t)) \ln u(b)} = \ln A_b(\mathbf{p}(t)) \tag{22}$$

Here and from now on, I write $A_b(\mathbf{p})$ rather than just $A(\mathbf{p})$, to mark the fact that this normalisation changes matrix A.

We can now use these results to evaluate the derivatives in (21) at the point $x = E(t,b)$, $\mathbf{p} = \mathbf{p}(t)$:

$$\eta_{i1}(t,b) = \left[\frac{\partial s_i}{\partial \ln x}\right]_{\substack{\mathbf{p}=\mathbf{p}(t)\\ x=E(t,b)}} = \beta_i + \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}(t)} \ln\left[\frac{x(t,b)}{A_b(\mathbf{p}(t))}\right]$$

$$= \beta_i$$

using (22) and

$$\eta_{i2}(t,b) = \left[\frac{\partial^2 s_i}{\partial [\ln x]^2}\right]_{\substack{\mathbf{p}=\mathbf{p}(t)\\ x=E(t,b)}} = \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}(t)}$$

Substituting these results into (7) we obtain

$$s_i(t,b) = s_i(t,t) - \beta_i \ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right] - \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right]\right\}^2, \tag{23}$$

$$i = 1,2,...,N; \;\; t = 0,1,...,T$$

and this Taylor series expansion is not an approximation but is exact for the generalised PIGLOG with the specification of (18) and (19).

As a further step towards putting the demand system into a form which can be estimated in practice, it is helpful to use (20) and (22) to write the equations for the observed shares at time t as:

$$s_i(t,t) = \frac{\partial \ln A_b(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i \ln\left[\frac{x(t,t)/x(b,b)}{A_b(\mathbf{p}(t))/A_b(\mathbf{p}(b))}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{A_b(\mathbf{p}(t))/A_b(\mathbf{p}(b))}\right]\right\}^2 \tag{24}$$

Here we have used the fact that, from (22), $x(b,b) = A_b(\mathbf{p}(b))$.

One further result involving the interpretation of the Konüs price index is also needed. From the definition of the Konüs price index, equation (2), and equation (22), we find that for the generalised PIGLOG system:

$$\begin{aligned} \ln P^K(t,b) &= \ln E(t,b) - \ln E(b,b) = \ln x(t,b) - \ln x(b,b) \\ &= \ln A_b(\mathbf{p}(t) - \ln A_b(\mathbf{p}(b)) \end{aligned} \tag{25}$$

ie $P^K(t,b) = A_b(\mathbf{p}(t))/A_b(\mathbf{p}(b))$. Substituting this into the share equations (24),

$$s_i(t,t) = \frac{\partial \ln A_b(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i \ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right]\right\}^2 \tag{26}$$

The QAIDS specification will now be used to show how the Konüs price index can be estimated in practice, when there are too few observations to estimate all the parameters of the expenditure function.[20]

[20] Lewbel and Pendakur (2009) have recently proposed a new demand system, the Exact Affine Stone Index (EASI) system. This has all the advantages of the generalised PIGLOG (and of the QAIDS) while allowing Engel curves to be still more flexible, eg polynomials of cubic or higher order. In principle the method developed here could be applied to the EASI system as well. However, I have not been able to develop tractable expressions for the derivatives of the share equations with respect to log expenditure (the η_{ik}). From the point of view of the present paper, the EASI system suffers from the disadvantage that exact aggregation does not hold. This does not matter when the system is fitted to individual data but does when fitted to aggregate data: see section 4 for discussion of aggregation over consumers who may differ in income and in other ways.

3.4 The estimation procedure

In order to implement the procedure outlined above for estimating the Konüs price index, we need to estimate only the N β_i parameters and the N λ_i parameters of equations (26); in both cases only $N-1$ of these are independent because these coefficients each sum to zero across the products. That is, $2(N-1)$ parameters in total need to be estimated or just two per share equation. These parameters determine the consumer's response to changes in real expenditure. We do *not* need to estimate the much more numerous parameters which determine the response to price changes. This is a huge reduction in the difficulty of the task.

Even if we need only the expenditure response parameters, how can we estimate these while avoiding estimating all the other parameters of the system at the same time? After all, if we just estimate the share equations with the price variables omitted then our estimates of the expenditure response will undoubtedly be biased, since relative prices and real expenditures are likely to be correlated over time (and across countries). The answer is to collapse the $N-1$ relative prices in the system into a smaller number of variables using principal components.[21] We can collapse the relative prices into (say) M principal components, where $M < N-1$ is to be chosen empirically.

The share equations (26) can now be written in a form suitable for econometric estimation by replacing the individual price variables by principal components and adding an error term:

$$s_i(t,t) = \alpha_i^b + \sum_{k=1}^{M} \theta_{ik} PC_k(t) + \beta_i z(t,b) + \lambda_i y(t,b) + \varepsilon_i(t), \qquad i = 1,\dots,N;\ t = 0,\dots,T \tag{27}$$

Here α_i^b is the base-year-dependent constant term ($\sum_i \alpha_i^b = 1$); $PC_k(t)$ is the kth principal component of the $N-1$ relative prices; the θ_{ik} are coefficients subject to the cross-equation restrictions $\sum_i \theta_{ik} = 0,\ \forall k$; ε_{it} is the error term; and we have put $z(t,b) = \ln[x(t,t) / P^K(t,b)]$ and $y(t,b) = [z(t,b)]^2 / \prod_k p_k^{\beta_k}(t)$. The presence of the principal components in equation (27)

[21] See Johnson and Wichern (2002) for a textbook exposition of principal components.

means that the estimates of the coefficients on *z* and *y* need not be biased as they would be if prices were simply omitted.[22]

We have now reduced the problem to estimating a system of $N-1$ independent equations, each of which contains only $M+3$ coefficients — the θ_{ik} (M in number), α_i, β_i and λ_i.[23] The success of this strategy will depend on whether the variation in relative prices can be captured by a fairly small number of principal components — small that is in relation to the number of time series observations, $T+1$. This is obviously an empirical matter. At one extreme, if there is little or no correlation between the prices over time (or space), then the use of principal components yields no benefit. At the other extreme, suppose that the demand system is specified in terms of the logs of prices and that all relative prices are just loglinear time trends, though the growth rate varies between prices. The evolution of relative prices can be written as:

$$\ln[p_j(t)/p_1(t)] = \mu_j t, \quad j = 2,...,N$$

where the μ_j are the growth rates and the first product is taken as the numeraire. Assume too that the matrix $A(\mathbf{p})$ takes the AIDS form:

$$\ln A(\mathbf{p}) = \alpha_0 + \sum_i \alpha_i \ln p_i + (1/2)\sum_i \sum_j \gamma_{ij} \ln p_i \ln p_j, \ \ \sum\nolimits_i \alpha_i = 1, \sum\nolimits_i \gamma_{ij} = \sum\nolimits_j \gamma_{ij} = 0, \ \gamma_{ij} = \gamma_{ji}$$

Then in the ith share equation (26) the price effects are

$$\frac{\partial \ln A(\mathbf{p}(t))}{\partial \ln p_i} = \alpha_i + \sum\nolimits_{j=1}^{N} \gamma_{ij} \ln p_j(t) = \alpha_i + \sum\nolimits_{j=2}^{N} \gamma_{ij} \ln[p_j(t)/p_1(t)]$$

$$= \alpha_i + t\left[\sum\nolimits_{j=2}^{N} \gamma_{ij}\mu_j\right] = \alpha_i + \delta_i t, \text{ say}$$

(Here we have used the fact that $\sum_j \gamma_{ij} = 0$). In this case the effect of changing relative prices is captured entirely by a time trend, with a different coefficient in each share equation (subject to the cross-equation restriction that $\sum_i \delta_i = 0$). So just one principal component

[22] The empirical flexibility of equation (27) could be increased by adding cubic and higher order terms in $z(t,b)$. (The coefficients on these additional terms must be constrained to sum to zero across products). The implied expenditure function could not now be written down in closed form but the share equations extended in this way could be regarded as polynomial approximations to the exact ones. However, in the presence of cubic and higher order terms the property of exact aggregation would no longer hold, making it hard to interpret the results in terms of individual welfare. See the next section for more on aggregation.

[23] This is not quite true since all the β_i appear in each equation via the denominator of *y*. We can handle this by an iterative procedure: see below.

captures the whole variation in relative prices (ie in this case $M = 1$). This is an extreme case and in practice we must expect that more than one principal component will be required to capture the variation in relative prices.[24]

The specification of the principal components depends on the demand system chosen. If we chose the AIDS (and QAIDS) form for $A(\mathbf{p})$, then it would be natural to estimate the principal components in terms of log relative prices, eg $\ln(p_j / p_1),\ j = 2,...,N$, taking the first product as the numeraire. Alternatively, we might use the normalised quadratic of Diewert and Wales (1988), in which case the principal components would be estimated in terms of relative prices (not in logs).

In estimating equations (27) econometrically, it is straightforward to impose the adding-up and homogeneity restrictions on the coefficients; homogeneity is imposed by using relative prices and adding-up is imposed by cross-section restrictions on the coefficients (these restrictions are automatically imposed by OLS though the latter is not necessarily the best method). But there is one loss from using principal components: we can no longer impose the symmetry restrictions.[25]

Equations (27) are nonlinear in the parameters of interest, since to measure both z and y correctly it is necessary to know the Konüs price index, the object of the whole exercise; in addition, to measure y we also need to know all the β_i and λ_i. The solution is an iterative process, similar to the one described in the previous section. Here the unknown parameters, the β_i and λ_i, are estimated jointly with the compensated shares and the Konüs price index. The system consists of equations (23), (27), and the equation for the Konüs price index, either equation (9) if we use a compensated Törnqvist to approximate the Konüs or equation (12) if we use a compensated Fisher. The iterative process for some particular choice of the base period is as follows:

---

[24] Oulton (2008) applied the method to 70 products covering the whole of the U.K.'s Retail Prices Index over 1974-2004. He found that six principal components were sufficient to capture 97.8% of the variation in the 69 log relative prices.

[25] For example, suppose that $N = 3$ and that the special case of all relative prices changing at constant rates applies. Then, dropping the third equation, taking the first product as the numeraire, and imposing all the constraints, the relationship between the δ_i and the γ_{ij} is as follows: $\delta_1 = \gamma_{12}\mu_2 - (\gamma_{11} + \gamma_{12})\mu_3$, $\delta_2 = \gamma_{22}\mu_2 - (\gamma_{12} + \gamma_{22})\mu_3$. These relationships imply no further restrictions on δ_1 and δ_2. So we cannot test whether $\gamma_{12} = \gamma_{21}$.

1. Obtain initial estimates of the Konüs price index $P^K(t,b)$ and of the β_i and λ_i coefficients. An initial estimate of $P^K(t,b)$ can be obtained from equation (9) or equation (12) by using actual instead of compensated shares (ie replace $s_i(t,b)$ by $s_i(t,t)$ in the formulas). And for an initial estimate of the β_i and λ_i, set $\beta_i = \lambda_i = 0, \forall i$.
2. Derive estimates of $z(t) = \ln[x(t)/P^K(t,b)]$ and of $y(t) = [z(t,b)]^2 / \prod_k p_k^{\beta_k}$, using the latest estimates of $P^K(t,b)$ and of the β_i. Using these new estimates of z and y, estimate equation (27) econometrically, to obtain new estimates of the β_i and λ_i.
3. Using the new estimates of the β_i and λ_i, estimate the compensated shares from equation (23). Then use the compensated shares to derive a new estimate of the Konüs price index $P^K(t,b)$ from equation (9) or equation (12).
4. If the estimate of the Konüs price index has changed by less than a preset convergence condition, stop. If not, go back to step 2.[26]

Finally, the estimates of the β_i and λ_i produced by the algorithm above can be plugged into the simpler algorithm of section 3.1 to generate Konüs price indices for any other base year.

3.6 Comparisons across space

The analysis carries over unchanged to the problem of estimating a cost of living index and hence the standard of living across countries at a point in time.[27] The solution for the Konüs price index given by equations (7) and (5) can be applied directly in the cross-country context. Initially we must imagine a continuum of countries indexed by t just as in section 3 we imagined a continuum of time periods. Then we consider discrete approximations; ie as before equation (5) can be approximated by either (9) or (12).

---

[26] This is the same as the Iterated Linear Least squares Estimator (ILLE) proposed by Blundell and Robin (1999). They prove that the limit values of these parameter estimates are consistent.

[27] See Hill (1997) for a survey of methods of making international comparisons. Caves *et al.* (1982) have applied chained superlative index numbers to cross-country comparisons. Hill (2004) also estimates a chain superlative index but employs the minimum-spanning tree approach to find the best links in the chain. Neary (2004) employed the World Bank's 1980 PPPs for 60 countries and 11 commodity groups to estimate a QAIDS; he then derived a measure of real GDP per capita for the 60 countries. The World Bank's current methodology for deriving PPPs at the aggregate level is set out in World Bank (2008).

One problem which is often said to arise in the cross-country but not the inter-temporal context is that, unlike time, countries have no natural order. In the present case this objection does not apply. Here the natural order for countries is the ranking by real income (or real expenditure) per capita. Adopting this order minimises the gap between country t and country $t-1$ and so should improve the discrete approximation. It is true that the rank order is not known for certain in advance, since the whole point of the exercise is to estimate the true standard of living. This suggests a refinement to the algorithm: at each step, re-order the countries (time periods) so as to put them in rank order of real expenditure per capita (where "real" means deflated by the algorithm's latest estimate of the Konüs price index). Alternatively, the ordering of countries could be determined by the minimum-spanning-tree method suggested by and implemented on cross-country data by Hill (1999). Then the links in the chain would be selected so as to minimise the (compensated) Paasche-Laspeyres spread.[28]

4. Extensions to the basic analysis

The preceding section 3 offered a solution to the problem of estimating a true cost-of-living index over time for a single representative consumer. In this section, I consider two extensions to the analysis. First, I consider the effect of relaxing the assumption of a single representative consumer. I now assume that the aggregate data is generated by heterogeneous consumers who differ in income. If the degree of inequality were constant the preceding analysis could stand unchanged. This may or may not be a reasonable approximation in a time series context over a few decades. But in a cross-country context the assumption is certainly problematic: countries differ widely in the extent of inequality (Anand and Segal, 2008). So we need to extend our framework to encompass this. Second, I consider aggregation over different types of household.

[28] Hill (2004) uses a different criterion, namely minimising a dissimilarity index suggested by Diewert (2002), but this seems less appropriate in the present context.

4.1 Aggregation over rich and poor consumers

Let the population be composed of G groups. The groups are assumed to be of equal size (eg percentiles, deciles or quintiles), with the first group being the poorest and the Gth group the richest. The fraction of households in each group is then $1/G$. Let x_g be mean expenditure per household in the gth group. Within a group, each household's expenditure is the same, namely the group mean. The share of product i in the expenditure of the gth group, s_{ig} ,is then

$$s_{ig} = \frac{p_i q_{ig}}{x_g}$$

where q_{ig} is the quantity per capita of the ith product purchased by each member of the gth group. The share of the ith product in aggregate expenditure is therefore

$$s_i = \frac{p_i q_i}{x} = \frac{\sum_{g=1}^{g=G} p_i q_{ig}}{Gx} = \sum_{g=1}^{g=G} \left[\frac{x_g}{Gx} \frac{p_i q_{ig}}{x_g} \right] = \sum_{g=1}^{g=G} w_g s_{ig} \tag{28}$$

where w_g is the share of the gth group in aggregate expenditure:

$$w_g = \frac{x_g}{Gx}, \quad \sum_{g=1}^{g=G} w_g = 1 \tag{29}$$

We assume that preferences have the Ernest Hemingway property: the rich are different from the poor but only because the rich have more money.[29] So the parameters of the expenditure function are the same for all households. All consumers are assumed to face the same prices. So from (20) and adopting the QAIDS formulation, the share of the ith product in expenditure by the gth group is:

$$s_{ig} = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln \left[\frac{x_g}{A(\mathbf{p})} \right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ \ln \left[\frac{x_g}{A(\mathbf{p})} \right] \right\}^2$$

Using (28), the aggregate share equations are weighted averages of the underlying equations for each group:

[29] The well-known (though apparently fictional (Clark, 2009)) dialogue runs as follows. Fitzgerald: "The rich are different from us, Ernest". Hemingway: "Yes, Scott, they have more money than we do".

$$s_i = \sum_{g=1}^{g=G} w_g s_{ig} = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \sum_{g=1}^{g=G} w_g \ln x_g - \beta_i \ln A_b(\mathbf{p})$$

$$+ \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left[\sum_{g=1}^{g=G} w_g (\ln x_g)^2 - 2 \ln A_b(\mathbf{p}) \sum_{g=1}^{g=G} w_g \ln x_g + [\ln A_b(\mathbf{p})]^2 \right] \quad (30)$$

The difference between this and our previous equation (20) is that instead of the log of aggregate expenditure per capita, $\ln x = \ln\left[\sum_{g=1}^{g=G} x_g / G\right]$, appearing on the right hand side, we now have the share-weighted average of log expenditure per capita in each group, $\sum_{g=1}^{g=G} w_g \ln x_g$; and instead of $(\ln x)^2$, we now have $\sum_{g=1}^{g=G} w_g (\ln x_g)^2$. The relationship between $\sum_{g=1}^{g=G} w_g \ln x_g$ and $\ln x$ is, from (29),

$$\sum_{g=1}^{g=G} w_g \ln x_g = \sum_{g=1}^{g=G} w_g \ln(w_g G x) = \sum_{g=1}^{g=G} w_g \ln w_g + \ln G + \ln x$$

The first term on the right hand side, $\sum_g w_g \ln w_g$, is the negative of entropy (ignoring an unimportant scale constant); it was suggested as a measure of inequality by Theil (1967), chapter 4. Define $I = -\sum_{g=1}^{g=G} w_g \ln w_g$ as entropy and define also the related inequality statistic $J = \sum_{g=1}^{g=G} w_g (\ln w_g)^2$. Substituting these into (30), we find after some manipulation (see section A.3 of the Appendix) that

$$s_i = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \left\{ W_1 + \ln\left[\frac{x}{A_b(\mathbf{p})}\right] \right\}$$

$$+ \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ W_2 + 2\left\{ W_1 \ln\left[\frac{x}{A_b(\mathbf{p})}\right] \right\} + \left\{ \ln\left[\frac{x}{A_b(\mathbf{p})}\right] \right\}^2 \right\} \quad (31)$$

where we have set $W_1 = \ln G - I$ and $W_2 = J - 2I \ln G + (\ln G)^2$. In the case of a perfectly equal distribution (when $w_g = 1/G$), note that $I = \ln G$, $J = (\ln G)^2$, and $W_1 = W_2 = 0$, so that (31) then reduces back down to the original QAIDS formulation, equation (20). Compared to (20), there are two additional variables in (31), W_1 and W_2 , though no additional parameters. These additional variables may help to explain changes in shares, to the extent that inequality varies either over time or across countries. Note too that in the simpler AIDS case (ie when all the λ_i are zero), equation (31) simplifies to

$$s_i = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \left\{ (\ln G - I) + \ln\left[\frac{x}{A_b(\mathbf{p})}\right] \right\} \qquad (32)$$

which contains just one additional variable (I).[30]

The upshot is that the QAIDS can be parsimoniously extended to capture the effect of income inequality. The additional empirical requirement is fairly modest: we need to know the shares of different groups in aggregate expenditure, at a reasonable level of detail.

4.2 Aggregation over different household types

Suppose there are a set of H characteristics that influence demand, in addition to income and prices. These could include household characteristics such as number of children, average age, and educational level, and also environmental characteristics such as climate. Now the share equations of the QAIDS for the gth income group could be written as:

$$s_{ig} = \alpha_i + \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln\left[\frac{x_g}{A(\mathbf{p})}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ \ln\left[\frac{x_g}{A(\mathbf{p})}\right] \right\}^2 + \sum_{n=1}^{n=H} \theta_{ih} K_{hg} \qquad (33)$$

where K_{hg} is the level of the hth characteristic in the gth group; I assume that each household in the gth group has the same level of each of the K_{hg} as all the other households in that group (this entails no loss of generality if there is only one household in each group). The θ_{in} coefficients must satisfy the adding-up restrictions:

$$\sum_{i=1}^{i=H} \theta_{in} = 0, \quad n = 1, 2, \ldots, H$$

(At some cost to parsimony, the model could be extended by interacting the characteristic variables with income). Again, underlying preferences are assumed to be the same but people's situations differ for various reasons, in the spirit of Stigler and Becker (1977):[31] at the same incomes and prices, people in cold climates buy more winter clothes. We can aggregate equation (33) over the income groups to obtain the same result as (31), but with an additional term:

$$+\sum_{h=1}^{h=H} \theta_{ih} K_h$$

---

[30] The role of Theil's inequality measure, entropy (I), was discussed in Deaton and Muellbauer (1980b) chapter 6, section 6.2. They derived a result equivalent to (32).

[31] This approach seems likely to be more fruitful in the present context than assuming that tastes may differ; the latter approach is taken by van Veelen and van der Weide (2008).

where $K_h = \sum_{g=1}^{g=G} w_g K_{hg}$. Now K_h is a weighted average of the level of the *h*th characteristic in a particular country (time period). The only difficulty from an empirical point of view is that it is an income-weighted, not a population-weighted, average. So for example if the rich have fewer children than the poor nowadays, then using the mean number of children per household as a measure would be a misspecification when estimating share equations from aggregate data.[32]

5. Cost functions: estimating input-biased scale economies and technical progress

In this section I look at the parallel problem of estimating an input price index and technical progress when the cost function is not homothetic. Now both economies of scale and technical progress may be input-biased. I assume that the typical firm is a price taker in input markets and wishes to minimise costs. We can write the cost function in general as:

$$x = C(\mathbf{p}, Y, t) \tag{34}$$

Here output (*Y*) plays the role of utility in the expenditure function. While formally this makes no difference, there is a big difference empirically since output is objectively and directly measurable (at least in principle) while utility is only indirectly measurable. The presence of time (*t*) as an indicator of technical progress in the cost function also has no counterpart in the theory of demand.[33]

[32] In Oulton (2010) I have applied the algorithm to estimate PPPs for 141 countries using 100 goods within the category of household consumption. This study allowed for both differences in income within countries and for a large number of household and country characteristics.

[33] The parallel between cost and expenditure functions would be complete if individuals were able to learn over time how to make better use of goods and services in order to generate more utility. In some cases there is very suggestive evidence of a social learning process. The death toll before the Second World War on the roads in Great Britain peaked in 1938 when 6,648 people were killed, of whom 3,046 were pedestrians. By 2006 the annual death toll had fallen to 3,172, of whom 673 were pedestrians, and the death rate per capita had dropped to a third of the earlier level, even though the number of vehicles per capita increased to more than 8 times its 1938 level. (Source: *Annual Abstract of Statistics*, 2008 and 1938-1948 editions). Of course, many things changed over this period but one of them was surely that the habit of looking both ways before stepping into the road became more deeply engrained. The decline in smoking rates as evidence about the health risks has accumulated might also be cited. An alternative justification for making time an argument of

By analogy with equation (16), we can use a generalised PIGLOG formulation:

$$\ln x = \ln C(\mathbf{p}, Y, t) = \ln A(\mathbf{p}) + \frac{B(\mathbf{p})\ln Y}{1-\lambda(\mathbf{p})\ln Y} + \beta_Y \ln Y + \mu(\mathbf{p})t + \mu_t t \tag{35}$$

where Y is output, $x = \sum_i p_i q_i$ is total expenditure on the inputs q_i, and as before $B(\mathbf{p}) > 0$ is homogeneous of degree one in prices and $\lambda(\mathbf{p}) \geq 0$ is homogeneous of degree zero in prices. There are two new elements here. First, the parameter β_Y measures overall economies of scale. When there are no input biases, ie $B(\mathbf{p}) = 1$ and $\lambda(\mathbf{p}) = 0$, then $\beta_Y = 0$ implies constant returns to scale and $\beta_Y < 0$ implies increasing returns. In this case the cost function is homothetic but not necessarily homogeneous of degree one in output. Second, the last two terms on the right hand side of (35) measure technical progress. Neutral technical progress is measured by the parameter μ_t ($\mu_t < 0$ implies that technical progress is positive); input-biased technical progress is measured by the function $\mu(\mathbf{p})$. By analogy with $\lambda(\mathbf{p})$, $\mu(\mathbf{p})$ could be specified as

$$\mu(\mathbf{p}) = \sum_{k=1}^{k=N} \mu_k \ln p_k, \quad \sum_{k=1}^{k=N} \mu_k = 0 \tag{36}$$

Under this specification, and with $B(\mathbf{p})$ and $\lambda(\mathbf{p})$ defined as earlier for the QAIDS (see (18) and (19)), the share equations are now given by:[34]

$$s_i = \frac{\partial \ln C}{\partial \ln p_i} = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \beta_i \left[\frac{B(\mathbf{p})\ln Y}{1-\lambda(\mathbf{p})\ln Y}\right] + \frac{\lambda_i}{B(\mathbf{p})}\left[\frac{B(\mathbf{p})\ln Y}{1-\lambda(\mathbf{p})\ln Y}\right]^2 + \mu_i t \tag{37}$$

The parameters β_i and λ_i now measure input bias in scale economies. If they are all zero there is no bias and the degree of returns to scale is measured just by β_Y. The parameter μ_i measures the bias in technical progress against input *i*: $\mu_i < 0$ would imply that technical progress is biased in favour of input *i*.

If our goal is to estimate the degree of economies of scale and the rate of technical progress, the parameters of interest in the cost function can be estimated by a simpler method than in the case of the expenditure function. We can consider equation (37) as a regression equation by adding an error term, in the same way as we did to obtain equation (27) above for

the expenditure function is preference change. But even though it is still possible to measure changes in the cost of living (Balk, 1989) when preferences change, it is difficult to see how changes in the standard of living can be measured.

[34] These are cost shares, not revenue shares. In the presence of economies of scale there may be monopoly power, so profit is above the competitive level. I assume that the competitive rate of return to capital is known so that it is possible to calculate competitive rental prices for capital inputs (see Oulton (2007) for alternative ways of doing this).

the expenditure shares. After replacing the price variables in (37) by principal components, we can then estimate the β_i, λ_i and μ_i by a similar iterative process to the one set out in section 3, while imposing the appropriate cross-equation restrictions. Next, the degree of scale economies and the rate of neutral technical progress can be estimated by differentiating the cost function (35) totally with respect to time, using (36), applying Shephard's Lemma, and rearranging:

$$\frac{d\ln x(t,t)}{dt} - \sum_{i=1}^{i=N} s_i(t,t)\left(\frac{d\ln p_i(t)}{dt}\right) - \sum_{i=1}^{i=N} \mu_i \ln p_i(t)$$
$$- \left[\frac{B(\mathbf{p}(t))}{\left[1-\lambda(\mathbf{p}(t)\ln Y(t)\right]^2}\right]\left(\frac{d\ln Y(t)}{dt}\right) = \mu_t + \beta_Y\left(\frac{d\ln Y(t)}{dt}\right) \tag{38}$$

Everything on the left hand side is now measurable and the only unknowns are the coefficients μ_t and β_Y on the right hand side. So (38) can be considered as a regression equation and used to estimate these remaining unknowns.[35]

The compensated shares, holding output constant at its level in period *b*, are

$$s_i(t,b) = \frac{\partial \ln A(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i\left[\frac{B(\mathbf{p}(t))\ln Y(b)}{1-\lambda(\mathbf{p}(t))\ln Y(b)}\right] + \frac{\lambda_i}{B(\mathbf{p}(t))}\left[\frac{B(\mathbf{p}(t))\ln Y(b)}{1-\lambda(\mathbf{p}(t))\ln Y(b)}\right]^2 + \mu_i t$$
$$= \frac{\partial \ln A(\mathbf{p}(t))}{\partial \ln p_i(t)} + \mu_i t \tag{39}$$

setting $\ln Y(b) = 0$. So the relationship between the actual and the compensated shares is

$$s_i(t,b) = s_i(t,t) - \beta_i\left[\frac{B(\mathbf{p}(t))\ln Y(t)}{1-\lambda(\mathbf{p}(t))\ln Y(t)}\right] - \frac{\lambda_i}{B(\mathbf{p}(t))}\left[\frac{B(\mathbf{p}(t))\ln Y(t)}{1-\lambda(\mathbf{p}(t))\ln Y(t)}\right]^2 \tag{40}$$

and the compensated shares can be used to construct a Konüs index of input prices.

The analysis of inequality in the preceding sub-section can also be applied to the cost functions of firms, if the size distribution varies over time or across countries. Entropy (*I*) and the related statistic *J* would now appear in the share equations (37), just as they do in (31).

Finally, an interesting question is whether anything useful can be concluded when output is not in fact measurable. In many private services, the inputs may be measured fairly easily but we don't know how to measure real output very well. This suggests that we might follow

[35] Actually, overall technical progress is not separately identifiable from biased technical progress. Any non-zero estimate for μ_t can be absorbed into the μ_i by relaxing the constraint that $\sum_i \mu_i = 0$.

the same strategy as in the case of consumer demand. In that case, we eliminated unmeasured utility from the right hand side of the share equations by substituting from the expenditure function. The shares thus became functions of deflated expenditure (see equations (17)). Could the same strategy work for cost functions? Unfortunately not. If we rearrange the cost function (35) we obtain:

$$\frac{B(\mathbf{p})\ln Y}{1-\lambda(\mathbf{p})\ln Y} = \ln\left[\frac{x}{A(\mathbf{p})}\right] - \left[\beta_Y \ln Y + \mu(\mathbf{p})t + \mu_t t\right]$$

If we substitute this expression into the share equations (37) we are still left with the problem of estimating the unknown coefficients μ_t and β_Y and we still need a measure of real output. The root of the problem is that real output is necessarily cardinal while utility is only ordinal. And for utility there is no counterpart to technical progress.[36]

6. Conclusions

An algorithm which generates Konüs price indices when demand is not homothetic has now been presented. We have shown that it can be applied in both time series and cross-section. It is not dependent on the assumption of a representative consumer but can be extended to the case where income levels and other characteristics differ between consumers. The same algorithm can be applied to the parallel problem of estimating a true index of a producer's input prices and of TFP in the presence of input-biased economies of scale. The algorithm involves some econometric estimation but uses exactly the same price and quantity data as are required for conventional index numbers. The advantage of the algorithm is that it does not require the estimation of a complete system of consumer (or producer) demand, but only the consumer's responses to expenditure changes. So it can be applied at a very disaggregated level. And no restrictive assumptions about preferences (such as separability) are needed.

It is now time to consider some limitations of the analysis and some unanswered questions. If we are trying to measure the standard of living, then our maintained hypothesis must be that tastes are identical. Otherwise the relative living standards of (say) Bangladeshi peasants and American investment bankers must be regarded as simply incommensurable.

[36] In special cases the problem is soluble. Mellander (1992) shows that we can deduce real output in the case where input demand is homothetic, there are decreasing returns to scale, and the mark-up of price over marginal cost is constant. Then the ratio of the value of output (assumed observable) to the value of total cost is an indicator of the degree of returns to scale.

But the assumption of identical tastes might be considered overly strong. Is an intermediate position possible, in which tastes are identical at some comparatively high level, but might differ at a lower one? For example, the taste for hot, non-alcoholic beverages might be universal even though (at identical incomes and prices) some people prefer tea and others coffee.

A related and unanswered question in the theory of demand and production is, at what level of aggregation is the analysis supposed to apply? It is hard to believe that there exists a stable structure of preferences (common to all time periods and all countries) at a very detailed level, such as individual brands of breakfast cereal. Equally, it is not obvious that "food" is the right level either, since food items range from necessities (bread) to luxuries (caviar). In practice, the level of aggregation is often chosen on pragmatic grounds, to obtain sufficient observations to estimate the parameters of interest.

Finally, the index numbers developed here are only "true" if the underlying theory is correct and also applicable to the problem at hand. The economic theory applied in this paper has been static. It is likely that agents' choices include an inter-temporal element: in deciding whether or not to purchase a line of cocaine, the consumer may consider the future consequences as well as current income and relative prices. Habit may be important even in the absence of addiction as the macro literature has emphasised. If so, index numbers should reflect an inter-temporal element too.

APPENDIX

A.1 Flexible functional forms: the homothetic case

A flexible functional form is one which provides a second order approximation to any expenditure function (or utility function), or to any cost function (or production function), which is acceptable to economic theory.[37] Note that these are local not global properties; a good approximation at the point in question does not guarantee a good approximation at some other point.

The flexible functional forms which Diewert (1976) studied were what he called quadratic means of order *s*, given by:

$$A(\mathbf{p};s) = \left[\sum_{i=1}^{i=N}\sum_{j=1}^{j=N} b_{ij} p_i^{s/2} p_j^{s/2}\right]^{1/s}, \ b_{ij} = b_{ji}, \forall i \neq j, s > 0 \qquad \text{(A1)}$$

where $A(\mathbf{p};s)$ is assumed concave and positive. For concreteness, in this section I interpret equation (A1) as referring to the consumer's problem of choosing amongst *N* products subject to a budget constraint but it could equally well refer to the producer's problem of allocating a given expenditure amongst *N* inputs. Under this interpretation, $A(\mathbf{p};s)$ is the cost per unit of utility and equation (A1) is part of an expenditure function of the following form:

$$x(t,b) = A(\mathbf{p}(t);s)u(b) \qquad \text{(A2)}$$

where $x = \sum_i p_i q_i$ is total expenditure, q_i is the quantity purchased of the *i*th product, and $x(t,b)$ is the minimum expenditure required to reach the utility level prevailing at time *b* when the consumer faces the prices of time *t*. Note that equation (A2) implies that demand is homothetic: all expenditure elasticities are equal to one.[38]

The Konüs price index for period *t* relative to period *b* corresponding to this expenditure function is then

$$P^K(t,b) = x(t,b) / x(b,b) = A(\mathbf{p}(t);s) / A(\mathbf{p}(b);s)$$

---

[37] A second order approximation is one for which the approximating function and the function approximated have the same value at a particular point, the first derivatives of the two functions are equal at that same point, and the second derivatives are also equal at that point.

[38] This follows from Shephard's Lemma which implies that the budget shares are given by $\partial \ln x / \partial \ln p_i$. These shares are independent of the level of utility and hence of expenditure when the expenditure function has the form of equation (A2). So a doubling of expenditure with all prices held constant doubles the quantity purchased of every product.

which is independent of the utility level. If the consumer maximises utility subject to the budget constraint $x(t,t)=\sum_i p_i(t)q_i(t)$, then Diewert showed that the Konüs price index for period t relative to period b which corresponds to (A1) is given by:

$$P_s(t,b)=\left[\frac{\sum_{i=1}^{N}\left(\frac{p_i(t)}{p_i(b)}\right)^{s/2}\left(\frac{p_i(b)q_i(b)}{\sum_{i=1}^{i=N}p_i(b)q_i(b)}\right)}{\sum_{i=1}^{N}\left(\frac{p_i(b)}{p_i(t)}\right)^{s/2}\left(\frac{p_i(t)q_i(t)}{\sum_{i=1}^{i=N}p_i(t)q_i(t)}\right)}\right]^{1/s} \tag{A3}$$

Note that base period (period b) expenditure shares appear in the numerator and current period (period t) ones in the denominator.

The importance of this result is that the formula for the price index requires knowledge only of prices and quantities (or equivalently, prices and budget shares). It does not require knowledge of any of the parameters of $A(\mathbf{p};s)$. The latter are very numerous and there may be insufficient observations available to estimate them econometrically. But Diewert's result tells us that we don't need to.

The quadratic mean of order s also includes the translog as a special case when $s = 0$; the Törnqvist is the corresponding superlative index. This can be seen by taking the limit as $s\to 0$ and applying de l'Hôpital's Rule. In the case where $s=2$ the corresponding superlative index is the Fisher (Diewert, 1976). The Fisher and the Törnqvist are the forms most commonly used in empirical economics. The Fisher index is widely used by national statistical agencies, including those of the U.S.

As stated above, the quadratic mean of order s is only guaranteed to be a good approximation locally. As we move farther away from the point on which the approximation is based, it may cease to be a good one. The solution now is chaining, since the index we seek to approximate, equation (5) or equivalently (6), has continuously changing weights. This means that we continue to believe that a quadratic mean of order s, with s assumed known, describes the data well, but the actual parameters can change over time. Eg, at time t the particular form given by (A1) may apply, but at some other time r a related but different form may be a better approximation to consumer behaviour:

$$A'(\mathbf{p};s)=\left[\sum_{i=1}^{i=N}\sum_{j=1}^{j=N}b'_{ij}p_i^{s/2}p_j^{s/2}\right]^{1/s},\ b'_{ij}=b'_{ji},\ \forall i\neq j, s>0 \tag{A4}$$

where each b'_{ij} may differ from the corresponding b_{ij}. So in measuring the change in the Konüs price index between time t and $t+1$ equation (A1) may apply, while from time r to time $r+1$ equation (A4) may be better. Underlying preferences may be unchanged (the true utility function is unchanged), it's just that at some periods equation (A1) may be a good approximation while at other periods equation (A4) may be better. We don't need to know whether this is the case or not, because both sets of parameters are captured by the superlative index of equation (A3). Hence chaining increases the flexibility of flexible functional forms by allowing parameters to change over time and this is consistent with preferences remaining unchanged.[39]

Hill (2006) has recently cast doubt on the optimistic conclusion that superlative indices solve the index number problem in the homothetic case. He argues that we have no good reason for picking one value of s over another and the value of the price index may be sensitive to the choice of s. He proves that as s is increased the value of the index approaches the geometric mean of the smallest and largest price relatives. Hence the index can be sensitive to outliers. He demonstrates this point using actual time series data for the US and cross country data for 43 countries and finds wide variations depending on the value of s. The spread between the largest and smallest values of a given index (for different values of s) often lies outside the Paasche-Laspeyres spread. However, there is not much variation in the indices as s increases from 0 (translog) to 2 (Fisher).

The optimistic conclusion can however be defended:

1. All Hill's comparisons are bilateral. He does not employ chain indices. But as argued above, chaining should substantially reduce the empirical uncertainty: the smaller the change between adjacent years (or between countries), the closer will be the values of all superlative indices, ie they become increasingly insensitive to the choice of s.
2. If we adopt the economic approach (to which Hill is not necessarily committed), then the use of superlative indices requires that demand is homothetic. However unrealistic this is as a description of demand, it is the maintained hypothesis. But then theory implies that the true index must lie between the Paasche and the Laspeyres (Konüs, 1939, Deaton and Muellbauer, 1980b, chapter 7). So to be consistent with the maintained hypothesis, we should reject any value for the order s which produces a result outside the Paasche-Laspeyres spread. This again reduces the empirical uncertainty about the value of s.[40]

---

[39] Diewert (1976) was well aware of this point: see his footnote 16.

[40] For Hill's time series data, the maximum (absolute) Paasche-Laspeyres spread was 5% and the average one was 1.2%. For his cross-section data, the spread was much larger:

A.2 Proofs of propositions in section 3

Proof of Proposition 1

From (1) and (3), the budget shares are functions of utility, but from (1) utility is a positive, monotonic function of expenditure when prices are held constant. So the budget shares are functions of expenditure, and therefore also of log expenditure, when prices are held constant. We now need the following assumption:

Assumption The function relating the budget shares of any product to log expenditure is *entire*: that is, it is infinitely differentiable (smooth) and its Taylor series converges to the value of the function at every point in the (economically relevant) domain.

In this case the economically relevant domain is $x > 0$. (Consumers with zero expenditure will not be observed; and though we may observe inactive firms we do not need to model their input choices). Note that polynomials, the exponential function, and the sine and cosine functions are entire. And sums, products and compositions of entire functions are also entire.[41]

Now consider the share function for the ith product, equation (3), at the point $s_i(t,t)$ and expand it in a Taylor series around the point $s_i(t,b)$: that is, hold prices constant at their levels at time *t* and vary expenditure (utility) from its level in the base period (period *b*), to obtain

$$s_i(t,t) = s_i(t,b) + \left(\frac{\partial s_i(\cdot,\cdot)}{\partial \ln E(\cdot,\cdot)}\right)_{\substack{\mathbf{p}=\mathbf{p}(t),\\ x=E(t,b)}} \cdot [\ln E(t,t) - \ln E(t,b)]$$

$$+\frac{1}{2!}\left(\frac{\partial^2 s_i(\cdot,\cdot)}{\partial \ln E(\cdot,\cdot)^2}\right)_{\substack{\mathbf{p}=\mathbf{p}(t),\\ x=E(t,b)}} \cdot [\ln E(t,t) - \ln E(t,b)]^2 \qquad \text{(A5)}$$

$$+\frac{1}{3!}\left(\frac{\partial^3 s_i(\cdot,\cdot)}{\partial \ln E(\cdot,\cdot)^3}\right)_{\substack{\mathbf{p}=\mathbf{p}(t),\\ x=E(t,b)}} \cdot [\ln E(t,t) - \ln E(t,b)]^3 + \ldots$$

173.5% and 33.7% respectively. (I subtract 1 from his figures since he gives the ratio of Paasche to Laspeyres).

[41] The logarithmic function is not entire since the Taylor series for $\ln x$ only converges within the range $1 < x \le 2$. This might cause a problem if budget shares were a function of x rather than of $\ln x$. But a specification in terms of $\ln x$ is more reasonable economically.

Note that $\ln E(t,t) - \ln E(t,b) = \ln[E(t,t) / E(t,b]$ is the log of the ratio of the expenditure needed to achieve the utility level of period *t* to the expenditure needed to achieve the level of period *b*, both evaluated at the prices of period *t*. In fact

$$\frac{E(t,t)}{E(t,b)} = \left[\frac{E(t,t)}{E(b,b)}\right] \cdot \left[\frac{E(b,b)}{E(t,b)}\right] = \frac{x(t,t) / x(b,b)}{P^K(t,b)} \tag{A6}$$

where $x(v,v)$ is actual money expenditure at time *v* and we have used the definition of the Konüs price index in equation (2).

Now substitute (A6) into (A5) and solve for the compensated shares $s_i(t,b)$:

$$s_i(t,b) = s_i(t,t) - \eta_{i1}(t,b) \ln\left[\frac{x(t,t) / x(b,b)}{P^K(t,b)}\right] - \frac{\eta_{i2}(t,b)}{2!}\left\{\ln\left[\frac{x(t,t) / x(b,b)}{P^K(t,b)}\right]\right\}^2$$

$$-\frac{\eta_{i3}(t,b)}{3!}\left\{\ln\left[\frac{x(t,t) / x(b,b)}{P^K(t,b)}\right]\right\}^3 - \dots, \tag{A7}$$

$$i = 1, \dots, N; \; t \in [0,T]$$

where

$$\eta_{ik}(t,b) = \left(\frac{\partial^k s_i(\cdot,\cdot)}{\partial \ln E(\cdot,\cdot)^k}\right)_{\substack{\mathbf{p}=\mathbf{p}(t),\\ x=E(t,b)}}, \quad k = 1,2,\dots; \; i = 1,\dots,N \tag{A8}$$

Q.E.D.

Convergence of the algorithm for finding compensated shares

We seek conditions under which the iterative process described informally in section 3 converges to the correct solution for the compensated shares. I assume that a *K*th order polynomial in deflated expenditure is an adequate representation of consumer demand. Write the system (7) in discrete time and adopt a discrete approximation for the Konüs price index (eg chained Fisher or chained Törnqvist: see section 3). Substitute the Konüs price index out of (7) using the discrete version of equation (5). Then the system can be written in matrix terms as

$$\mathbf{s}(t,b) = f_t(\mathbf{s}(t,b)), \quad t = 0,1,\dots,T \tag{A9}$$

where $\mathbf{s}(t,b) = [s_i(t,b)]$ is an *N* x 1 vector of the compensated shares at time *t*. The form of the functions $f_t(\cdot)$ can be seen from equations (7), with the Taylor series truncated after *K* terms. Then the solution we seek is a fixed point of the system (A9). A common way to find the fixed point is by functional iteration:

$$\mathbf{s}^{m+1}(t,b) = f_t(\mathbf{s}^m(t,b)), \qquad t = 0,1,...,T \tag{A10}$$

Here the superscript denotes the iteration number and the initial guess is $\mathbf{s}^1(t,b) = \mathbf{s}^1(t,t)$. On certain assumptions this process can be shown to converge. What follows is based on Judd (1998, chapter 5). His Theorem 5.4.1 states that if $f_t(\cdot)$ is a differentiable contraction map on a closed, convex and bounded set *D*, then (1) the fixed point problem has a unique solution and (2) the sequence defined in (A10) converges to this solution. Note that this is a sufficient but not a necessary condition. Here *D* is the set $\left\{s_i \mid 0 \le s_i \le 1, \sum_i s_i = 1\right\}$. For $f_t(\cdot)$ to be a differentiable contraction map requires that the values of each element of the Jacobian of $f_t(\cdot)$ at all points in the set *D* be less than one in absolute value.

The Jacobian of $f_t(\cdot)$ is

$$\mathbf{J}_t(s) = \begin{bmatrix} \frac{\partial f_{1t}}{\partial s_1(t,b)} & \frac{\partial f_{1t}}{\partial s_2(t,b)} & \cdots & \frac{\partial f_{1t}}{\partial s_N(t,b)} \\ \frac{\partial f_{2t}}{\partial s_1(t,b)} & \frac{\partial f_{2t}}{\partial s_2(t,b)} & \cdots & \frac{\partial f_{2t}}{\partial s_N(t,b)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{Nt}}{\partial s_1(t,b)} & \frac{\partial f_{Nt}}{\partial s_2(t,b)} & \cdots & \frac{\partial f_{Nt}}{\partial s_N(t,b)} \end{bmatrix}, \; t = 0,1,...,T$$

and the requirement for $f_t(\cdot)$ to be a contraction map is that

$$\left| \frac{\partial f_{it}}{\partial s_j(t,b)} \right| < 1, \; \forall i, j \tag{A11}$$

If the compensated Törnqvist form is employed as the discrete version of the Konüs, then the elements of the Jacobian contain terms like $(\eta_{i1}/2)\Delta\ln(p_j(t)$, $(\eta_{i2}/2\cdot 2!)\Delta\ln(p_j(t)$,…, $(\eta_{ik}/2\cdot K!)\Delta\ln(p_j(t)$ and sums and products of these terms. If the $\Delta\ln p_j(t)$ are sufficiently small, which depends in part on the size of the time interval (or the gap between countries), then the requirement of (A11) can be satisfied, since we expect the absolute values of the $\eta_{i1},\eta_{i2},...,\eta_{iK}$ to be less than one. But even if it is not satisfied a weaker requirement for convergence may suffice. Theorem 5.4.2 of Judd (1998, page 166)) states that if the Jacobian at a fixed point is a contraction when viewed as a linear map in R^n, then iterating $f_t(\cdot)$ will converge if the initial guess is good. This theorem requires $f_t(\cdot)$ to be Lipschitz at the fixed point, which is the case for the functions considered here, and that the spectral radius of the Jacobian matrix at the fixed point be less than one.

An explicit solution of equations (4) *and* (7) *when the Engel curves are log-linear*

In this case $\eta_{i2} = \dots \eta_{iK} = 0$ and the system of equations (A7) in discrete time can be written as

$$s_i(t,b) = s_i(t,t) - \eta_{i1}(t,b)\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b)}\right] \qquad i = 1,\dots,N;\ t = 1,\dots,T$$

Set $t = b+1$ so we have

$$s_i(b+1,b) = s_i(b+1,b+1) - \eta_{i1}\ln\left[\frac{x(b+1,b+1)/x(b,b)}{P^K(b+1,b)}\right] \qquad i = 1,\dots,N$$

We also assume that the Konüs price index takes the compensated Törnqvist form of equation (9):

$$\ln P^T(t,t-1,b) = \sum_{i=1}^{i=N}\left(\frac{s_i(t,b)+s_i(t-1,b)}{2}\right)\ln\left(\frac{p_i(t)}{p_i(t-1)}\right)$$

Substitute this into the preceding equation to obtain:

$$\begin{aligned} s_i(b+1,b) = {} & s_i(b+1,b+1) - \eta_{i1}(b+1,b)\ln\left[x(b+1,b+1)/x(b,b)\right] \\ & + \eta_{i1}\sum_{i=1}^{i=N}\left(\frac{s_i(b+1,b)+s_i(b,b)}{2}\right)\left[\ln p_i(b+1) - \ln p_i(b)\right] \\ & \qquad i = 1,\dots,N \end{aligned}$$

In matrix notation this becomes

$$\begin{aligned} \mathbf{s}(b+1,b) &= \{\mathbf{s}(b+1,b+1) - \ln\left[x(b+1,b+1)/x(b,b)\right]\boldsymbol{\eta}_1(b+1,b) + \frac{1}{2}\boldsymbol{\eta}_1\Delta\ln\mathbf{p}'(b+1)\mathbf{s}(b,b)\} \\ &\quad + \frac{1}{2}\boldsymbol{\eta}_1\Delta\ln\mathbf{p}'(b+1)\mathbf{s}(b+1,b) \\ &= \mathbf{K}(b+1,b) + \frac{1}{2}\boldsymbol{\eta}_1\Delta\ln\mathbf{p}'(b+1)\mathbf{s}(b+1,b) \text{ say} \end{aligned}$$

where $\mathbf{s} = [s_i]$, $\boldsymbol{\eta}_1 = [\eta_{i1}]$, $\Delta\ln\mathbf{p}(b+1) = [\ln p_i(b+1) - \ln p_i(b)]$, and $\mathbf{K}(b+1,b)$ are $N\times 1$ column vectors; note that all components of $\mathbf{K}(b+1,b)$ are assumed known. Solving for $\mathbf{s}(b+1,b)$,

$$\left[\mathbf{I} - \frac{1}{2}\boldsymbol{\eta}_1\Delta\ln\mathbf{p}'(b+1)\right]\mathbf{s}(b+1,b) = \mathbf{K}(b+1,b)$$

whence

$$\mathbf{s}(b+1,b)=\left[\mathbf{I}-\frac{1}{2}\boldsymbol{\eta}_1\Delta\ln\mathbf{p}'(b+1)\right]^{-1}\mathbf{K}(b+1,b)$$

Everything on the right hand side is known so this yields an explicit solution for the compensated shares at time $b+1$. Proceeding in a similar way we can get an explicit solution for the compensated shares at time $b+2$, and so on. Similarly, we can work backwards from b to find the compensated shares at time $b-1$, $b-2$, etc.

Proof of Proposition 2: the Konüs price index lies between the compensated Laspeyres and the compensated Paasche indices

This proposition follows from the well known inequalities for the Laspeyres and Paasche derived by Konüs (1939); see also Deaton and Muellbauer (1980), chapter 7. By definition of the expenditure function, we have:

$$\sum_{i=1}^{i=N} p_i(t)q_i(t-1)\geq E[\mathbf{p}(t),u(t-1)]$$

Denoting the Laspeyres price index for year t with base year $t-1$ by $P^L(t,t-1)$, it follows that

$$P^L(t,t-1)=\frac{\sum_{i=1}^{i=N} p_i(t)q_i(t-1)}{\sum_{i=1}^{i=N} p_i(t-1)q_i(t-1)}\geq\frac{E[\mathbf{p}(t),u(t-1)]}{E[\mathbf{p}(t-1),u(t-1)]}$$

since $\sum_i p_i(t-1)q_i(t-1)=E[\mathbf{p}(t-1),u(t-1)]$. By definition of the expenditure function again,

$$\sum_{i=1}^{i=N} p_i(t-1)q_i(t)\geq E[\mathbf{p}(t-1),u(t)]$$

whence

$$P^P(t,t-1)=\frac{\sum_{i=1}^{i=N} p_i(t)q_i(t)}{\sum_{i=1}^{i=N} p_i(t-1)q_i(t)}\leq\frac{E[\mathbf{p}(t),u(t)]}{E[\mathbf{p}(t-1),u(t)]}$$

where $P^P(t,t-1)$ is the Paasche price index for year t with base year $t-1$. Now in the present case utility is being held constant at the level of period b, ie $u(t-1)=u(t)=u(b)$, so we have

$$P^L(t,t-1)\geq P^K(t,b)/P^K(t-1,b)\geq P^P(t,t-1) \tag{A12}$$

where $P^K(t,b)=\frac{E[\mathbf{p}(t),u(b)]}{E[\mathbf{p}(b),u(b)]}$ is the Konüs price index with base year b. QED.

In recognition of the fact that this proposition holds when utility is held constant at the level of period b, in the text we refer to these Laspeyres and Paasche indices as compensated ones and write them as $P^L(t,t-1,b)$ and $P^P(t,t-1,b)$ respectively.

Since the compensated Fisher index is the geometric mean of the compensated Laspeyres and the compensated Paasche, like the Konüs it must always lie between the Laspeyres and the Paasche:

$$P^L(t,t-1) \geq P^F(t,b)/P^F(t-1,b) \geq P^P(t,t-1) \tag{A13}$$

A.3 Aggregating over unequal incomes in the QAIDS

As given in equation (30), repeated here for convenience, the share of product i in aggregate expenditure is a weighted average of the shares of the various income groups:

$$\begin{aligned} s_i = \sum_{g=1}^{g=G} w_g s_{ig} = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \sum_{g=1}^{g=G} w_g \ln x_g - \beta_i \ln A_b(\mathbf{p}) \\ + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left[\sum_{g=1}^{g=G} w_g (\ln x_g)^2 - 2\ln A_b(\mathbf{p}) \sum_{g=1}^{g=G} w_g \ln x_g + [\ln A_b(\mathbf{p})]^2 \right] \end{aligned} \tag{A14}$$

From (29),

$$\sum_{g=1}^{g=G} w_g \ln x_g = \sum_{g=1}^{g=G} w_g \ln(w_g G x) = \sum_{g=1}^{g=G} w_g \ln w_g + \ln G + \ln x$$

Also, from (29) again,

$$\begin{aligned} \sum_{g=1}^{g=G} w_g (\ln x_g)^2 &= \sum_{g=1}^{g=G} w_g (\ln w_g + \ln(Gx))^2 \\ &= \sum_{g=1}^{g=G} w_g \left[(\ln w_g)^2 + 2(\ln G + \ln x) \ln w_g + (\ln G + \ln x)^2 \right] \\ &= \sum_{g=1}^{g=G} w_g (\ln w_g)^2 + 2\ln G \sum_{g=1}^{g=G} w_g \ln w_g + 2\ln x \sum_{g=1}^{g=G} w_g \ln w_g \\ &\quad + \left[(\ln G)^2 + 2\ln G \ln x + (\ln x)^2 \right] \sum_{g=1}^{g=G} w_g \\ &= \sum_{g=1}^{g=G} w_g (\ln w_g)^2 + 2\ln G \sum_{g=1}^{g=G} w_g \ln w_g + 2\ln x \left[\sum_{g=1}^{g=G} w_g \ln w_g + \ln G \right] \\ &\quad + (\ln G)^2 + (\ln x)^2 \end{aligned} \tag{A15}$$

Therefore, plugging these results into (A14):

$$s_i = \sum_{g=1}^{g=G} w_g s_{ig} = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \sum_{g=1}^{g=G} w_g \ln x_g - \beta_i \ln A_b(\mathbf{p})$$

$$+ \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left[\sum_{g=1}^{g=G} w_g (\ln x_g)^2 - 2\ln A_b(\mathbf{p}) \sum_{g=1}^{g=G} w_g \ln x_g + [\ln A_b(\mathbf{p})]^2 \right]$$

$$= \left[\frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln G \right] + \beta_i \sum_{g=1}^{g=G} w_g \ln w_g + \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln \left[\frac{x}{A_b(\mathbf{p})} \right]$$

$$+ \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left[(\ln x)^2 - 2\ln A_b(\mathbf{p}) \ln(x) + [\ln A_b(\mathbf{p})]^2 \right]$$

$$+ \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left[\sum_{g=1}^{g=G} w_g (\ln w_g)^2 + 2\ln G \sum_{g=1}^{g=G} w_g \ln w_g + 2\ln x \sum_{g=1}^{g=G} w_g \ln w_g \right.$$

$$\left. + (\ln G)^2 + 2\ln x \ln G - 2\ln A_b(\mathbf{p}) \left[\sum_{g=1}^{g=G} w_g \ln w_g + \ln G \right] \right]$$

Therefore

$$s_i = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \left\{ (\ln G - I) + \ln \left[\frac{x}{A_b(\mathbf{p})} \right] \right\}$$

$$+ \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ \left[J - 2I \ln G + (\ln G)^2 \right] + 2 \left\{ (\ln G - I) \ln \left[\frac{x}{A_b(\mathbf{p})} \right] \right\} + \left[\ln \left[\frac{x}{A_b(\mathbf{p})} \right] \right]^2 \right\}$$

where we have set $I = -\sum_{g=1}^{g-G} w_g \ln w_g$ and $J = \sum_{g=1}^{g-G} w_g (\ln w_g)^2$. Now after defining $W_1 = \ln G - I$ and $W_2 = J - 2I \ln G + (\ln G)^2$ we obtain equation (31) of the main text.

References

Anand, Sudhir, and Paul Segal (2008). "What Do We Know About Global Income Inequality?". *Journal of Economic Literature*, 46:1, 57-94.

Apostol, Tom M. (1957). *Mathematical Analysis*. Reading, Mass.: Addison-Wesley.

Balk, Bert M. (1995). "Axiomatic Price Index Theory: A Survey." *International Statistical Review*, 63 (1): 69-93.

Balk, Bert M. (1989). "Changing consumer preferences and the cost-of-living index: theory and nonparametric expressions". *Journal of Economics*, 50(2), 157-169.

Balk, Bert M. (2004). "Direct and Chained Indices: A Review of Two Paradigms." Paper presented to the SSHRC International Conference on Index Number Theory and the Measurement of Prices and Productivity, Vancouver, 2004; accessed on 17 December 2007 at http://www.ipeer.ca/papers/Balk,%20Sept.15,2004,%20Vancouver%20rev.pdf.

Balk, Bert M.)2005). "Divisia Price and Quantity Indices: 80 Years After." *Statistica Neerlandica*, 59: 119-158.

Banks, James, Richard Blundell and Arthur Lewbel (1997). "Quadratic Engel Curves and Consumer Demand." *The Review of Economics and Statistics*, 79(4): 527-539.

Blackorby, Charles, Daniel Primont, and R. Robert Russell (2008). "Separability". In *The New Palgrave Dictionary of Economics*, edited by Steven N. Durlauf and Lawrence Blume. Palgrave Macmillan. [DOI:10.1057/9780230226203.1511].

Blow, Laura, Andrew Leicester, and Zoë Oldfield. (2004). *Consumption Trends in the UK, 1975-99*. London: The Institute for Fiscal Studies.

Blundell, Richard and Jean Marc Robin (1999). "Estimation in Large and Disaggregated Demand Systems: An Estimator for Conditionally Linear Systems". *Journal of Applied Econometrics*, 14(3), 209-232.

Blundell, Richard and Jean-Marc Robin (2000). "Latent Separability: Grouping Goods Without Weak Separability". *Econometrica*, 68(1), 53-84.

Blundell, Richard, Xiaohong Chen and Dennis Christensen (2007). "Semi-nonparametric IV Estimation of Shape-invariant Engel Curves". *Econometrica*, 75(6): 1613-1669.

Caves, Douglas W., Laurits R. Christensen, and W. Erwin Diewert (1982). "Multilateral Comparisons of Output, Input, and Productivity Using Superlative Index Numbers." *Economic Journal*, 92: 73-86.

Chai, Andreas and Alessio Moneta (2010). "Retrospectives: Engel curves". *Journal of Economic Perspectives*, 24(1), Winter, 225-240.

Christensen, Laurits R., Dale W. Jorgenson and Lawrence J. Lau (1975). "Transcendental Logarithmic Utility Functions". *American Economic Review*, 65 (June): 367-383.

Clark, Gregory (2008). "In defense of the Malthusian interpretation of history". *European Review of Economic History*, 12(2:August), 175-199.

Crawford, Ian, and J. Peter Neary (2008). "Testing for a Reference Consumer In International Comparisons of Living Standards". *American Economic Review*, 98(4): 1731-1732.

Deaton, Angus, and John Muellbauer (1980a). "An Almost Ideal Demand System." *American Economic Review*, 70: 312-336.

Deaton, Angus, and John Muellbauer (1980b). *Economics and Consumer Behaviour*. Cambridge: Cambridge University Press.

Diewert, W. Erwin (1976). "Exact and Superlative Index Numbers." *Journal of Econometrics*, 4: 115-46.

Diewert, W. Erwin (1981). "The Economic Theory of Index Numbers: A Survey". In *Essays in Index Number Theory Volume 1*, edited by W. Erwin Diewert and Alice O. Nakamura, 1993. Amsterdam: North-Holland.

Diewert, W. Erwin (2002). "Similarity and Dissimilarity Indexes: An Axiomatic Approach". Department of Economics, University of British Columbia, working paper no. 10-02.

Diewert, W. Erwin (2008). "Index Numbers". In *The New Palgrave Dictionary of Economics*, edited by Steven N. Durlauf and Lawrence Blume. Palgrave Macmillan.

Diewert, W. Erwin (2009). "Cost of Living Indexes and Exact Index Numbers". In *Quantifying Consumer Preferences*, edited by Daniel Slottje. Emerald Publishing Group.

Diewert, W. Erwin, and Terence J. Wales (1988). "A Normalized Quadratic Semiflexible Functional Form". *Journal of Econometrics*, 37: 327-342.

Divisia, François (1925-1926). "L'indice Monétaire et la Théorie de la Monnaie." *Revue d'Economie Politique*, 39(4): 842-864; 39(5): 980-1008; 39(6): 1121-1151 [1925]; 40(1): 49-81 [1926].

Dowrick, Steve and John Quiggin (1997). "True Measures of GDP and Convergence". *American Economic Review*, 87(1), March: 41-64.

Engel, Ernst (1857) [1895]. "Die Productions- und Consumptionverhältnisse des Königsreich Sachsen". Reprinted in *Bulletin de l'Institut International de Statistique*, IX.

Feenstra, Robert C. and Marshall B. Reinsdorf (2000). "An Exact Price Index for the Almost Ideal Demand System." *Economics Letters*, 66: 159-162.

Gorman, William M. (1981). "Some Engel Curves". In *Essays in the Theory and Measurement of Consumer Behaviour: In Honour of Richard Stone*, edited by Angus Deaton, 7-30. Cambridge: Cambridge University Press.

Hill, Robert J. (1997). "A Taxonomy of Multilateral Methods for Making International Comparisons of Prices and Quantities." *The Review of Income and Wealth*, 43(1), March: 49-69.

Hill, Robert J. (1999). "Comparing Price Levels Across Countries Using Minimum Spanning Trees". *Review of Economics and Statistics*, 81(1): 135-142.

Hill, Robert J. (2004). "Constructing Price Indexes Across Space and Time: The Case of the European Union." *American Economic Review*, 94 (5): 1379-1410.

Hill, Robert J. (2006). "Superlative Index Numbers: Not All of Them Are Super". *Journal of Econometrics*, 130: 25-43.

Houthakker, Hendrik S. (1957). "An International Comparison of Household Expenditure Patterns Commemorating the Centenary of Engel's Law". *Econometrica*, 25: 532-551.

Hulten, Charles R. (1973). "Divisia Index Numbers." *Econometrica*, 41: 1017-1025.

Johnson, Richard A. and Dean W. Wichern (2002). *Applied Multivariate Statistical Analysis*. New Jersey: Prentice-Hall Inc.

Judd, Kenneth L. (1998). *Numerical Methods in Economics*. Cambridge, MA.: The MIT Press.

Konüs, Alexander A. (1939). "The Problem of the True Index of the Cost-of-Living." *Econometrica*, 7(1): 10-29. (English translation; first published in Russian in 1924).

Lewbel, Arthur (1991). "The Rank of Demand Systems: Theory and Nonparametric Estimation". *Econometrica*, 59(3): 711-730.

Lewbel, Arthur and Krishna Pendakur (2009). "Tricks with Hicks: the EASI Demand System". *American Economic Review*, 99:3 (June): 827-863.

Marshall, Alfred (1920) [1966]. *Principles of Economics* (8^{th} edition). London: Macmillan.

Mellander, Erik (1992). "An Indirect Approach to Measuring Productivity in Private Services". *Scandinavian Journal of Economics*, 94, Supplement: 229-244.

Muellbauer, John (1976). "Community Preferences and the Representative Consumer." *Econometrica*, 44: 524-543.

Neary, J. Peter (2004). "Rationalizing the Penn World Table: True Multilateral Indices for International Comparisons of Real Income." *American Economic Review*, 94(5): 1411-1428.

Office for National Statistics (1998). *The Retail Prices Index: Technical Manual: 1998 Edition*. London: The Stationery Office. Available at www.statistics.gov.uk.

Office for National Statistics (2006). *Consumer Price Indices: Technical Manual (2006 Edition)*. London: Office for National Statistics. Available at www.statistics.gov.uk.

Oulton, Nicholas (2007). "Ex Post Versus Ex Ante Measures of the User Cost of Capital". *Review of Income and Wealth*, Series 53, No. 2 (June): 295-317.

Oulton, Nicholas (2008). "Chain Indices of the Cost of Living and the Path-Dependence Problem: An Empirical Solution." *Journal of Econometrics*, 144: 1 (May): 306-324. [http://dx.doi.org/ 10.1016/j.jeconom.2008.02.001].

Oulton, Nicholas (2010). "The Wealth and Poverty of Nations: True PPPs for 141 countries". Centre for Economic Performance. Mimeo.

Samuelson, Paul A. and Subramanian Swamy (1974). "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis." *American Economic Review*, 64: 566-593.

Stigler, George J., and Gary S. Becker (1977). "De Gustibus Non Est Disputandum". *American Economic Review*, 67(2): 76-90; stable URL: http://www.jstor.org/stable/1807222.

Theil, Henri (1967). *Economics and Information Theory*. Amsterdam: North-Holland.

Tobin, James (1987). "Fisher, Irving". In *The New Palgrave: A Dictionary of Economics*, edited by John Eatwell, Murray Milgate, and Peter Newman. Basingstoke: Palgrave. Reprinted in *The New Palgrave Dictionary of Economics*, edited by Steven N. Durlauf and Lawrence Blume, 2008. Palgrave Macmillan.

U.S. Bureau of Labor Statistics (2007). "The Consumer Price Index (updated 06/2007)". Chapter 17 of *BLS Handbook of Methods*. Accessed at http://www.bls.gov/opub/hom/pdf/homch17.pdf on 4 June 2009.

van Veelen, Matthijs (2002). "An Impossibility Theorem Concerning Multilateral International Comparison of Volumes". *Econometrica* 70(1), 369-75.

van Veelen, Matthijs, and Roy van der Weide (2008). "A Note on Different Approaches to Index Number Theory." *American Economic Review*, 98(4): 1722-1730.

World Bank (2008). *Global Purchasing Power Parities and Real Expenditures: 2005 International Comparison Program*. Available at http://siteresources.worldbank.org/ICPINT/Resources/icp-final.pdf.