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Solving the Index-Number Problem in a Historical Perspective

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Abstract

The problem of price and quantity indexes is a typical problem of aggregation. Even when the aggregation conditions are not rejected on the basis of the observed data, there still remains a certain degree of uncertainty regarding the point estimate of the index number. Following the truly constructive Afriat's method, we can reinterpret this uncertainty by reverting the problem and asking: (i) whether the available data can be rationalized and aggregated with well-behaved "true" index functions *independently* from how they have been actually determined, (ii) if yes, what are the upper and lower numerical values of all the alternative "true" index functions aggregating the data? (iii) if the answer is no, then either the data are not generated by a rational behaviour (and in this case a correction for inefficiency can be made), or else the data are generated by a rational behaviour within a different or wider set of determinants to be considered in a different or extended accounting framework. Since any "true" price index function satisfies all Fisher's tests by construction, including the transitivity requirement at least locally, also the reconstructed upper and lower numerical values of the set of all admissible "true" price indexes must respect those tests. This solution is valid irrespective of the existence of such non-observable objects as for example utility and production functions governing the observed behaviour. The purpose of this paper is to present a full solution of the index-number problem in the perspective of the theoretical developments occurred during the last century. The proposed solution is built on Afriat's method and consist in defining for the first time the "true" bounds of the set of possible aggregating indexes which are fully consistent with all Fisher's tests, including transitivity, and invariance to the change of bases. An empirical application on Irving Fisher's data illustrates the method.

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Quotations

"The fundamental and well-known theorem for the existence of a price index that is invariant under change in level of living is that each dollar of income be spent in the same way by rich or poor, with all income elasticities exactly unity (the homothetic case). Otherwise, a price change in luxuries could affect only the price index of the rich while leaving that of the poor relatively unchanged. This basic theorem was well known already in the 1930's, but is often forgotten and is repeatedly being rediscovered".

"[…] Although most attention in the literature is devoted to price indexes, when you analize the use to which price indexes are generally put, you realize that quantity indexes are actually most important. Once somehow estimated, price indexes are in fact used, if at all, primarily to 'deflate' nominal or monetary totals in order to arrive at estimates of underlying 'real magnitudes' (which is to say, *quantity* indexes!)".

"[…] The fundamental point about an economic quantity index, which is too little stressed by writers, Leontief and Afriat being exceptions, is that it must itself be a cardinal indicator of ordinal utility".

P.A. Samuelson and S. Swamy (1974, pp. 567-568)

"Index numbers have pervaded economics for a long time; early ways of thinking survive almost mindlessly, and the subject is petrified with its history".

Sydney N. Afriat (1975, p. 369)

"Men of science love to study the treasures gathered by others, in order to know well the value of their own."

Vito Volterra (quoted by E.S. Allen, 1941, p. 516)

"Each maker of index-numbers is free to retain his conviction that his own plan is the very best. I only ask him to think it possible that others may not be entirely mistaken."

F.Y. Edgeworth (1925, p. 388)
(quoted by G. Stuvel , 1989, p. *v*)

1. Introduction

"Economists can't even report the past, let alone predict the future". This spiteful announcement came recently from the title of an article by Simon Carr published on a major British newspaper[1]. This article was referring to the analyses of current unemployment in the UK by a would-be independent but allegedly government influenced body. "How many times did they adjust the dates of the economic cycles in the 1990s and 2000s?", the columnist asks. The message is not new, however. Apart from many statistical problems, analysing data has always systematically challenged and put on trial the economic profession. One of the most heated debates regarded the systematic bias and the unacceptable intransitivity of social cost-of-living indexes and GDP deflators (see, for example, the famous Boskin commission report[2] in the US and the ensuing prolonged discussions on the press and academic journals worldwide) or even the existence itself of the unobservable "objects" that these indexes are intended to (approximately) measure (many seminal papers as early as those by Keynes, 1909 and Leontief, 1936 have discussed these points). And, alas, the bias in the provided measures inevitably turned out to produce real effects throughout the economy by means of institutional indexation mechanisms and financial and economic expectations!

The index-number construction is typically a procedure of aggregation of changes in heterogeneous elements. It is often intended to measure a metaphysical object that is never observable. Even the general level of prices is a purely theoretical concept from the economic point of view, which cannot be directly observed or measured. Mathematically, the index-number problem consists in reducing the relative changes of the elements of a vector into changes in one single numerical value, a scalar. At best, this is possible only under very restrictive conditions. In his famous *Econometrica* survey of general economic theory dedicated to the problem of index numbers, Ragnar Frisch (1936, p. 1) described it in these terms: "The index-number problem arises whenever we want a quantitative expression for a *complex* that is made up of individual measurements for which non common *physical* unit exists. The desire

[1] *The Independent*, 14 July 2010, p. 6.

[2] See Boskin *et al*.(1996) and, for the debate on this report, see for example Triplett (2006).

to unite such measurements and the fact that this cannot be done by using physical or technical principles of comparison only, constitute the essence of the index-number problem and all the difficulties center here". In economics, the solution of this problem is typically required to subdivide changes of total nominal values into meaningful price and quantity components.

The national accountants are asked to provide a split of the changes of nominal economic aggregates into a deflator and a volume figure. Similarly, monitoring monetary policies usually entails a decomposition of money supply into inflation and volume indexes representing the purchasing power of circulating money. At firm level, changes in nominal profits can be accounted for by decomposing them into productivity and price components. It turns out that this is possible only under very restrictive conditions. In the general case, every attempt of forcing the application of any specific index number formulas is doomed to yield misleading results (see, *e.g.*, McCusker, 2001, Derks, 2004, Officer and Williamson, 2006 on intertemporal comparisons of the purchasing power of money and Leontief, 1936 and Samuelson, 1947, p. 162, who warned us against "the tendency to attach significance to the numerical value of the index computed").

Even when the aggregation conditions are not rejected on the basis of the observed data, there still remains a certain degree of uncertainty regarding the point estimate of the index number. Following the truly constructive method established by Afriat (1981), we can bypass this uncertainty by reverting the problem and asking: (*i*) whether the available data can be rationalized by well-behaved "true" index functions, (*ii*) if yes, what are the upper and lower bounds of the region containing all numerical values of possible "true" index functions? (*iii*) if the data cannot be rationalized by well behaved index functions, then either the data are not generated by a rational behaviour (and a correction for inefficiency may be made), or else the data are generated within a different set of variables to be considered in an alternative or extended accounting framework.

Since all "true" economic index functions respect, by construction, all Fisher's tests (see Samuelson and Swamy, 1974), also the reconstructed upper and lower values belonging to the set of possible "true" indexes respect those tests, and so does a geometric mean of those bounds, which may be required for practical needs of point estimation. This solution is purely constructive and is obtainable irrespective of the actual existence or non-existence of the underlying utility of production functions governing the observed economic behaviour on the markets.

The purpose of this paper is to present a solution of the index number problem in the perspective of the theoretical developments occurred during the last century. This does not intend by any means to be a history of thought on index numbers, on which a number of sources are available[3]. Further references to the current state of the theory and applications of index numbers can be found in Vogt and Barta (1997), von der Lippe (2001)(2007), Balk (2008), and the manuals on consumer price indices (CPIs), producer price indices (PPIs), and import-export price indices (XMPIs) published jointly by ILO, IMF, OECD, UN, Eurostat, and The World Bank (2004a)(2004b)(2008). The proposed solution represents a further step forward with respect to Afriat's (1981)(2005)(2008) method which was used also in Afriat and Milana (2009). It builds on this method to define and construct chain-consistent (transitive) tight bounds of the "true" index number. Although, for brevity reasons, we shall concentrate mainly on the price index, many results are applicable to the quantity index, which may also be obtained implicitly as the ratio between the index of nominal values and the price index.

The rest of the papers proceeds as follows: The second section recalls the classical search for the ideal index number formula with particular reference to Irving Fisher's contribution. The third section presents the essential concept of "true" (constant-utility) cost-of-living index starting from Konüs' pioneering formulation and the search for its possible computable bounds. The fourth section considers one possible solution of the bounding problem based on the "exact" correspondence between index numbers formulas and functional forms of the "true" aggregating functions. Given the inevitable strong limitations of the "exact" index-number approach and the indeterminacy of the "true" index, the fifth section aims at showing the conditions derived from the Antonelli's integrability approach under which the observable market data can be used to construct the two-sided Laspeyres-Paasche (L-P) bounds. Since the L-P bounds offer reliable numerical values only locally as they are essentially "first-order" linear approximations, they can never be "true" limits (and part of) the set of all possible "true" indexes under the hypothesis of strongly convex preferences which are necessarily assumed in the integrability approach. The sixth, seventh, and eighth sections offer

[3] For historial accounts of the development of index number theory, see for example, in chronological order, Walsh (1901)(1921), Fisher (1922, pp. 458-460), Frisch (1936), Ferger (1946), Chance (1966), Kendall (1969), Rothenberg (1979), Aldrich (1992), Diewert (1993), Vogt and Barta (1997, pp. 9-67), Persky (1998), and Balk (2008, pp. 1-52).

useful alternatives developed in the literature allowing the L-P bounds to be within the set of "true" indexes. Sections 9, 10, and 11 present Afriat's "giant leap" based on his revolutionary reformulation of revealed preference theory allowing for multi-valued demand correspondences (many quantity solutions may be possible in correspondence with certain relative prices) and, by this way, to reconstruct piecewise linear bounds for the "true" price index. Using several observation points simultaneously, this approach also enables us to define and measure bounds that are tighter than those of the traditional bilateral indexes. However, these tight bounds may still turn out to be intransitive and cannot be considered "true" indexes themselves. Section 12 contains our proposal to find the full solution with chain-consistent (transitive) "true" bounds. Section 13 offers a practical illustration on the same data that were used by Irving Fisher's to perform his well-known tests on index-number formulas. Section 14 concludes.

2. Irving Fisher and the "ideal" index number formula

In Fisher's (1911) book *The Purchasing Power of Money. Its Determination to Credit, Interest and Crisis*, the theory of the price level was related to the quantity theory of money. Let M = stock of money, V = the velocity of circulation of money; p_i = price level of the ith transaction, T_i = volume of the ith transaction carried out using money. The starting (infamous) equation of exchange is

$$MV = p_1T_1 + p_2T_2 + \ldots + p_nT_n, \tag{2.1}$$

In order to make the foregoing equation workable, the following version is usually considered

$$MV = PT \tag{2.2}$$

where P is the general price level and T is the volume of all transactions, which have been replaced with the aggregation Q of real outputs $q_1, q_2, \ldots, q_n$, often measured by real GDP, that is $MV = PQ$ (see Fisher, 1911, Ch. 2). Equation (2.1) does not necessarily imply equation (2.2). While the former is based, at least in principle, on observable variables, the latter contains non-observable aggregates and relies on computation techniques in order to "correctly" construct them. It is in this vein that Irving Fisher dedicated energies and efforts in the search of his "ideal" index number formula satisfying as many desired properties as possible. This search culminated in his famous book *The Making of Index Numbers* published in 1922 (3rd edition 1927), where he recognized that no index number would satisfy all the desired properties, but he chose the geometric mean of the Laspeyres and Paasche indices as his "ideal" index number formula. Applied to the price index between the points of observation 0 and 1, this "ideal" index number formula is

$$(2.3) \qquad P_F^{0,1} \equiv \sqrt{P_L^{0,1} \cdot P_P^{0,1}} \quad \text{where } P_L^{0,1} \equiv \frac{\sum_i p_i^1 q_i^0}{\sum_i p_i^0 q_i^0} = \frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0} \text{ and } P_P^{0,1} \equiv \frac{\sum_i p_i^1 q_i^1}{\sum_i p_i^0 q_i^1} = \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1}$$

where $\mathbf{p}^t \equiv [p_1^t p_2^t ... p_n^t]$ and $\mathbf{q}^t \equiv [q_1^t q_2^t ... q_n^t]$ are the price and quantity vectors and, $P_L^{0,1}$, $P_P^{0,1}$, and $P_F^{0,1}$ are the Laspeyres, Paasche, and Fisher's "ideal" price indices. This last formula had been previously considered by Bowley (1899) and others even before (see also Bowley, 1923, p. 252) and recommended by Walsh (1901) and Pigou (1912), although it does not generally satisfy the transitivity or circularity property, that is $P_F^{0,2} \neq P_F^{0,1} \cdot P_F^{1,2}$ and the equivalent invariance with changes in bases, that is $P_F^{0,2} \neq P_F^{1,2} / P_F^{1,0}$. (by contrast, any price level, if any, is transitive by construction ($P^2/P^0 = (P^2/P^1)(P^1/P^0)$). Surprisingly, Fisher dropped the requirement of this property and deemed it as unimportant compared to other properties which his "ideal" formula always satisfies.

In their article dedicated to economic index numbers, Samuelson and Swamy (1974) commented Fisher's choice in these terms: "Indeed, so enamoured did Fisher become with his so-called Ideal index that, when he discovered it failed the circularity test, he had the hubris to declare '…, therefore, a *perfect* fulfilment of this so-called circular test should really be taken as proof that the formula which fulfils it is erroneous' (1922, p. 271). Alas, Homer has nodded; or, more accurately, a great scholar has been detoured on a trip whose purpose was obscure from the beginning" (p. 575). By contrast, in order to avoid strong discrepancies in the results obtained, the subsequent developments in this field have been devoted to satisfy, among the other tests, the transitivity property that ensures consistency in multilateral comparisons.

3. Konüs' constant-utility index numbers

Bennet (1920) introduced a method "by which a change of expenditure can be analysed into two parts, one corresponding to changes in cost of living and the other to changes in standard of living" (p. 455). This decomposition was proposed in terms of absolute differences. Konüs (1924) and Allen (1949) have, respectively, introduced the concepts of constant-utility indexes of prices and quantities in terms of ratios. Konüs price index is defined as $P_K \equiv \dfrac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,\bar{u})}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,\bar{u})}$, which takes into account the price-induced adjustments in quantities for a *given* level of utility $\bar{u}$.

Setting $\bar{u}=u^0$ yields the Laspeyres-type Konüs price index $P_{K-L}=\dfrac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0)}{\mathbf{p}^0\mathbf{q}^0}$, where $\mathbf{p}^0\mathbf{q}^0=\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^0)$, while setting $\bar{u}=u^1$ yields the Paasche-type Konüs price index $P_{K-P}=\dfrac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1)}$, where $\mathbf{p}^1\mathbf{q}^1=\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^1)$.

It must be noted that the constant-utility index numbers P_K^0 and P_K^1 cannot be computed directly since the respective expenditures $\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0)$ and $\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1)$ are not observed. Unless the demand quantities $\mathbf{q}(\mathbf{p},u^0)$ and $\mathbf{q}(\mathbf{p},u^1)$ are somehow estimated and simulated with prices $\mathbf{p}^1$ and $\mathbf{p}^0$ respectively at the given utility levels (by following, for example, the econometric approach), a way to proceed with Konüs' constant-utility index numbers is to work with a theoretical relationship between them and the known index number formulas. Some of these may be used to establish upper and lower limits, when possible, for the Konüs' constant-utility index numbers, which remain unknown. In the general (non-homothetic) case, Konüs had established the following *one-sided* bounds with the price index from the point of view of demand (on the supply side, the algebraic signs are reversed)

(3.1) $$P_{K-L} < \frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0} \equiv P_L \qquad \text{and} \qquad P_P \equiv \frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1} < P_{K-P}$$

Since, with strictly convex preferences, $\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0) < \mathbf{p}^1\mathbf{q}^0$ and $\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1) < \mathbf{p}^0\mathbf{q}^0$ because the left-hand sides of these last inequalities are those actually consistent with a cost-miminizing behaviour at the prices $\mathbf{p}^1$ and $\mathbf{p}^0$ respectively.

Allen (1949) observed that the economic (utility-constant) quantity index could be obtained directly, for given reference prices $\bar{\mathbf{p}}$, as

(3.2)
$$Q_A \equiv \frac{\bar{\mathbf{p}} \cdot \mathbf{q}(\bar{\mathbf{p}}, u^1)}{\bar{\mathbf{p}} \cdot \mathbf{q}(\bar{\mathbf{p}}, u^0)}$$

Setting $\bar{\mathbf{p}} = \mathbf{p}^0$ yields the Laspeyres-type "true" Allen quantity index $Q_{A-L} = \frac{\mathbf{p}^0 \mathbf{q}(\mathbf{p}^0, u^1)}{\mathbf{p}^0 \mathbf{q}^0}$, where $\mathbf{p}^0 \mathbf{q}^0 = \mathbf{p}^0 \mathbf{q}(\mathbf{p}^0, u^0)$, and setting $\bar{\mathbf{p}} = \mathbf{p}^1$ yields the Paasche-type "true" Allen quantity index $Q_{A-P} = \frac{\mathbf{p}^1 \mathbf{q}^1}{\mathbf{p}^1 \mathbf{q}(\mathbf{p}^1, u^0)}$, where $\mathbf{p}^1 \mathbf{q}^1 = \mathbf{p}^1 \mathbf{q}(\mathbf{p}^1, u^0)$.

The Laspeyres- and Paasche-type "true" Allen quantity index numbers can also be obtained by deflating the nominal income ratio between the two observation points by the Paasche- and Laspeyres-type "true" Konüs price index numbers, that is:

(3.3)
$$Q_{A-L} \equiv \frac{\mathbf{p}^0 \mathbf{q}(\mathbf{p}^0, u^1)}{\mathbf{p}^0 \mathbf{q}^0} = \frac{\mathbf{p}^1 \mathbf{q}^1}{\mathbf{p}^0 \mathbf{q}^0} / \frac{\mathbf{p}^1 \mathbf{q}^1}{\mathbf{p}^0 \mathbf{q}(\mathbf{p}^0, u^1)} = \frac{\mathbf{p}^1 \mathbf{q}^1}{\mathbf{p}^0 \mathbf{q}^0} / P_{K-P}$$

(3.4)
$$Q_{A-P} \equiv \frac{\mathbf{p}^1 \mathbf{q}^1}{\mathbf{p}^1 \mathbf{q}(\mathbf{p}^1, u^0)} = \frac{\mathbf{p}^1 \mathbf{q}^1}{\mathbf{p}^0 \mathbf{q}^0} / \frac{\mathbf{p}^1 \mathbf{q}(\mathbf{p}^1, u^0)}{\mathbf{p}^0 \mathbf{q}^0} = \frac{\mathbf{p}^1 \mathbf{q}^1}{\mathbf{p}^0 \mathbf{q}^0} / P_{K-L}$$

The theory of bounds with respect to the quantity index numbers is similar to that of the price index numbers. Following Konüs' suggestion, any point of the numerical interval between these two index numbers could correspond to the "true" quantity index with a certain level of relative prices.

Konüs (1924) also considered various situations in relation to the ranking between the Laspeyres and Paasche indices. In summary, from the point of view of demand, the following alternative cases are possible:

Case 1: Laspeyres < Paasche

(3.5)
$$P_{K-L} < P_L < P_P < P_{K-P}$$

Case 2: Laspeyres ≥ Paasche

(3.6) $$P_{K-L} < P_P \le P_L < P_{K-P}$$

(3.7) $$P_{K-L} \le P_P < P_{K-P} \le P_L$$

(3.8) $$P_P \le P_{K-L} < P_L \le P_{K-P}$$

(3.9) $$P_P < P_{K-P} \le P_{K-L} < P_L$$

(3.10) $$P_P < P_{K-L} \le P_{K-P} < P_L$$

The indeterminacy of the numerical value of "true" index with respect to the Laspeyres-Paasche interval seemed to be eliminated, at least in practice, by various alternative ways. Konüs observed that it is always possible to find a reference utility level, say u^*, such that the cost of living index falls between the Laspeyres and Paasche indexes, that is

(3.11) $$P_P > P_K^* > P_L \quad \text{in case 1}$$

or

(3.12) $$P_P < P_K^* < P_L \quad \text{in case 2.}$$

Konus claimed that these results would suggest that we can work with the Laspeyres and Paasche bounds and take an average of the two to approximate the "true" price index P_K^*. This solution is valid, however, only locally with reference to a particular (unknown) level of utility. Moreover, it has implications also on the Allen quantity index derived implicitly by deflation: the resulting quantity index would fail, in general, to satisfy the homogeneity requirements (if the elementary quantities are multiplied by λ, the resulting index fails to be equal to the same factor λ, as required).

4. "Exact" and "superlative" index numbers

A second alternative to solve the indeterminacy of "true" index number was offered by Byushgens (1924) and Konüs and Byushgens (1926) by introducing the concept of "exact" index numbers for the true aggregator function. They showed that the Fisher "ideal" index formula (the geometric mean of the Laspeyres and Paasche index numbers) may be numerically equal to the ratios of values taken by a quadratic aggregator function. If the observed data were generated by a demand governed by such function, then the transitivity o circularity property would be satisfied by Fisher “ideal” index formula. Following the modern generalization of their proposition, let us assume a utility function consistent with the following minimum expenditure function has the *quadratic mean-of-order-r* functional form $C(\mathbf{p},u) \equiv c_{Q^r}(\mathbf{p},u)\cdot u$, where $c_{Q^r}(\mathbf{p},u) = (\mathbf{p}^{r/2}\mathbf{A}(u)\mathbf{p}^{r/2})^{1/r}$ with $-\infty \le r < 0,\ 0 < r \le \infty$, and the matrix $\mathbf{A}(u)$ is a normalized symmetric matrix of positive coefficients $a_{ij}(u) = a_{ji}(u)$ satisfying the restriction $\sum_i \sum_j a_{ij}(u) = 1$, so that $c_{Q^r}(\mathbf{p},u) = 1$ if $\mathbf{p} = [11...1]$.

The functional form c_{Q^r} can be seen as a generalization of a CES functional form, to which it collapses if all $a_{ij} = 0$ for $i \neq j$ (see McCarthy, 1967 and Kadiyala, 1972), and it reduces to the Generalized Leontief functional form with $r = 1$ (Denny, 1972, 1974) and the Konüs-Byushgens (1926) functional form with $r = 2$ (Diewert, 1976, p. 130).

We have, in fact,

$$(4.1)\qquad \frac{c_{Q^r}(\mathbf{p}_1,u_1)}{c_{Q^r}(\mathbf{p}_0,u_0)} = \frac{\left(\mathbf{p}_1^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}\right)^{1/r}}{\left(\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}\right)^{1/r}}$$

$$= \left[\frac{\mathbf{p}_1^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}} \cdot \frac{\mathbf{p}_1^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}\right]^{1/r} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}}\right]^{1/r} \quad \text{since } \mathbf{p}_1^{r/2}\mathbf{A}(u_t)\mathbf{p}_0^{r/2} = \mathbf{p}_0^{r/2}\mathbf{A}(u_t)\mathbf{p}_1^{r/2}$$

with a symmetric $A(u_t)$

$$= \left[\frac{\mathbf{p}_1^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\hat{\mathbf{p}}_1^{1-r/2}\hat{\mathbf{p}}_1^{r/2-1}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}} \cdot \frac{\mathbf{p}_1^{r/2}\hat{\mathbf{p}}_0^{1-r/2}\hat{\mathbf{p}}_0^{r/2-1}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}}\right]^{1/r} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}}\right]^{1/r}$$

where ^ denotes a diagonal matrix formed with the elements of a vector

$$= \left[\frac{\sum_i \frac{p_{1i}^{r/2}}{p_{0i}^{r/2}} s_{0i}}{\sum_i \frac{p_0^{r/2}}{p_1^{r/2}} s_{1i}}\right]^{\frac{1}{r}} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}}\right]^{1/r}$$

where $s_{ti} \equiv \dfrac{p_{ti}q_{ti}}{\sum_j p_{tj}q_{tj}} = \dfrac{p_{ti}^{r/2}\sum_j a_{ij}(u_t)\,p_{tj}^{r/2}}{\mathbf{p}_t^{r/2}\mathbf{A}(u_t)\mathbf{p}_t^{r/2}}$, which is the observed value share of the ith quantity

$q_{ti} = \partial C(\mathbf{p}_t,u_t)/\partial p_{ti} = \dfrac{\partial c(\mathbf{p}_t,u_t)}{\partial p_{ti}} \cdot u_t = \dfrac{p_{ti}^{\frac{r}{2}-1}\sum_j a_{ij}(u_t)\,p_{tj}^{r/2}}{(\mathbf{p}_t^{r/2}\mathbf{A}(u_t)\mathbf{p}_t^{r/2})^{1-\frac{1}{r}}} \cdot u_t$ by Shephard’s lemma, with a_{ij} being the (i,j) element of matrix $\mathbf{A}$. Thus, the index number yields exactly (is “exact” for) the same numerical value that would be obtained as a ratio of the values of the underlying function in the two compared situations. Diewert (1976) called "superlative" the index numbers that are exact for flexible functional forms and described them as approximating each other up to the second order. However, it has been noted that, in practice, these index numbers are far from being second-order approximations and even differ from each other more than Laspeyers and Paasche indexes themselves (see Milana, 2005 and Hill, 2006a). This terminology diverges in meaning from that used by Fisher (1922), who has defined "superlative" those index numbers that simply performed numerically very closely to his "ideal" index formula with his dataset.

Since all the price variables and utility are considered here at their current levels, the shares s_{ti} are those actually observed. As we shall see below, in the homothetic case, we have $C(\mathbf{p},u) = c(\mathbf{p}) \cdot u$ and, consequently, the observed shares s_{ti} are equal to the theoretical weights that are functions only of prices (with $\mathbf{A}(u_0) = \mathbf{A}(u_1) = \mathbf{A}$).

The first multiplicative bracketed element of the last line of (4.1) can be considered as a candidate price index number

(4.2) $$P_{Q^r}(\mathbf{p_0},\mathbf{p_1},\mathbf{q_0},\mathbf{q_1}) \equiv \left[\frac{\sum_i \frac{p_{1i}^{r/2}}{p_{0i}^{r/2}} s_{0i}}{\sum_i \frac{p_0^{r/2}}{p_1^{r/2}} s_{1i}} \right]^{\frac{1}{r}}$$

which corresponds to Diewert's (1976, p.131) *quadratic mean-of-order-r* price index number. If r = 2, then the price index P_{Q^r} is to the "ideal" Fisher index.

It should be noted that, in the general non-homothetic case, the price index number defined by the right-hand side of (4.2) is ill-defined as a "true" price index. In fact, from (4.1), we have:

(4.3) $$\left[\frac{\sum_i \frac{p_{1i}^{r/2}}{p_{0i}^{r/2}} s_{0i}}{\sum_i \frac{p_0^{r/2}}{p_1^{r/2}} s_{1i}} \right]^{\frac{1}{r}} = \frac{c_{Q^r}(\mathbf{p}_1,u_1)}{c_{Q^r}(\mathbf{p}_0,u_0)} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}} \right]^{1/r}$$
$$= \left[\frac{\mathbf{p}_1^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_1)\mathbf{p}_1^{r/2}} \right]^{1/r} \cdot \left[\frac{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2}\mathbf{A}(u_0)\mathbf{p}_0^{r/2}} \right]^{1/r}$$

where the index obtained is clearly not a pure function of elementary price variables because it is also a function of the utility levels at the two compared points. This has a number of consequences and makes transitivity of a bilateral formula impossible in the context of multilateral comparisons. All the existing multilateral approaches based on index numbers are not really multilateral being based *de facto* on "star systems" consisting in indirect comparisons made by linking bilateral indexes of each observation point with respect to a common reference point. This is the case of the G-EKS and CCD methods using, respectively, bilateral Fisher ideal index formulas (with $P_{Q^r}(\mathbf{p_0},\mathbf{p_1},\mathbf{q_0},\mathbf{q_1})$ where r = 2) and Törnqvist index number (with $P_{Q^r}(\mathbf{p_0},\mathbf{p_1},\mathbf{q_0},\mathbf{q_1})$ where $r \to 0$). The Geary-Khamis procedure uses bilateral Laspeyres-type index formulas where the weights are those of the common reference point.

Let us consider the following weighted geometric average of the direct bilateral price indexes of an arbitrary functional form $P(\mathbf{p}_i,\mathbf{p}_j,\mathbf{q}_i,\mathbf{q}_j)$ of the observation point i relative to all the observation points $j = 0,M$ (including itself):

(4.4) $$\bar{P}^i(\mathbf{P},\mathbf{Q}) \equiv \left\{ \prod_{j=0}^{M} [P(\mathbf{p}_i,\mathbf{p}_j,\mathbf{q}_i,\mathbf{q}_j)]^{\sigma_j} \right\}^{\frac{1}{\sum_k \sigma_k}}$$

where $\mathbf{P} \equiv [\mathbf{p_0},\mathbf{p_1},...,\mathbf{p_M}]$ and $\mathbf{Q} \equiv [\mathbf{q_0},\mathbf{q_1},...,\mathbf{q_M}]$, and σ_j denote the appropriate weight for country j such that $\sum_k \sigma_k = 1$. It can be noted that $\bar{P}^i(\mathbf{P},\mathbf{Q})$ has the meaning of absolute price relative to an average price calculated over all the observation points.

In this context, the bilateral comparisons are made indirectly using the ratios

(4.5) $$\bar{P}^{i,j}(\mathbf{P},\mathbf{Q}) \equiv \bar{P}^i(\mathbf{P},\mathbf{Q}) / \bar{P}^j(\mathbf{P},\mathbf{Q})$$

(4.6) $$\tilde{\bar{Q}}^{i,j}(\mathbf{P},\mathbf{Q}) \equiv (\mathbf{p}_i\mathbf{q}_i / \mathbf{p}_j\mathbf{q}_j) / \bar{P}^{i,j}(\mathbf{P},\mathbf{Q})$$

The definition of the methodology of multilateral interspatial comparisons is completed by specifying the weights σ_j and the functional form of the direct bilateral indexes used as building blocks. We note that:

i) If $\sigma_j = 1/M$ (with M being the number of the examined points of observation) and all the direct bilateral indexes $P(\mathbf{p}_i,\mathbf{p}_j,\mathbf{q}_i,\mathbf{q}_j)$ are Fisher ideal price indexes P_{Q^r} with r =2, then (4.4) corresponds to the so-called *G-EKS* method (Gini, 1924, 1931; Elteto and Koves, 1964; Szulc, 1964); whereas if $\sigma_j = 1/M$ and $P(\mathbf{p}_i, \mathbf{p}_j, \mathbf{q}_i, \mathbf{q}_j)$ are Tömqvist input price indices P_{Q^r} with *r* → 0, then (4.4) corresponds to the so-called *CCD* method (Caves, Christensen and Diewert, 1982). The weights $\sigma_j = 1/M$ have been called "democratic", since they are the same for all the observation points;

ii) If $\sigma_j = \mathbf{p_j q_j} / \sum_r \mathbf{p_r q_r}$*,* then (4.4) corresponds to the "plutocratic" weighting system;

iii) If $\sigma_j = u_j / \sum_r u_r$ *,* then (4.4) corresponds to the "own share" system in the case *u* is an observed and measurable variable (as, for example, gross output in production).

Aggregation consistency over countries requires that the weights be based on the relative importance of the examined countries. If the weights are proportional to the production size of the countries, then the empirical results must be affected by a hypothetical or real splitting or aggregation of the countries. For this reason, the weighted systems *ii)* and *iii)* are preferable. The "plutocratic" weight system *ii),* however, is not invariant to scale changes in the prices of any one observation point.

In the general non-homothetic case, the multilateral indexes are transitive by construction, that is $\bar{P}^2(\mathbf{P},\mathbf{Q}) / \bar{P}^0(\mathbf{P},\mathbf{Q}) = [\bar{P}^2(\mathbf{P},\mathbf{Q}) / \bar{P}^1(\mathbf{P},\mathbf{Q})] \cdot [\bar{P}^1(\mathbf{P},\mathbf{Q}) / \bar{P}^0(\mathbf{P},\mathbf{Q})]$ and do not coincide with the respective direct bilateral indexes, that is $\bar{P}^2(\mathbf{P},\mathbf{Q}) / \bar{P}^0(\mathbf{P},\mathbf{Q}) \neq P^{0,2}(\mathbf{p_0}, \mathbf{p}_2, \mathbf{q_0}, \mathbf{q}_2)$.

All these methods are, however, seriously flawed in the general non-homothetic case because the resulting indexes fail to be pure price and quantity indexes. Let us see this critical aspect in some more detail. In the general non-homothetic case, it is easy to verify that the volume index obtained by deflating the nominal expenditure by means of the price index does not satisfy the linear homogeneity property (if the elementary quantities change proportionally by a common factor *λ*, the aggregating index fail, in general, to change by the same factor *λ*, as expected). This can be immediately seen if the compared quantity vector $\mathbf{q_1}$ happens to be proportional to the base quantity vector $\mathbf{q_0}$ by *λ*. In general, we might obtained the unacceptable result that the volume index does not result to be equal to *λ*. In the case of c_{Q^r}, if $\mathbf{A}(u_0) \neq \mathbf{A}(u_1)$, while $\mathbf{p_1} \neq \mathbf{p_0}$ and these prices are such that $\mathbf{q_1} = \lambda \mathbf{q_0}$, then the aggregate volume index measured implicitly by deflating the index of nominal expenditure by the index $P_{Q^r}(\mathbf{p_0}, \mathbf{p_1}, \mathbf{q_0}, \mathbf{q_1})$ is

$$(\mathbf{p_1 q_1} / \mathbf{p}_0\mathbf{q}_0) / P_{Q^r}(\mathbf{p_0}, \mathbf{p_1}, \mathbf{q_0}, \mathbf{q_1}) = (\mathbf{p_1}\lambda\mathbf{q}_0 / \mathbf{p}_0\mathbf{q}_0) / P_{Q^r}(\mathbf{p_0}, \mathbf{p_1}, \mathbf{q_0}, \mathbf{q_1}) \neq \lambda$$

This means that $P_{Q^r}(\mathbf{p_0}, \mathbf{p_1}, \mathbf{q_0}, \mathbf{q_1})$ is a spurious price index that incorporates a non-price component, which in this case is

(4.7)
$$P_{Q^r}(\mathbf{p_0}, \mathbf{p_1}, \mathbf{q_0}, \mathbf{q_1}) / (\mathbf{p_1} \cdot \mathbf{q_1} / \mathbf{p_0} \cdot \mathbf{q_0}) / \lambda$$

$$= \left[\frac{\mathbf{p}_1^{r/2} \mathbf{A}(u_1) \mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2} \mathbf{A}(u_1) \mathbf{p}_1^{r/2}} \right]^{1/r} \left[\frac{\mathbf{p}_0^{r/2} \mathbf{A}(u_0) \mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2} \mathbf{A}(u_0) \mathbf{p}_0^{r/2}} \right]^{1/r} / (\mathbf{p_1} \cdot \mathbf{q}_0 / \mathbf{p_0} \cdot \mathbf{q_0})$$

$$= \left[\frac{\mathbf{p}_1^{r/2} \mathbf{A}(u_1) \mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2} \mathbf{A}(u_1) \mathbf{p}_1^{r/2}} \right]^{1/r} \left[\frac{\mathbf{p}_0^{r/2} \mathbf{A}(u_0) \mathbf{p}_1^{r/2}}{\mathbf{p_1} \hat{\mathbf{p}}_0^{r/2-1} \mathbf{A}(u_0) \mathbf{p}_0^{r/2}} \right]^{1/r} = (\mathbf{p_1} \cdot \mathbf{q}_1 / \mathbf{p}_1 \cdot \mathbf{q_0}) / \left[\frac{\mathbf{p}_0^{r/2} \mathbf{A}(u_1) \mathbf{p}_1^{r/2}}{\mathbf{p}_0^{r/2} \mathbf{A}(u_0) \mathbf{p}_1^{r/2}} \right]^{1/r} \cdot \frac{u_1}{u_0}$$

If we attempt to correct $P_{Q^r}(\mathbf{p_0}, \mathbf{p_1}, \mathbf{q_0}, \mathbf{q_1})$ for this non-price component, we might end up to an index that is no longer linear homogeneous in prices, which is nonsense for a price index, which should be linearly homogeneous by construction. This shows that price and quantity indexes both satisfying the linear homogeneity property do not exist in the general non-homothetic case. Any attempt to use these indexes both in bilateral and multilateral context is doomed to yield seriously distorted results.

The same unacceptable result with the implict measure of volume index may occur by using the Törnqvist price index number as a deflator. This corresponds to the limit of P_{Q^r} as r tends to 0, that is:

(4.8)
$$\lim_{r \to 0} P_{Q^r} = P_T \equiv \exp[\frac{1}{2} \sum_i (s_{0i} + s_{1i})(\ln p_{1i} - \ln p_{0i})]$$

which is exact for the *translog* unit cost function

(4.9)
$$c_T(\mathbf{p}, u) = \exp(\alpha_0 + \alpha_u \ln u + \sum_i \alpha_i \ln p_i + \sum_i \alpha_{iu} \ln p_i \ln u + \frac{1}{2} \sum_i \sum_j \alpha_{ij} \ln p_i \ln p_j)$$

We note that, in the homothetic case, if the observed data were generated by a demand consistent with a minimum quadratic cost function $c_{Q^r}(\mathbf{p}, u) = c_{Q^r}(\mathbf{p}) \cdot u$ with specific parameter values over the set of examined data on prices and quantities, then we would have

(4.10) $$\frac{c_{Q^r}(\mathbf{p}_N)}{c_{Q^r}(\mathbf{p}_0)} = \frac{c_{Q^r}(\mathbf{p}_1)}{c_{Q^r}(\mathbf{p}_0)} \cdot \frac{c_{Q^r}(\mathbf{p}_2)}{c_{Q^r}(\mathbf{p}_1)} \cdot \ldots \cdot \frac{c_{Q^r}(\mathbf{p}_N)}{c_{Q^r}(\mathbf{p}_{N-1})}$$

and, therefore,

(4.10a) $$P_{Q^r}(p_0, p_N, q_0, q_N) = P_{Q^r}(p_0, p_1, q_0, q_1) \cdot P_{Q^r}(p_1, p_2, q_1, q_2) \ \ldots \cdot P_{Q^r}(p_{N-1}, p_N, q_{N-1}, q_N),$$

that is the exact index number P_{Q^r} would satisfy the transitivity test as well as all the other Fisher's tests between the observation points taken into exam[4]. If the transitivity property is not satisfied, then either the demand is not governed by a rational behaviour or the index number formula is not consistent with a cost function or utility function that rationalizes the data.

At time of the "discovery" of Konüs and Byushgens (1926), the concept of homotheticity of indifference curves and its relationship with existence of a pure price (and quantity) index was not widely known. The concept of homotheticity was explicitly spelled out by Shephard (1953) and Malmquist (1953) in the field of production technology and independently by Afriat (1972) using the terminology of "conical functions" in the field of consumer utility. Earlier contributions dating back at least from Antonelli (1886) and including Frisch (1936, p. 25) and Samuelson (1950, p. 24) have dealt with it implicitly.

When the "true" price index defined by Konüs is not independent of the utility level, as in the general non-homothetic case, the corresponding Allen "true" quantity index fails to be linearly homogeneous (if all the elementary quantities are multiplied by a factor λ, then the index number fails to be proportional by the same factor λ).

In Allen's (1949, p. 199) words, "[t]he index has no meaning unless we make the assumption that the preference map is the same in the two situations". This affects, in a way,

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[4] In this (homothetic) case, by defining $P_{Q^r}(p_0, p_1, \ldots, p_N, q_0, q_1, \ldots, q_N) \equiv P_{Q^r}(p_0, p_1, q_0, q_1) \cdot P_{Q^r}(p_1, p_2, q_1, q_2) \ \ldots \cdot P_{Q^r}(p_{N-1}, p_N, q_{N-1}, q_N)$, a "true" and "exact" bilateral index can be rewritten in an equivalent multilateral form so that $P_{Q^r}(p_0, p_N, q_0, q_N) = P_{Q^r}(p_0, p_1, \ldots, p_N, q_0, q_1, \ldots, q_N)$. The right-hand side of this equivalence is a multilateral form similar, *mutatis mutandis*, to that considered for the quantity index by van Veelen (2008) for the homothetic case. It is perfectly consistent with the classic definition of a bilateral true and exact index defined originally by Konüs and Byushgens (1926) (see also Afriat, 1972, Samuelson and Swamy, 1974, p. 573, fn. 9, Diewert, 1976, pp. 132-133, Diewert and Hill, 2009, p. 5 for further discussion). In the non-homothetic case, both bilateral and multilateral index numbers are ill-defined as "true" indexes.

also the price index: although this index is always linearly homogeneous by construction in the non-homothetic case it results to be a *spurious* price index whose weights are functions not only of prices but also of the utility level and, then, of the demanded relative quantities. This has been often overlooked even in the current literature on economic index numbers.

In the quadratic function considered above, the weights be functions only of prices if and only if $A(u_1) = A(u_0) = A$. In the application of indexes defined by Divisia (1925), this is called "path independence" since the index is independent of the path taken with respect to the reference quantity variables. Hulten (1973) has shown that the Divisia index is path-independent if and only if the underlying function is homothetic (tastes do not change). This can be seen immediately related to the limit of infinitesimal changes in the Törnqvist index number

(4.11) $$d \ln P_{Div} = \lim_{\Delta t \to 0} \left[\frac{1}{2} \sum_i (s_{ti} + s_{(t+\Delta t)i}) \cdot \frac{\ln p_{(t+\Delta t)i} - \ln p_{ti}}{\Delta t} \right] = \sum_i s_{ti} \frac{d \ln p_{ti}}{dt}$$

hence

(4.12) $$P_{Div}^{0,1} = \exp\left(\int_{t=0}^{t=1} \sum_i s_{ti} \frac{d \ln p_{ti}}{dt} dt\right)$$

which is the Divisia price index. If the weights s_{ti} are not functions of the prices alone, but depend also on relative levels of the reference quantities (as in the case with changes in tastes), then the Divisia price index is not a "pure" price index and depends on the particular path taken by the shares.

These considerations were already implicit in the analysis of contributors in the early part of last century, who were well aware of the importance of homothetic tastes for the existence of economic aggregate index numbers. Bowley (1899), for example, in search of a constant-utility price index had been among the first proponent of the geometric mean of the Laspeyres and Paasche indexes (which had later become famous as Fisher "ideal" index). He also devised another index as an approximation to the constant-utility price index given by the following formula, previously proposed by Edgeworth:

(4.13) $$P_E \equiv \frac{\mathbf{p}^1(\mathbf{q}^0 + \mathbf{q}^1)}{\mathbf{p}^0(\mathbf{q}^0 + \mathbf{q}^1)}$$

to be applied under the hypothesis of *no changes in tastes*. He, in fact, wrote: “Assume that our records represent the expenditure of an average man, and that the satisfaction he derives from his purchases is a function of the quantities bought only, say $u(\mathbf{q})$, are the numbers of units bought of the *n* commodities. Further, suppose that the form and constants of this function are unchanged over the period considered. The last condition limits the measurement to an interval of time in which customs and desires have not changed and to a not very wide range of real income. The analysis and conclusions do not apply to comparisons between citizens of two countries, nor over, say, 60 years in one country” (Bowley, 1928, pp. 223-224). As it will be seen more extensively below, identical preferences, implying a homothetic utility function, have been noted as early as the work of Antonelli (1886) as a necessary and sufficient condition for the recoverability of a utility function from the observed market demand data.

It is remarkable, however, that also the foregoing Bowley-Edgeworth index number does not satisfy the requirement of transitivity. In general, the lack of transitivity would signal the poor approximation given by the formulas chosen. This is the situation encountered particularly in interspatial comparisons, where the alternative measures could differ more than 100% even with “superlative” index numbers (see, *e.g.*, Hill, 2006a, 2006b). Given the discouraging results obtained with specific index-number formulas, we now turn to alternative lines of thought have roots in the contributions to the fields of index number theory by Keynes, Hicks, Samuelson, and Afriat.

5. Antonelli’s integrability conditions

A third alternative approach to solve the practical indeterminacy of the two-sided bounds of the unknown “true” index number can be associated with the conditions of integrability of the market demand functions. We may observe that two-sided limits could always be advocated for a Konus-type “true” index numbers as follows:

(5.1) $$\underbrace{\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_1,u_0)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_1,u_0)}}_{?}<\underbrace{\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_1,u_0)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_0)}}_{P_{K-L}}<\underbrace{\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_0,u_0)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_0)}}_{P_L}$$

(5.2) $$\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_1,u_1)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_1,u_1)}<\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_1,u_1)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_1)}<\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_0,u_1)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_1)}$$

$$\underbrace{\qquad\qquad}_{P_P}\quad\underbrace{\qquad\qquad}_{P_{K-P}}\quad\underbrace{\qquad\qquad}_{?}$$

As Hicks (1939, 1946, p. 330) pointed out, if price movements do not occur on the same indifference level, that is $u_0 \neq u_1$, then there is a change in real income. The consequent income effect may distort the orthodox relation between the index numbers, that is it may happen $\frac{\mathbf{p}_1 \cdot \mathbf{x}(\mathbf{p}_1, u_0)}{\mathbf{p}_0 \cdot \mathbf{x}(\mathbf{p}_1, u_0)} \neq P_P$ and $\frac{\mathbf{p}_1 \cdot \mathbf{x}(\mathbf{p}_0, u_1)}{\mathbf{p}_0 \cdot \mathbf{x}(\mathbf{p}_0, u_1)} \neq P_L$. We can note that $\mathbf{x}(\mathbf{p}_0, u_1)$ and $\mathbf{x}(\mathbf{p}_1, u_0)$ are compensated demanded quantities that in general are not observable, whereas $\mathbf{x}(\mathbf{p}_0, u_0) = \mathbf{x}(\mathbf{p}_0, y_0)$ and $\mathbf{x}(\mathbf{p}_1, u_1) = \mathbf{x}(\mathbf{p}_1, y_1)$, with y_0 and y_1 representing nominal income, are the observed demanded quantities at time 0 and 1, respectively. If the consumption choices are optimal, all these quantities can be referred to the first-order conditions of utility maximization under budget constraint.

The necessary and sufficient "mathematical" conditions for the recoverability of the utility function were stated for the first time in the extraordinary Antonelli's (1886) memoir[5] which had remained difficult to find for a long time. They consist simply in the "symmetry" of all pairs of cross price-quantity effects on the uncompensated demand. These conditions are to be complemented with the so-called "economic" integrability condition reflecting the convexity properties of the utility function due to the *negative* correlation between relative prices and demanded quantities of which Antonelli (1886, subsection 2.12) was aware[6] (the Antonelli matrix should therefore be not only symmetric, as required by the "mathematical" integrability, but also negative-semidefinite). Antonelli's conditions were defined on the *indirect* demand equations (where prices are determined in relation with the "observed"

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[5] This memoir was exceptional in many ways, including the fact that it was the only known contribution to economic theory (apart from another study on compulsory amortization of capital) by this Italian author, who soon undertook a successful career of entrepreneurial engineer. This amazing memoir anticipated by more than sixty years the major developments of modern economic theory of demand and many basic concepts of duality theory (see, for example, Chipman, 1971 and Martina, 2000 for detailed accounts of these anticipations). Privately printed (not really published) at Pisa, Italy, this document was still fortuitously known to a small circle of economists when it was eventually popularized in a famous review article by Samuelson (1950), who admittedly based his account of Antonelli's integrability conditions only on second- and third-hand citations (for a modern reformulation of these conditions, see Hurwitcz, 1971, Hurwicz and Uzawa, 1971, and Debreu, 1972).

[6] On this point, the original text contains, however, a lapse indicating a positive sign instead of a negative price effect. This has been noted by G. Ricci, *Giornale degli Economisti*, N.S., Vol. 10, 1951, p. 291 and in the commentary on the English translation published in Chipman *et al*. (1971, p. 350, fn. 57).

uncompensated demand quantities) leading to the recovery of the *direct* utility function defined in the space of demanded quantities.

A useful dual version of integrability conditions was fully described by Hurwicz and Uzawa (1972) for the recovery of *indirect* utility as a function of relative prices and real income starting from *direct* demand equations (Antonelli (1886) was evidently aware also of this second approach since he had also redefined explicitly the relevant elements of his analysis in terms of the indirect utility function and direct demand equations). This second version can be immediately related to the Konüs-type index numbers and their limits by applying the mathematical and economic integrability conditions to the more familiar Slutsky matrix of second-order derivatives.

We start by postulating a continuous twice differentiable convex direct utility function $u = U(\mathbf{x})$. The utility maximization problem (UMP) is to maximize $U(\mathbf{x})$ under the condition of budget constrain $\mathbf{p} \cdot \mathbf{x} = y$. The Lagrangian is $L(\mathbf{x}, \mu) = U(\mathbf{x}) + \mu(y - \mathbf{p} \cdot \mathbf{x})$, where μ is the Lagrange multiplier that, as shown below, represents the "marginal utility of money". The first-order conditions are that $\partial L / \partial x_i = \partial U / \partial x_i - \mu p_i = 0$ which implies

(5.3) $$p_i = \frac{\partial U}{\partial x_i} \cdot \frac{1}{\mu} = \frac{\partial U}{\partial x_i} \cdot \frac{1}{\mu} \cdot \frac{y}{\sum_r p_r x_r} \quad \text{given the budget constraint}$$

hence, the uncompensated inverse demand (ordinary demand-price) functions can be obtained:

(5.4) $$p_i^m(\mathbf{x}) = \frac{y \cdot \partial U(\mathbf{x}) / \partial x_i}{\sum_r x_r \cdot \partial U(\mathbf{x}) / \partial x_r}$$

This relation has become known as Hotelling-Wold's Identity after Hotelling (1935, p.71, eq. 3.4) and Wold (1944, pp. 69-71)(1953, p. 145). In terms of budget shares, it becomes:

(5.5) $$x_i \cdot \frac{p_i^m}{y} = s_i^m(\mathbf{x}) = \frac{x_i \cdot \partial U(\mathbf{x}) / \partial x_i}{\sum_r x_r \cdot \partial U(\mathbf{x}) / \partial x_r} = \frac{x_i \cdot \partial U(\mathbf{x}) / \partial x_i \cdot \dfrac{1}{U}}{\sum_r x_r \cdot \partial U(\mathbf{x}) / \partial x_r \cdot \dfrac{1}{U}} = \frac{\dfrac{\partial \ln U}{\partial \ln x_i}}{\sum_r \dfrac{\partial \ln U}{\partial \ln x_i}}$$

The indirect utility function[7] (defined by Antonelli, 1886, subsections 10 and 11 [Engl. transl. 1971, pp. 348-350], and later rediscovered by Hotelling, 1932, p. 594 (implicitly), Court, 1941, Hicks, 1942, p. 129, and Roy, 1943, 1947, pp. 205-225) can be derived from the budget constrained utility maximization problem (UMP) using the envelop theorem:

(5.6) $$V(\mathbf{p}, y) \equiv \max_{\mathbf{x}} \{U(\mathbf{x}) : y - \mathbf{p}\cdot\mathbf{x} \geq 0; \mathbf{x} \geq 0\}$$

The indirect utility function $V(\mathbf{p}, y)$ is homogeneous of degree zero in $\mathbf{p}$ and y, decreasing in $\mathbf{p}$ and increasing in y.

The Lagrangian is $L(\mathbf{x}, \mu) = U(\mathbf{x}) + \mu(y - \mathbf{p}\cdot\mathbf{x})$ where μ is the Lagrange multiplier and, by the envelope theorem,

(5.7) $$\partial V(\mathbf{p}, y) / \partial p_i = \partial L(\mathbf{x}^*, \mu^*) / \partial p_i = \mu^* x_i^* \quad \text{for all } i\text{'s}$$

$$\frac{\partial V(\mathbf{p}, y)}{\partial y} = \frac{\partial L(\mathbf{x}^*, \mu^*)}{\partial y} = \mu^*$$

And, by solving the two foregoing equations for $\mathbf{x}^*$, we obtain the "observable" uncompensated direct demand (demand-quantity) functions (known as Marshallian demand functions):

(5.8) $$x_i^m(\mathbf{p}, y) = -\frac{\partial V(\mathbf{p}, y) / \partial p_i}{\partial V(\mathbf{p}, y) / \partial y}$$

$$= x_i^{*m}(\boldsymbol{\upsilon}) = \frac{\partial V^*(\boldsymbol{\upsilon}) / \partial \upsilon_i}{\sum_r \upsilon_r \cdot \partial V^*(\boldsymbol{\upsilon}) / \partial \upsilon_i}$$

[7] The designations "direct" and "indirect" for utility functions were introduced by Houthakker (1952-1953, p. 157). The indirect utility function was called by Hotelling (1932, p. 594) *the price potential* "on the basis of physical analogies", thus revealing to know the concepts that Antonelli and Volterra had derived from the physical sciences.

since $\partial V(\mathbf{p}, y)/\partial y$ = $-\frac{1}{y}\sum_r p_r \frac{\partial V}{\partial p_r}$ (by Euler's theorem on homogeneous functions[8]) and, by defining $\boldsymbol{\upsilon} \equiv \mathbf{p} \cdot y^{-1}$, we can rewrite $V(\mathbf{p}, y)$ as $V^*(\boldsymbol{\upsilon}) \equiv V(\boldsymbol{\upsilon}, 1)$ and $x_i^m(\mathbf{p}, y)$ as $x_i^{*m}(\boldsymbol{\upsilon}) \equiv x_i^{*m}(\boldsymbol{\upsilon}, 1).$[9] This is Antonelli's (1886) equation (24) stating that the demand for a commodity is equal to the ratio between "minus the marginal indirect utility of its price" and "the marginal indirect utility of income". Today, it is better known as *Ville-Roy's Identity* after its rediscovery by Ville (1946) and Roy (1947, p. 218-220)(1949, p. 180, eq. A4). The resulting function $x_i^m(\mathbf{p}, y)$ has the property of being continuously once differentable, *i.e.* it is of class C^1 (since V is assumed to be continuously differentiable at least twice) and homogeneous of degree 0 in prices and income, that is $\mathbf{x}^m(\mathbf{p}, y) = \mathbf{x}^m(\lambda\mathbf{p}, \lambda y)$ for every real number λ. The foregoing identity has been also expressed by Roy (1947, p. 222) in terms of budget shares as functions of price elasticities of indirect utility by multiplying $x_i^m(\mathbf{p}, y)$ through by p_i / y, that is

(5.9) $$s_i^m(\mathbf{p}, y) \equiv \frac{p_i x_i^m}{y} = -\frac{p_i \cdot \partial V(\mathbf{p}, y)/\partial p_i}{y \cdot \partial V(\mathbf{p}, y)/\partial y} = \frac{p_i \cdot \partial V(\mathbf{p}, y)/\partial p_i \cdot \frac{1}{V}}{\sum_r p_r \cdot \partial V(\mathbf{p}, y)/\partial p_i \cdot \frac{1}{V}} = \frac{\frac{\partial \ln V^*}{\partial \ln \upsilon_i}}{\sum_r \frac{\partial \ln V^*}{\partial \ln \upsilon_r}}$$

Provided that the direct utility function is strictly increasing in $\mathbf{x}$, $V(\mathbf{p}, y)$ is monotonically and continuously increasing in *y* and can be inverted to yield the expenditure function

(5.10) $$y = C(\mathbf{p}, u)$$

The function $C(\mathbf{p}, u)$ can be also derived from the consumer's expenditure minimization problem (EMP):

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[8] Since the indirect utility function *V* is homogeneous of degree zero in **p** and *y*, by Euler's theorem on homogeneous functions, $y\frac{\partial V}{\partial y} + \sum_r p_r \frac{\partial V}{\partial p_r} = 0.$ Hence, $\frac{\partial V}{\partial y} = -\frac{1}{y}\sum_r p_r \frac{\partial V}{\partial p_r}.$

[9] Since the indirect utility function $V(\mathbf{p}, y)$ is homogeneous of degree 0 in $\mathbf{p}$ and y, the corresponding function $V^*(\boldsymbol{\upsilon})$ is homogeneous of degree -1, hence by Euler's theorem, $\sum_r \upsilon_r \cdot \partial V^*(\boldsymbol{\upsilon})/\partial \upsilon_i = -V^*(\boldsymbol{\upsilon}).$

(5.11) $$C(\mathbf{p},u) \equiv \min_{\mathbf{x}} \{\mathbf{p}\cdot\mathbf{x} : u(\mathbf{x}) \geq U; \mathbf{x} \geq 0\}$$

It indicates how much income is required to achieve the utility level U at prices $\mathbf{p}$. We have the following identities:

(5.12) $$C(\mathbf{p},V(\mathbf{p},y)) = y \quad \text{and} \quad V(\mathbf{p},C(\mathbf{p},u)) = u$$

The compensated (Hicksian) direct demand (demand-quantity) functions are obtained for an arbitrary utility level using Hicks (1939, 1946, p. 331)-Shephard's (1953) lemma, that is

(5.13) $$x_i^h(\mathbf{p},u) = \frac{\partial C(\mathbf{p},u)}{\partial p_i}$$

They have the property of being homogeneous of degree 0 in prices, that is $\mathbf{x}^h(\mathbf{p},u) = \mathbf{x}^h(\lambda\mathbf{p},u)$ for every real number λ.

In terms of budget shares, the foregoing relation becomes

(5.14) $$s_i^h(\mathbf{p},u) = \frac{p_i}{C}\cdot x_i^h(\mathbf{p},u) = \frac{p_i\cdot\partial C(\mathbf{p},u)/\partial p_i}{\sum_r p_i\cdot\partial C(\mathbf{p},u)/\partial p_r} = \frac{(p_i/y)\cdot\dfrac{\partial C}{\partial(p_i/y)}\cdot\dfrac{1}{C}}{\sum_r (p_r/y)\dfrac{\partial C}{\partial(p_r/y)}\cdot\dfrac{1}{C}} = \frac{\dfrac{\partial \ln C^*}{\partial \ln(\upsilon_i)}}{\sum_r \dfrac{\partial \ln C^*}{\partial \ln(\upsilon_r)}}$$

since, by Walras' Law, $y = C(\mathbf{p},u)$.

The compensated inverse demand (demand-price) functions can be derived from the direct utility function written implicitly as

(5.15) $$T(u,\mathbf{x}) \equiv \max_t \{t : U(\mathbf{x}/t) \geq u, \quad t \in R_+^1\}$$

Which is decreasing in u for fixed $\mathbf{x}$. The direct utility function is implicitly defined in $T(u,\mathbf{x})$=1. The compensated inverse demand (demand-price) functions can be derived as follows:

(5.16) $$\frac{p_i}{y} = v_i^h(\mathbf{x},u) = \frac{\partial T(\mathbf{x},u)/\partial x_i}{\sum_r x_r\cdot\partial T(\mathbf{x},u)/\partial x_i}$$

Differently from the compensated demand-quantity functions $x_i^h(\mathbf{p},u)$ that are homogeneous of degree 0 in its price arguments, the compensated demand-price functions $v_i^h(\mathbf{x},u)$ are not necessarily homogeneous of degree 0 in its quantity arguments.

(5.17) $$\frac{p_i x_i}{y} = s_i^h(\mathbf{x},u) = \frac{x_i \cdot \partial T(\mathbf{x},u)/\partial x_i}{\sum_r x_r \cdot \partial T(\mathbf{x},u)/\partial x_i} = \frac{x_i \cdot \dfrac{\partial T}{\partial x_i} \cdot \dfrac{1}{T}}{\sum_r x_r \dfrac{\partial T}{\partial x_i} \cdot \dfrac{1}{T}} = \frac{\dfrac{\partial \ln T}{\partial \ln x_i}}{\sum_r \dfrac{\partial \ln T}{\partial \ln x_r}}$$

which has become later known as Hotelling-Hanoch's lemma after its rediscovery by Hotelling (1935) and Hanoch (1970, 1978) (see Deaton, 1979 and Weymark, 1980 for use of this lemma).

At initial equilibrium condition the Hicksian and Marshallian demand functions are related to each other as follows:

(5.18) $$\mathbf{x}^h(\mathbf{p},u) = \mathbf{x}^m(\mathbf{p},C(\mathbf{p},u))$$

(5.19) $$\mathbf{x}^h(\mathbf{p},V(\mathbf{p},y)) = \mathbf{x}^m(\mathbf{p},y)$$

Differentiating the foregoing equation totally with respect to $\mathbf{p}$ yields, for all pairs of commodities with indexes $i,j = 1, 2, \ldots N$,

(5.20) $$x_{ij}^h = x_{ij}^m + x_j \cdot x_{iy}^m$$

with $x_{ij}^h \equiv \dfrac{\partial x_i^h(\mathbf{p},u)}{\partial p_j}$; $x_{ij}^m \equiv \dfrac{\partial x_i^m(\mathbf{p},y)}{\partial p_j}$; $x_{iy}^m \equiv \dfrac{\partial x_i^m(\mathbf{p},y)}{\partial y}$; and, in the initial equilibrium state, $x_j = x_j^m = x_j^h \equiv \dfrac{\partial C(\mathbf{p},u)}{\partial p_j}$.

Since $x_i^h = \dfrac{\partial C(\mathbf{p},u)}{\partial p_i}$ by Hicks-Shephard's Lemma, then $x_{ij}^h \equiv \dfrac{\partial x_i^h}{\partial p_j} = \dfrac{\partial^2 C(\mathbf{p},u)}{\partial p_i \partial p_j}$. The matrix formed by all $\dfrac{\partial^2 C(\mathbf{p},u)}{\partial p_i \partial p_j}$ (the Hessian of the cost function) is always symmetric by Young's theorem (the order of taking partial derivatives is irrelevant). Hence the "substitution" matrix $[x_{ij}^h]$ is therefore always symmetric.

The matrix $[x_{ij}^m]$ is also symmetric provided that the matrix $[x_j \cdot x_{iy}^m]$ is symmetric. In fact, $x_{ij}^m = x_{ji}^m$ if and only if $x_j \cdot x_{iy}^m = x_i \cdot x_{jy}^m$ or, equivalently,

$$x_{ij}^h - x_j \cdot x_{iy}^m = x_{ji}^h - x_i \cdot x_{jy}^m \quad \text{for all } i,j\text{'s} \tag{5.21}$$

The relation $x_{ij}^m = x_{ij}^h - x_j \cdot x_{iy}^m$ has become known as the Slutsky equation and is regarded as the Fundamental Equation of Value Theory. The symmetry of the matrix formed by the Slutsky equations across all pairs of demand functions, that is $[x_{ij}^m]$=$[x_{ij}^m]$' (with prime indicating a transposed matrix), represents the "mathematical integrability" conditions that have become widely known as "Antonelli conditions" (see Antonelli, 1886, equation (21b)).

As a corollary, Hicks (1939, 1946, p. 310) noted that the symmetry of matrix $[x_j \cdot x_{iy}^m]$ implies that all income elasticities of demand are equal. Moreover, when income elasticities of demand are all equal, these must be necessarily equal to 1 under budget constraint and nonsatiated utility (see, for example, Carey, 1977, p. 1970). In fact, multiplying each equality $x_j \cdot x_{iy}^m = x_i \cdot x_{jy}^m$ through by $y / x_i x_j$ yields equal income elasticities of demand $[y / x_i] x_{iy} = [y / x_j] x_{jy} = \varepsilon_y$. Because of the budget constraint under nonsatiation, by Walras' law, $\sum_r p_r x_r = y$, and differentiating this constraint with respect to y gives

$$\sum_r p_r x_{ry} = 1 \tag{5.22}$$

Hence, in view of this result,

$$\varepsilon_y = \sum_r \varepsilon_y \frac{p_r x_r}{y} = \sum_r \frac{y}{x_r} \cdot x_{ry} \cdot \frac{p_r x_r}{y} = \sum_r p_r x_{ry} = 1 \tag{5.23}$$

This in turn implies that, under the necessary integrability conditions on the observed demand functions, the Engels curves are linear.

It can be noted that, in this case, the utility function can be written in the more general form $U = U[u(\mathbf{x}^m(\mathbf{p}, y)]$, where U is any arbitrary homothetic (homogeneous of degree *r*) function and $u(\mathbf{x}^m(\mathbf{p}, y)$ is the aggregating quantity index function that is linear homogeneous in $\mathbf{x}^m$. Therefore, the income elasticity of the quantity index u is equal to 1. Differentiating U with respect to income, when all income elasticities of demand are equal to 1, that is $x^m_{iy} \cdot y / x^m_i = 1$ and then $x^m_{iy} = x^m_i / y$ for all i's, we have

$$U_y = \sum_i U_i \cdot x^m_{iy} = \sum_i U_i \cdot x^m_i / y = U \cdot r / y \tag{5.24}$$

with $U_y \equiv \partial U / \partial y$, $U_i \equiv \partial U / \partial x_i$, and $x^m_{iy} \equiv \partial x^m_i / \partial y$ since, by Euler's theorem on homogeneous functions of degree *r*, $\sum_i U_i \cdot x^m_i = U_u \cdot \sum_i u_i \cdot x^m_i = U_u \cdot u(\mathbf{x}^m) = U \cdot r$. with $U_u \equiv \partial U / \partial u$.

In the homothetic case, because of budget constraint and nonsatiated utility, hence $U_y = U \cdot r / y$. Since $U(\mathbf{x}) = y \cdot U_y / r = u(\mathbf{x}) \cdot U_u / r$, the quantity index function $u(\mathbf{x})$ is related to the utility level by a multiplicative factor equal to U_u / r, where $U_u = 1$ if $r = 1$.

The level of U is an indeterminate homothetic transformation of quantity level of $u(\mathbf{x})$. However, if income elasticity of utility is equal to 1, that is $U_y = U / y$, and all income elasticities of demand are equal to 1, then $r = 1$ and $U(\mathbf{x}) = u(\mathbf{x})$ and the indirect utility function $V(\mathbf{p}, y)$ must be written as $V(\mathbf{p}, y) = y / a(\mathbf{p})$. in this case, by inversion, the expenditure function $y = C(\mathbf{p}, u)$ can be factored into a price and a quantity component as $C(\mathbf{p}, u) = a(\mathbf{p}) \cdot u(\mathbf{x})$ (since $U(\mathbf{x}) = V(\mathbf{p}, y)$). This factorization of the expenditure function was later independently rediscovered by Shephard (1953) and Afriat (1970, 1972, p. 36) and reproposed by Samuelson (1972) and Samuelson and Swamy (1974).

The sufficiency of Antonelli's "mathematical integrability" conditions for the recovery of the aggregating utility function starting from the observable direct or inverse uncompensated demand functions can be demonstrated by making use of Green's theorem, on which Antonelli (1882) himself had previously devoted special attention. The matrix [x^m_{ij}] of the "observable" uncompensated (Marshallian) demand functions is postulated to be symmetric (implying that, under budget constraint, all income elasticities of the observed demand functions are equal to 1), that is

(5.25) $\dfrac{\partial x_i^m(\mathbf{p}, y)}{\partial p_j} = \dfrac{\partial x_j^m(\mathbf{p}, y)}{\partial p_i}$ (implying $[y/x_i]x_{iy} = [y/x_j]x_{jy} = \varepsilon_y = 1 \text{ with } \sum_r p_r x_r = y$)

The mathematical integrability problem is the following: given the observed data on $\mathbf{x} = \mathbf{x}^m(\mathbf{p}, y)$, does there exist a "potential" function from which this vector can be derived[10]? Antonelli (1886) found that the answer is positive up to an arbitrary constant under the foregoing symmetry conditions. The "economic" part of the integrability problem is the following: how can we recognize the recovered potential function, if any, as a non-satiated utility function? Antonelli (1886) indicated that the elements of this matrix should have the proper algebraic sign (with a lapse, however, he indicated a wrong positive sign instead of a negative one and he did not specify that the substitution matrix has to be negative semi-definite).

There are no known general methods to find a solution of a system of partial differential equations. Frequently, necessary conditions (the so-called "integrability conditions") can be found, but in the very special case of the foregoing symmetry conditions these indeed turn out to be not only necessary (as we have seen above) but also sufficient. To show this, let us describe the easier case of only two goods taken as a reference model. Antonelli (1986) has shown that in the case of two goods, integrability is always possible, but the symmetry conditions lead to the additional homogeneity property.

For mathematical convenience, we rescale all prices by dividing them by y, that is $\bar{p}_r = p_r / y$, since the Marshallian demand functions are homogeneous of degree 0 in prices *and* income (Antonelli normalized prices with the price of the first commodity). By integrating the difference $x_{ij}^m - x_{ji}^m = 0$, we obtain

(5.26) $$\int_R \left(\frac{\partial x_2^m}{\partial \bar{p}_1} - \frac{\partial x_1^m}{\partial \bar{p}_2}\right) d\bar{p}_1 d\bar{p}_2 = 0$$

which, by Green's theorem (see, for example, Verblunsky, 1949), is derivable from the following integral equation

[10] The theory of integrability of total differential equations is referred to as Frobenius Theorem.

(5.27) $$\int_{\gamma-\tau} x_1^m d\overline{p}_1 + x_2^m d\overline{p}_2 = 0$$

hence

(5.28) $$\int_{\gamma} x_1^m d\overline{p}_1 + x_2^m d\overline{p}_2 - \int_{\tau} x_1^m d\overline{p}_1 + x_2^m d\overline{p}_2 = 0$$

The functions $\int_{\gamma} x_1^m d\overline{p}_1 + x_2^m d\overline{p}_2$ and $\int_{\tau} x_1^m d\overline{p}_1 + x_2^m d\overline{p}_2$ have the same value $a(\overline{p}_1, \overline{p}_2)$. The obtained function is well-defined, in the sense that it is independent of the path chosen (γ or τ in this case)[11]. This allows us to show that $\partial a(\overline{p}_1, \overline{p}_2) / \partial \overline{p}_r = a_r\left(\overline{p}_1, \overline{p}_2\right)$. In fact

(5.29) $$\frac{\partial a(\overline{p}_1, \overline{p}_2)}{\partial \overline{p}_1} = \lim_{\overline{p} \to \overline{p}_1} \frac{a(\overline{p}, \overline{p}_2) - a(\overline{p}_1, p_2)}{\overline{p} - \overline{p}_1} = \lim_{\overline{p} \to \overline{p}_1} \frac{\int_{p_1}^{p} a_1(t, \overline{p}_2) dt}{\overline{p} - \overline{p}_1} = a_1(\overline{p}_1, \overline{p}_2)$$

and similarly

(5.30) $$\frac{\partial a(\overline{p}_1, \overline{p}_2)}{\partial \overline{p}_2} = a_2(\overline{p}_1, \overline{p}_2).$$

Thus, the symmetry conditions imposed on the uncompensated demand functions lead to a homogeneous function which is factorised into separable functions of prices and quantities.

In the case of more than two goods, a model example can be developed using a more complex theory using the results derivable from an extended Frobenius Theorem (see, Hartman, 1970 and Afriat, 1977a, 1980a). By rescaling prices, $p_r = \overline{p}_r \cdot y$, we may rewrite the expenditure function as $C(\mathbf{p}, u) = a(\mathbf{p}) \cdot u(\mathbf{x})$ and, therefore, $u = V(\mathbf{p}, y) = y / a(\mathbf{p})$ as seen above. But any linear transformation of utility is also a valid solution, for example $U[u(\mathbf{x})] = C(\overline{\mathbf{p}}, u) = a(\overline{\mathbf{p}}) \cdot u(\mathbf{x})$ provided the price component is rescaled appropriately so that $C(\mathbf{p}, U) = [a(\mathbf{p}) / a(\overline{\mathbf{p}})] \cdot [a(\overline{\mathbf{p}}) \cdot u(\mathbf{x})]$. This is the case of a homogeneneous utility function where the change in income or utility levels by an arbitrary factor brings about the change in all the demanded quantities by exactly the same proportion at given relative prices. It can be said that a necessary and sufficient condition for the utility function to be homogeneous is that the expenditure function can be factorable into a price and quantity level aggregates, that is $C(\mathbf{p}, u) = a(\mathbf{p}) \cdot u(\mathbf{x})$ (see, for example, Afriat, 1970, 1972, p. 36).

[11] This is a result that was later put in evidence by Hulten (1973).

In subsection 4 of his work, Antonelli (1886) developed his analysis further by introducing a linear variation across individuals by dropping the hypothesis of all income elasticities of demand being equal to 1 but maintaining the hypothesis of linear relation of utility with income. For the n^{th} individual the expenditure function is of the type $C_n(\mathbf{p},u_n)=a(\mathbf{p})u_n+b_n(\mathbf{p})$ (where the additive term $b_n(\mathbf{p})$ reflects the n^{th} individual's "subsistence" component of the cost of living). The corresponding indirect utility function is therefore $V_n(\mathbf{p},y)=[y_n-b_n(\mathbf{p})]/a(\mathbf{p})$. The resulting aggregated demand is still a linear function of income and can be transformed to $a(\mathbf{p})u=y^*$, where $y^*=y-b(\mathbf{p})$. This is the approach to aggregation over individuals (or social groups) that have been later reproposed by Afriat (1953)(1953-1956), Gorman (1953), and Nataf (1953).

With a differentiable homothetic function $f^t(\mathbf{z})$, that has the following general form $f^t(\mathbf{z})=F^t[\phi(\mathbf{z})]$ with different parameters or functional form of $F^t(\cdot)$ across the observation points t's where the function $\phi(\cdot)$ is homogenous of degree 1 in $\mathbf{z}$, then the *homothetic share equations* defined as

$$(5.31)\qquad s_i^t=\frac{z_i\cdot\partial f^t(\mathbf{z})/\partial z_i}{\sum_j z_i\cdot\partial f^t(\mathbf{z})/\partial z_i}=\frac{\partial\ln f^t(\mathbf{z})/\partial\ln z_i}{\sum_j\partial\ln f^t(\mathbf{z})/\partial\ln z_i}$$

can be used to construct the Laspeyres- and Paasche-type bounds of the index $\phi(z^1)/\phi(z^0)$, respectively given by $\sum_i s_i^0\frac{z_i^1}{z_i^0}$ and $\left[\sum_i s_i^1\frac{z_i^0}{z_i^1}\right]^{-1}$, since

$$(5.32)\quad s_i^t=\frac{z_i\cdot\partial f^t(\mathbf{z})/\partial z_i}{\sum_j z_i\cdot\partial f^t(\mathbf{z})/\partial z_i}=\frac{z_i\cdot(F^t/d\phi)\cdot\partial\phi(\mathbf{z})/\partial z_i}{(dF^t/d\phi)\cdot\sum_j z_i\cdot\partial\phi(\mathbf{z})/\partial z_i}=\frac{z_i\cdot\partial\phi(\mathbf{z})/\partial z_i}{\phi(\mathbf{z})}=\partial\ln\phi(\mathbf{z})/\partial\ln z_i$$

Therefore, the index $\phi(z^1)/\phi(z^0)$ can be constructed using the observed data that are compatible with infinite number of homothetic transformations. In Antonelli's (1886, [Engl. Tr. 1971, p. 337]) words and mathematical symbols, "there are an infinite family of functions and if U is one of these, all the others are of the form $s(U)$ where s is an arbitrary function" (Engl.

Tr. 1971, p. 337)[12]. However, under the hypothesis of unitary income elasticities of utility and all demanded quantities, the compensated and uncompensated demand-quantity functions become, respectively, $x_i^h(\mathbf{p},u)=\dfrac{\partial a(\mathbf{p})}{\partial p_i}\cdot u(\mathbf{x})$ and $x_i^m(\mathbf{p},y)=\dfrac{\partial a(\mathbf{p})}{\partial p_i}\cdot\dfrac{y}{a(\mathbf{p})}$. The aggregating weights of prices are exactly the same for the compensated and uncompensated demand functions:

(5.33) $$s_i^m(\mathbf{p},y)\equiv\frac{p_i x_i^m}{y}=s_i^h(\mathbf{p},y)\equiv\frac{p_i x_i^h}{y}=\frac{p_i\cdot\partial a(\mathbf{p})/\partial p_i}{\sum_r p_r\cdot\partial a(\mathbf{p})/\partial p_r}=\partial\ln a(\mathbf{p})/\partial\ln p_i.$$

This implies

(5.34) $$\frac{\mathbf{p}_1\cdot\mathbf{x}^h(\mathbf{p}_1,u_0)}{\mathbf{p}_0\cdot\mathbf{x}^h(\mathbf{p}_1,u_0)}=\frac{\mathbf{p}_1\cdot\nabla_{\mathbf{p}_1}a(\mathbf{p}_1)\cdot u(\mathbf{x}_0)}{\mathbf{p}_0\cdot\nabla_{\mathbf{p}_1}a(\mathbf{p}_1)\cdot u(\mathbf{x}_0)}=\frac{\mathbf{p}_1\cdot\nabla_{\mathbf{p}_1}a(\mathbf{p}_1)\cdot u(\mathbf{x}_1)}{\mathbf{p}_0\cdot\nabla_{\mathbf{p}_1}a(\mathbf{p}_1)\cdot u(\mathbf{x}_1)}=\frac{\mathbf{p}_1\cdot\mathbf{x}^h(\mathbf{p}_1,u_1)}{\mathbf{p}_0\cdot\mathbf{x}^h(\mathbf{p}_1,u_1)}=P_P\equiv\frac{\mathbf{p}_1\cdot\mathbf{x}_1^m}{\mathbf{p}_0\cdot\mathbf{x}_1^m}$$

(5.35) $$\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_0,u_1)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_1)}=\frac{\mathbf{p}_1\cdot\nabla_{\mathbf{p}_0}a(\mathbf{p}_0)\cdot u(\mathbf{x}_1)}{\mathbf{p}_0\cdot\nabla_{\mathbf{p}_0}a(\mathbf{p}_0)\cdot u(\mathbf{x}_1)}=\frac{\mathbf{p}_1\cdot\nabla_{\mathbf{p}_0}a(\mathbf{p}_0)\cdot u(\mathbf{x}_0)}{\mathbf{p}_0\cdot\nabla_{\mathbf{p}_0}a(\mathbf{p}_0)\cdot u(\mathbf{x}_0)}=\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_0,u_0)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_0)}=P_L\equiv\frac{\mathbf{p}_1\cdot\mathbf{x}_0^m}{\mathbf{p}_0\cdot\mathbf{x}_0^m}$$

(5.36) $$P_{K-L}=\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_1,u_0)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_0)}=\frac{a(\mathbf{p}_1)\cdot u(\mathbf{x}_0)}{a(\mathbf{p}_0)\cdot u(\mathbf{x}_0)}=\frac{a(\mathbf{p}_1)\cdot u(\mathbf{x}_1)}{a(\mathbf{p}_0)\cdot u(\mathbf{x}_1)}=\frac{\mathbf{p}_1\cdot\mathbf{x}(\mathbf{p}_1,u_1)}{\mathbf{p}_0\cdot\mathbf{x}(\mathbf{p}_0,u_1)}=P_{K-P}$$

where $\nabla_{\mathbf{p}_t}a(\mathbf{p}_t)$ denotes the differentiation of $a(\mathbf{p}_t)$ with respect to the elements of the vector $\mathbf{p}_t$. while, the "economic" integrability condition of non-concave utility imposes the LP-inequality

(5.37) $$P_P<P_L$$

in the case of demand, or $P_P>P_L$ in the case of supply[13]. In view of the results reported above, this inequality becomes the long-sought observable double-bounded interval of the admissible numerical values of the "true" (Konüs-type) index number of consumer cost of living index under the integrability conditions, that is

[12] On this point see also Allen (1932, p. 223).

[13] The strict equality $P_P=P_L$ is ruled out since the demand functions considered in this context are not kinked and the quantity-price correspondence is uniquely determined (both demand-quantity and demand-price functions are assumed to be single-valued). It must be stressed, at this point, that ruling out the possibility of equality between *L* and *P* constitutes an obstacle to the definition of the "true" bounds when price changes are not local.

(5.38) $$P_P < P_{K-P} = P_{K-L} < P_L$$

These double-sided bounds do not define, however, a closed set of possible numerical values of the Konüs "true" price index numbers as the bounds in the above inequalities cannot be considered "true" index numbers themselves. Therefore the "true" index numbers that correspond to the two extreme limits of all possible values of the "true" index numbers, which we may call the "true" bounds, remain unknown. The size of the difference between these "true" bounds and the bounds of the above inequalities depends on the "(local) degree of accuracy of index numbers", using the words of Samuelson,'s (1974, p. 600, fn. 1)[14]. This indeterminateness is inherent in the integrability approach, which needs to assume that the market quantity and price data are generated along uncompensated demand functions that are single-valued and Lipschitz-continuous in prices and total income. In other words, these demand functions are consistent only with smooth strongly convex tastes which exclude linear subset of values implied by the Laspeyers and Paasche index numbers[15]. By contrast, with the general demand correspondence where a set of alternative quantities may be determined for a certain level of relative prices (or vice versa), the demand-quantity or demand-price functions, if any, are multi-valued. This is the case of flat/piece-wise linear or L-shaped indifference curves, respectively. In this general setting, in our context, we may obtain *weak* inequalities (where in (5.38) the strong inequality sign < is replaced with the weak inequality sign $\leq$) defining the double bounds as *"true" index numbers* under the integrability conditions, but only in certain special cases of multivalued demand functions (see, for example, Hurwicz, 1971, p. 210-212).

[14] The degree of accuracy of L and P as approximating measures of the "true" bounds varies with the distance of the price and quantity observations under comparison. For infinitesimal changes in price and quantities, of course, L and P coincide with the "true" bounds.

[15] Noting this fact, Swamy (1984, p. 43, fn. 10) wrote: "This is not to dissuade scholars from using the Laspeyres and Paasche indexes, but merely to urge them to restrict the use of these indexes to local changes in **p**. These indexes can be used to determine bounds for the true index which may not be known".

6. John Maynard Keynes' method of limits

Under the influence of Marshall (1887), who doubted that a unique and true measure of the price index could ever be founded, in his famous paper on index numbers winning the Adam Smith Prize, Keynes (1909) reached the following conclusion with reference to Walsh (1901): "If there was a perfect measure of general exchange value, Mr. Walsh would certainly have found it; but the method of exhaustion is barren, if the object of search has no real existence"(p. 135). If individual preferences are not of the same kind, tastes change over time or tastes differ across space, then aggregation problems may arise because the object of measure (the aggregate price index) does not exist .

In the *Treatise on Money*, Keynes (1930, Vol.I, ch. 8) made in fact no explicit reference to the idea of a price index. Rather, he compared the purchasing power of money in two situations of consumption with different relative prices. The comparison was made by using the so-called "method of limits" (p. 98). *No change in taste* and *proportionality of composite quantities (and prices)* with respect to total real expenditure are assumed. These hypotheses imply monotonicity along a beam line where, at given relative prices, all individual quantities change proportionally. Two alternative ratios of real expenditures can be calculated at constant relative prices of the base and the current situations, respectively. It turns out that these ratios are the upper and lower limits (bounds) of the index of the real expenditure. Leontief (1936, pp. 46-47) and Afriat (1977b, pp. 108-115, 2005, pp. 91-98) noted that these limits correspond, respectively, to the Laspeyres and Paasche index numbers of real expenditure.

Keynes (1930, p. 99) observed: "This conclusion is not unfamiliar […]. It is reached, for example, by Professor Pigou (*Economics of Welfare*, part I, chapter VI). The matter is also very well treated by Harberler (*Der Sinn der Indexzahlen*, pp. 83-94). *The dependence of the argument, however, on the assumption of uniformity of tastes, etc., is not always sufficiently emphasised*" (italics added). He added here, the following footnote: "Dr. Bowley in his 'Notes on Index Numbers' published in the *Economic Journal*, June 1928, may be mentioned amongst those who have expressly introduced this necessary condition". Similar methods were used by other authors. In his famous review article, Frisch (1936, p. 17-27) mentioned Pigou, Haberler, Keynes, Gini, Konüs, Bortkiewicz, Bowley, Allen, and Staehle and discussed them briefly.

Keynes' method of limits has not been widely used, probably because it has not been immediately understood in its fundamental reasoning. Frisch (1936, p. 26), for example, while conceding the correctness of Keynes' proof, overlooked the real sense of his proceeding by observing: "If we know that $\mathbf{q}_0$ and $\mathbf{q}_1$ are adapted *and equivalent*, the indifference-defined [price] index can be computed exactly, namely, as the ratio $\mathbf{p}^1 \cdot \mathbf{q}^1 / \mathbf{p}^0 \cdot \mathbf{q}^0$ [since it is assumed that $\mathbf{q}^t = \mathbf{q}(\mathbf{p}^t, \overline{u})$ with t = 0,1]. In these circumstances, to derive *limits* for it is to play hide-and-seek. It was Staehle who first pointed this out". In fact, Keynes did not assume that $\mathbf{q}_0$ and $\mathbf{q}_1$ were necessarily on the *same* indifference curve, but on *homothetic* indifference curves on the hypothesis of uniformity of tastes. This implies monotonicity along a *beam* (a line where all individual quantities change proportionately) along which the purchasing power of money can be compared at different prices. This reasoning was later recovered and further developed by Afriat (1977b, pp. 108-115).

7. Hicks' Laspeyres-Paasche inequality condition

In his *Value and Capital,* Hicks (1940, p. 329) considered the Laspeyres (L) > Paasche (P) strong inequality for a consumer who remains on the same indifference level. However, in a chapter entitled "The Index-Number Theorem" of his *Revision of Demand Theory* (hereafter cited as *R.D.T.*), Hicks (1956, pp. 180-188) established a proposition based on the weak version the "Laspeyres-Paasche inequality" on the demand side

(7.1) Laspeyres (L) ≥ Paasche (P) (for both price or quantity indexes)

(see also Hicks, 1958, p. 140). We note, in passim, that the use of the "weak" rather than "strong" form of the L-P inequality was not much discussed. In a footnote, however, Hicks (1958) wrote that the rules of L-P inequalities that he was discussing "are much nothing more than a restatement of the familiar rules of Revealed Preference theory. They should, therefore, be completed by some discussion of the limiting cases when one or both of L and P are equal to 1—cases which may have some importance in practice, but which are especially interesting because of the role which they play in discriminating between Strong and Weak Ordering in that other application (*R.D.T.*, ch. 6)" (p. 140, fn. 2).

The (non-negative) difference between Laspeyres and Paasche indicates a substitution effect (S) in the case the points of observation are on the same indifference curve or the sum of substitution effect and a *certain* income effect (I) in the case they are not on the same indifference curve.

In the more general case, we have

(7.2) $$L - P = I + S$$

where, L and P are the Laspeyres and Paasche indexes (here we use Hicks' original notation). If the income-elasticities of all commodities are the same (that is the preferences are homothetic), then I is equal to zero. In this case, the proportion of demanded quantities do not change as real income changes.

We have the following possible results:

Case 1: $L - P < 0$ (Hicks' index-number theorem breaks down) meaning either that demand is not governed by rational behaviour and/or the preferences are non-homothetic with a negative and strong enough income effect so that real-income change induces a relative expansion in demand for those goods whose prices have relatively risen. A strong negative income effect offsets a positive substitution effect ($I + S < 0$)

Case 2: $L - P > 0$ (Hicks' index-number theorem holds), meaning either that preferences are homothetic (so that $I = 0$ and $S > 0$) or preferences are non-homothetic (with $I \neq 0$ and $I + S > 0$). If preferences are homothetic, implying that the income-elasticities of all commodities are the same then the proportion of demanded quantities do not change as real income changes. and I is equal to zero.

The Hicks' index-number theorem pointing to a positive LP difference (case 2) is a necessary and sufficient condition for using the observed data on prices and quantities to reconstruct "true" index numbers based on *hypothetical homothetic preferences.* These, however, do not necessarily coincide with the actual criteria governing the observed behaviour. In other words, the LP inequality might be the result of the concomitant "non-proportional" effects of real income changes as well as substitution effects under non-homothetic preferences (if any), but *the observed data could always be rationalized by a hypothetical homothetic preference field if* $L - P > 0$. Under this condition we could always reconstruct "true" price and quantity index numbers that are consistent with that homothetic preference field and, as such, always respect all Fisher's requirement, including transitivity. This is, in fact, (as Keynes had recalled) the only condition under which it is possible to make such construction.

8. Samuelson's considerations on the Laspeyres-Paasche inequality

Samuelson (1974)(1984), Samuelson and Swamy (1974), and Swamy (1984), independently from Hicks (1956) and consistently with their "index-number theorem", they have considered the following cases admitting L-shaped non-smooth indifference contours (see for example Samuelson, 1974, pp. 599-600):

Case 1: $L-P$ < 0, so that the observed relative prices are not negatively correlated with the observed relative quantities (as expected with homothetic changes). In such an anomalous case, we might obtain the following ranking:

(8.1)
$$P_{K-L} \equiv \underbrace{\frac{\mathbf{p}^1\mathbf{q}(\mathbf{p}^1,u^0)}{\mathbf{p}^0\mathbf{q}^0}}_{\text{Laspeyres-type Konüs}} \leq \underbrace{\frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0}}_{\text{Laspeyres}} \leq \underbrace{\frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1}}_{\text{Paasche}} \leq \underbrace{\frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}(\mathbf{p}^0,u^1)}}_{\text{Paasche-type Konüs}} \equiv P_{K-P}$$

with $\mathbf{q}(\mathbf{p}^0,u^1)$ and $\mathbf{q}(\mathbf{p}^1,u^0)$ being the vectors of *non-observed* (theoretical) quantities that would have been demanded at the price-utility combinations $(\mathbf{p}^0,u^1)$ and $(\mathbf{p}^1,u^0)$, respectively. This is a rather problematic case, where aggregation is not possible. Even Fisher's "ideal" index, which consists in the geometric mean of the Laspeyres and Paasche indexes, falling between these two indexes, is farther than the ones from both "true" economic indexes! (See the numerical example given by Samuelson and Swamy, 1974, where "the Ideal index cannot give high-powered approximation to the true index in the general, nonhomothetic case", p. 585.)

Case 2: $L-P>0$. If preferences are homothetic, then I = 0 and $S>0$. If preferences are non-homothetic *with* real-income changes inducing a relative expansion in demand for those goods whose prices have relatively fallen (a case considered by Samuelson, 1974, 1984, Swamy, 1984 and others under the name of "Engel-Gerschenkron effect"), then $I>0$, which reinforces the positive substitution effect $S>0$. In these two cases, we can rely on the following ranking

$$(8.2) \qquad \underset{\text{Paasche}}{\frac{\mathbf{p}^1\mathbf{q}^1}{\mathbf{p}^0\mathbf{q}^1}} \leq \underset{\text{Paasche-type Konüs}}{P_{K-P}} \; \substack{\leq \\ > } \; \underset{\text{Laspeyres-type Konüs}}{P_{K-L}} \leq \underset{\text{Laspeyres}}{\frac{\mathbf{p}^1\mathbf{q}^0}{\mathbf{p}^0\mathbf{q}^0}}$$

The Laspeyres and Paasche index numbers correspond to alternative utility functions governing piece-wise linear demand correspondences.

9. Afriat's "giant leap" for index number theory: "Any point in the Laspeyres-Paasche interval, if any"

Along the lines open by Hicks (1956), the joint information given by the Laspeyres and Paasche indexes could provide us with an alternative information concerning two limiting functions allowing substitution effects. These two limiting functions are piece-wise linear boundaries of a set of possible homothetic utility functions which can rationalize the observed data. Even though these data have been actually generated under non-homothetic preferences, the Hicks' (1956) Laspeyres-Paasche inequality condition is necessary and sufficient for constructing index *homothetic* functions that are "true" for an homothetic function. It is in this vein that Afriat (1977b, pp. 108-115) recovered Keynes' (1930) reasoning on the purchasing power of money under the hypothesis of unchanged tastes and translated it into the construction of the bounds of a "true" price index.

As Samuelson and Swamy (1974, p. 570) have recognized, "[t]he invariance of the price index [from the reference quantity base] is seen to imply and to be implied by the invariance of the quantity index from the reference price base". This conclusion was anticipated in Afriat (1977b, pp. 107-112). A pure price index is consistent with a conical (homothetic) utility function rationalizing the observed prices and quantities in different situations. The conical (homothetic) utility condition which permits this determination, for arbitrary $\mathbf{p}_0 \text{ and } \mathbf{p}_1,$ is a non-observational object, which may remain a purely hypothetical and "metaphysical" concept.

In Afriat (1977b, p. 110) words: "The conclusion [...] is that the price index is bounded by the Paasche and Laspeyres indices. [...] The Paasche index does not exceed the Laspeyers index. [...] The set of values [of the "true index"] is in any case identical with the Paasche-Laspeyres interval. The "true" points are just the points in that interval and no others; and none is more true than another. There is no sense to a point in the interval being a better approximation to "the true index" than others. There is no proper distinction of 'constant utility' indices, since all these points have that distinction".

The same conclusion is replicated in Afriat (2005, p. *xxiii*): "Let us call the *LP* interval the closed interval with *L* [Laspeyers index] and *P* (Paasche index] as upper and lower limits, so the *LP*-inequality is the condition for this to be non-empty. While every true index is recognized to belong to this interval, it can still be asked what points in this interval are true? The answer is all of them, all equally true, no one more true than another. When I submitted this theorem to someone notorious in this subject area it was received with complete disbelief.

"Here is a formula to add to Fisher's collection, a bit different from the others.

"**Index Formula**: Any point in the LP-interval, if any."

In my review article (Milana, 2005), it is shown that any price index number that is exact for a continuously differentiable cost function $C(\mathbf{p}_t, u_t)$ can be translated into the following form

(9.1) $$P_{0,1} \equiv \frac{\theta + (1-\theta)\sum s_{0i} \dfrac{p_{1i}}{p_{0i}}}{(1-\theta) + \theta \sum s_{1i} \dfrac{p_{0i}}{p_{1i}}} = \left[\sum_{i=1}^{N} \frac{p_{1i}q_{0i}}{p_{0i}q_{0i}}\right]^{\lambda(\theta)} \cdot \left[\sum_{i=1}^{N} \frac{p_{1i}q_{1i}}{p_{0i}q_{1i}}\right]^{1-\lambda(\theta)} = P_L^{\lambda(\theta)} \cdot P_P^{1-\lambda(\theta)}$$

where, for t = 0,1, $s_{ti} \equiv p_{ti} \dfrac{\partial C(\mathbf{p}_t, u_t)}{\partial p_{ti}} / \sum_j p_{tj} \dfrac{\partial C(\mathbf{p}_t, u_t)}{\partial p_{tj}}$

$= p_{ti} q_{ti} / \sum_j p_{tj} q_{tj}$ using Shephard's lemma ($q_{ti} = \dfrac{\partial C(\mathbf{p}_t, u_t)}{\partial p_{ti}}$)

and θ is an appropriate parameter whose numerical value depends on the remainder terms of the two first-order approximations of $C(\boldsymbol{p},u)$ around the base and current points of observations.

The index $P_{0,1}$ is linearly homogeneous in p (that is, if $p_1 = \lambda\, p_0$, then $P_{0,1} = \lambda$). With $\theta = 0$, it reduces to a Laspeyres index number, whereas, with $\theta = 1$, it reduces to a Paasche index number.

The "true" exact index number, if any, is numerically equivalent to $P_{0,1}$. If the functional form of $C(\mathbf{p}, u)$ is square root quadratic in $\mathbf{p}$, then $P_{0,1}$ can be transformed into a Fisher “ideal” index number. In this case, the index $P_{0,1}$ is numerically equivalent to a quadratic mean-of-order-2 index number.

As we have already seen, the price index is invariant with respect to the reference utility level if and only if $C(\mathbf{p},u)$ is homothetically separable and can be written $C(\mathbf{p},u)=c(\mathbf{p})\cdot u$, so that

(9.2) $$s_{ti} = p_{ti}q_{ti} / \sum_j p_{tj}q_{tj} = p_{ti}\frac{\partial c(\mathbf{p}_t)}{\partial p_{ti}} / \sum_j p_{tj}\frac{\partial c(\mathbf{p}_t)}{\partial p_{tj}}.$$

Moreover, $Q_{0,1}=[C(\mathbf{p}_1,u_1)/C(\mathbf{p}_0,u_0)]/P_{0,1}$ is the quantity index measured implicitly by deflating the index of the functional value with the price index $P_{0,1}$. *It has the meaning of a pure quantity index if and only if $P_{0,1}$ is a pure price index.*

The parameter θ, however, remains unknown. For this reason, it is concluded that “it would be more appropriate to construct a range of alternative index numbers (including those that are not superlative), which are all equally valid candidates to represent the true index number, rather than follow the traditional search for only *one* optimal formula” (Milana, 2005, p. 44). Previous attempts in this direction using non-parametric approaches based on revealed preference techniques include Banker and Maindiratta (1988), Manser and McDonald (1988), Chavas and Cox (1990)(1997), Dorwick and Quiggin (1994)(1997), Hill (2000), but these do not provide, in general, stringent tests for homotheticity and, more importantly, the derived index numbers fail to satisfy the transitivity requirement.

An alternative approach to the Afriat methodology would be that of the econometric estimation of the function $c(\mathbf{p})$ in order to eliminate the indeterminacy of the “true” index number (see, among the first attempts, Goldberger and Gamaletsos, 1970 and Lloyd (1975), and, among the most recent contributions, Blundell *et al*. 2003, Neary, 2004, and Oulton, 2005), but this implies the imposition of a subjective choice of *a priori* functional forms where stochastic components of the derived demand functions are also included. The theory of bounds becomes more complex with the addition of the stochastic term to each demand function (see Phlips, 1983, pp. 145-148 for a numerical example). Moreover, critical remarks on this approach could be made regarding the non-identifiability of the elasticities of substitution and the bias in changes in technology or consumer tastes if no a priori information is available (see, for example, Diamond, McFadden and Rodriguez, 1978).

10. Afriat's price indices between several observation points

The approach outlined in the previous section can be enhanced by considering more than two observation points simultaneously. This idea had been advanced during the debate on index numbers in the early part of last century. Frisch (1936, p. 36), commenting the "iso-expenditure method" of Staehle (1935), wrote: "The comparison between two paths will be more exact if made via an intermediate path. The closer the individual paths the better. Knowing a very close path-system is equivalent to knowing the indifference surfaces themselves. In this case the indifference index can be computed exactly". Similar statements were written also by Samuelson (1947, ch. VI). It is worth quoting Samuelson and Swamy's (1974, p. 576) own words: "[…] Fisher missed the point made in Samuelson (1947, p. 151) that knowledge of a third situation can add information relevant to the comparison of two given situations. Thus Fisher contemplates Georgia, Egypt, and Norway, in which the last two each have the same price index relative to Georgia :

> "'We might conclude, since 'two things equal to the same thing are equal to each other,' that, therefore, the price levels of Egypt and Norway must equal, and this would be the case if we thus compare Egypt and Norway *via* Georgia. But, evidently, if we are intent on getting the very best comparison between Norway and Egypt, we shall not go to Georgia for our weights … [which are], so to speak, none of Georgia's business.' [1922, p. 272].

"This simply throws away the transitivity of indifference and has been led astray by Fisher's unwarranted belief that only fixed-weights lead to the circular's test's being satisfied (an assertion contradicted by our P_i / P_j and Q_i / Q_j forms."

Samuelson (1947, p. 160) had in fact clearly stated: "Knowledge of a third point may be utilized by the methods of the previous section; as may also knowledge of any intermediate expansion paths. In fact, in the limit as all intermediate expansion paths are known, *i.e.* as we know the functions $x_i = h^i(p_1,...,p_n,I)$ (i = 1, …, n) the indifference map itself may be solved for implicitly".

One of Afriat's main contribution in index number theory has been the development an original approach of constructing aggregating index numbers using all the data simultaneously (see Afriat, 1967, 1981, 1984, 2005). He also has developed an efficient algorithm to find the minimum path of *chained upper limit index numbers* (the chained Laspeyres indices on the side of demand). In the following section this algorithm is briefly described. From these chained upper limit index numbers can be derived directly the *chained lower limit index numbers* (the chained Paasche indices on the side of demand).

11. Afriat's computational method

In this section, for expositional convenience, some notation is changed with respect to the previous sections. The matrices of bilateral Laspeyres (**L**) and Paasche (**K**) index numbers comparing aggregate prices at the point of observation i relative to those at point j, for i,j =1, 2, …, N, are respectively

$$\mathbf{L} \equiv \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1N} \\ L_{21} & L_{22} & \dots & L_{2N} \\ \dots & \dots & \dots & \dots \\ L_{N1} & L_{N2} & \dots & L_{NN} \end{bmatrix} \quad \text{and} \quad \mathbf{K} \equiv \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1N} \\ K_{21} & K_{22} & \dots & K_{2N} \\ \dots & \dots & \dots & \dots \\ K_{N1} & K_{N2} & \dots & K_{NN} \end{bmatrix}$$

where $L_{ij} \equiv \dfrac{\mathbf{p}^i\mathbf{q}^j}{\mathbf{p}^j\mathbf{q}^j}$, and $K_{ij} \equiv \dfrac{\mathbf{p}^i\mathbf{q}^i}{\mathbf{p}^j\mathbf{q}^i} = \dfrac{1}{L_{ji}}$. Obviously, $K_{ij} = \dfrac{1}{L_{ji}}$ and $L_{ii} = K_{ii} = 1$.

The Laspeyres and Paasche index numbers are usually considered as two alternative measures of the unknown "true" index number P_{ij} which can be seen as an aggregation of the elementary price ratios p_r^i / p_r^j or, alternatively, as a ratio of aggregate price levels, *i.e.* $P_{ij} \equiv P_i / P_j$, where P_i and P_j are "true" aggregate price levels at the *i*th and *j*th points of observation. The price level ratio, always respects, by construction, the "base reversal" test, that is $P_{ij} = 1/P_{ji}$, and the "circularity" test, that is $P_{it} \cdot P_{tj} = P_{ij}$. By contrast, in the general case where the elementary price ratios *and* the relative quantity weights change, the Laspeyres and Paasche indices fail to be "base-" and "chain-consistent", that is $L_{ij} \neq 1/L_{ji} = K_{ij}$, $L_{it} \cdot L_{tj} \neq L_{ij}$ and $K_{it} \cdot K_{tj} \neq K_{ij}$. Even more unacceptable is well-known failure of chained indexes to return on the previous levels if all elementary prices go back to their older levels (the so-called "drift effect"): $L_{it} \cdot L_{ti} \neq L_{ii} = 1.$ and $K_{it} \cdot K_{ti} \neq K_{ii} = 1.$ These failures make the two index number formulas, like all the other alternative formulas, unsuitable to represent a price index.

Nevertheless, as we shall see below, they are useful for testing the existence of the "true" price index and constructing its consistent bounds.

The so-called *LP-inequality* condition is that $L_{ij} \geq K_{ij}$ on the purchaser's side ($L_{ij} \leq K_{ij}$ on the supplier's side) is necessary and sufficient for the existence of a "true" price index number P_{ij} with a numerical value falling between the Laspeyres and Paasche indices. If this condition is not satisfied for all pairs of observation, then a correction of the data for possible inefficiency can be devised and/or an alternative more general model using a wider or different set of variables could be considered.

If the LP-inequality condition is satisfied for all pairs of points of observation, let us define, in the purchaser's case (following Afriat, 1981, 1984, p. 47, 2005, p. 167, 2008),

(11.1) $$M_{ij} \equiv \min_{kl...m} L_{ik} L_{kl} ... L_{mj}$$ (minimum chained Laspeyres price index number)

(11.2) $$H_{ij} \equiv \max_{kl...m} K_{ik} K_{kl} ... K_{mj} = \frac{1}{M_{ji}}$$ (maximum chained Paasche price index number)

so that we have tighter bounds with $L_{ij} \geq M_{ij} \geq P_{ij} \geq H_{ij} \geq K_{ij}$ for $i \neq j$ and $L_{ii} = M_{ii} = P_{ii} = H_{ii} = K_{ii} = 1$. In the case of supplier, the inequality signs and the "min/max" problems are reversed.

If the LP-inequality condition is not satisfied for some or all pairs of points of observation, then we could "correct" the data for inefficiency. Diagonal elements $M_{ii} < 1$ and $H_{ii} < 1$ tell the inconsistency of the system.

A critical efficiency parameter e^* can be found for correction of the L matrix. For any element $M_{ii} < 1$, let d_i represent the number of nodes in the path $i...i$, then

(11.3) $$e_i = (M_{ii})^{\frac{1}{d_i}}$$

If $M_{ii} \geq 1$, let e_i take the value of 1 and then the critical efficiency parameter is determined as

(11.4) $$e^* = \min_i e_i$$

The adjusted Laspeyres matrix is obtained as

(11.5) $$L^*_{ij} = L_{ij} / e^* \qquad \text{for } i \neq j$$

and the procedure goes on as before with L^* in place of the original L.

Our correction for inefficiency starts by applying Edmunds' (1973) minimum path and Bainbridge's (1978) power algorithm to the Laspeyres matrix L for all six compared years as promptly adapted by Afriat (1979)(1980b)(1981)(1982) for identifying the optimized chained indexes efficiently. It consists in raising the Laspeyres matrix to powers N times, with N being the number of the compared observation points (6 years in the case of Fisher's data), in a modified arithmetic where + means *min.* In this special arithmetic, the resulting matrix M (corrected for inefficiency) remains unchanged if multiplied further by L, that is $M \equiv L^N = L^{N+1} = M \cdot L$.

However, the optimized chained Laspeyres and Paasche indexes (the elements of the matrices L and M, respectively) are still intransitive – like any other chained index – since they exhibit the *triangle inequalities* $M_{it}M_{tj} \geq M_{ij}$ and $H_{it}H_{tj} \leq H_{ij}$. The matrix of the geometric mean elements $(M_{ij} \cdot H_{ji})^{1/2}$ proposed by Afriat (2008) and used by Afriat and Milana (2009) in practical illustrations may turn out to be only approximately transitive.

12. Proposed solution

Chain-consistent (transitive) tight bounds that are "true" index numbers themselves can be derived by adopting the following new procedure. Let us assume, without loss of generality, that all prices are normalized with an arbitrary aggregate price level, say for example P_1, and define the maximum and minimum price levels

(12.1) $$\widehat{p}_i = (\max_t M_{it} / M_{(i-1)t}) \cdot \widehat{p}_{i-1} = (\max_t M_{it} \cdot H_{t(i-1)}) \cdot \widehat{p}_{i-1} \quad \text{for } i = 2, 3, ..., N;\ t = 1, 2, ..., N$$

(12.2) $$\breve{p}_i = (\min_t H_{it} / H_{(i-1)t}) \cdot \breve{p}_i = (\min_t H_{it} \cdot M_{t(i-1)}) \cdot \breve{p}_i \quad \text{for } i = 2, 3, ..., N;\ t = 1, 2, ..., N$$

with $\widehat{P}_1$ and $\breve{P}_1$ being equal to 1.

The chain-consistent bounds of the "true" index numbers are therefore obtained as

(12.3)
$$\widehat{P}_{ij} = \widehat{p}_i / \widehat{p}_j \quad \text{and} \quad \breve{P}_{ij} = \breve{p}_i / \breve{p}_j$$

With only to observation points $(N = 2)$, the index-number problem of a consumer is solved by finding the following bounds:

(12.4)
$$\widehat{\mathbf{P}} \equiv \left[\widehat{P}_{ij} \right] = \begin{bmatrix} 1 & K_{12} \\ L_{21} & 1 \end{bmatrix} \quad \text{and} \quad \breve{\mathbf{P}} \equiv \left[\breve{P}_{ij} \right] = \begin{bmatrix} 1 & L_{12} \\ K_{21} & 1 \end{bmatrix}$$

With 4 observation points, after reordered their sequence of comparison conveniently, we might obtain

(12.5)
$$\widehat{\mathbf{P}} = \begin{bmatrix} 1 & K_{12} & K_{12}K_{23} & K_{12}K_{23}K_{34} \\ L_{21} & 1 & K_{23} & K_{23}K_{34} \\ L_{32}L_{21} & L_{32} & 1 & K_{34} \\ L_{43}L_{32}L_{21} & L_{43}L_{32} & L_{43} & 1 \end{bmatrix}$$

and

(12.6)
$$\breve{\mathbf{P}} = \begin{bmatrix} 1 & L_{12} & L_{12}L_{23} & L_{12}L_{23}L_{34} \\ K_{21} & 1 & L_{23} & L_{23}L_{34} \\ K_{32}K_{21} & K_{32} & 1 & L_{34} \\ K_{43}K_{32}K_{21} & K_{43}K_{32} & K_{43} & 1 \end{bmatrix}$$

Chain-consistent bounds of *quantity indices* can be obtained by using a similar procedure directly or implicitly by deflating the nominal total expenditure by means of the respective consistent bounds $\widehat{P}_{ij}$ and $\breve{P}_{ij}$.

In fact, the single tight bounds in the matrices $\widehat{\mathbf{P}}$ and $\breve{\mathbf{P}}$ satisfy *all Fisher's tests*, that is

$\widehat{P}_{ii} = 1$ and $\breve{P}_{ii} = 1$ for every i *Identity test*

$\widehat{P}_{ij} = \lambda$ and $\breve{P}_{ij} = \lambda$ if $p_i = \lambda p_j$ *General mean of price relatives or proportionality test* (linear homogeneity in price levels) from which the identity test can be derived as a special case with $\lambda = 1$)

$\widehat{P}_{ij}\widehat{P}_{ji} = 1$ and $\breve{P}_{ij}\breve{P}_{ji} = 1$ for every i, j *Time-reversal test*

$\widehat{P}_{ij}\widehat{P}_{jk}=\widehat{P}_{ik}$ and $\breve{P}_{ij}\breve{P}_{jk}=\breve{P}_{ik}$ for every i, j, k *Chain (Circular-reversal) or transitivity test*

$\widehat{P}_{ij}=\widehat{P}_{ij}^{*}$ and $\breve{P}_{ij}=\breve{P}_{ij}^{*}$ where $p_t^{*}=\alpha p_t$ and $q_t^{*}=q_t/\alpha$ for $t=i, j$
Dimensional invariance test

$\widehat{P}_{ij}\breve{Q}_{ij}=M_i/M_j$ and $\breve{P}_{ij}\widehat{Q}_{ij}=M_i/M_j$ for every *i,j* , where M_t is nominal total expenditure at $t = i,j$ *(Weak) factor-reversal test*[16]

This is a remarkable result, since we have achieved the solution of the index-number problem following Samuelson and Swamy (1974), who have noted: "Although Ragnar Frisch (1930) has proved that, when the number of goods exceeds unity, it is impossible to find well-behaved formulae that satisfy *all* of these Fisher criteria, we derive here canonical index numbers of price and quantity that do meet the spirit of all of Fisher's criteria in the only case in which a single index number of the price of cost of living makes economic sense—namely, the ("homothetic") case of unitary income elasticities in which at all levels of living the calculated price change is the same. This seeming contradiction with Frisch is possible because the price and quantity variables are not here allowed to be arbitrary independent variables, but rather are constrained to satisfy the observable demand functions which optimize well-being" (emphasis in the original text) (p. 566). A diagrammatical explanation is given in the Appendix.

The critical remarks made by Pfouts (1966) on the excess rigidity imposed on the "true" index number *formula* with all Fisher's requirements do not apply here. Since the matrix of bilateral ratios of price (or quantity) levels is singular by construction, that is its determinant is zero since the matrix rows are linearly dependent, this would require too much a restrictive condition for an index number formula to exist (see also von der Lippe, 2007, pp. 76-77). The foregoing matrices of bounds are not defined by imposing the same *mathematical formula* to each element, but are derived by finding directly *numerical values* within certain conditions to be satisfied with the data.

As clarified also by the recent theoretical literature (see, in particular, van Veelen, 2002, Quiggin and van Veelen, 2007, van Veelen and van der Weide, 2008, Crawford and Neary, 2008), the apparent contradiction between the impossibility theorem and the solution of the

[16] Samuelson and Swamy (1974, p. 575) have introduced the concept of the *weak* factor-reversal test, as opposed to the *strong* factor-reversal test: "we drop the *strong* requirement that the *same* formula should apply to *q* as to p. A man and wife should be properly matched; but that does not mean I should marry my identical twin!"

index-number problem reflects essentially the conflict between *changing tastes* that are consistent with traditional index number formulas and *constant tastes* that are implied in the construction of a “well-behaved” (homothetic) index.

The usual undesirable properties of chained index number formulas, in particular, the “drift” effect and intransitivity (see for example von der Lippe, 2001 for a critical position against the use of such indices) are not met with the algorithm proposed here, which constructs *chained numbers* rather than *chained formulas*. Moreover, other methods based on linking bilateral index numbers in a multilateral context, such as those based on a tree structure of chained bilateral comparisons according to the minimum distance in the weights (as, for example, the “minimum spanning tree” used by Hill, 1999, 2004), do not guarantee the minimum or maximum chaining paths needed to define the tightest bounds.

Most of the OECD countries currently use chained Laspeyres production volume indexes on a year-to-year basis in the national accounts statistics (see the survey by Schreyer, 2004). These do not coincide with the tight bounds defined here. The proposed procedure is based on replacing the idea of “ideal” formula that is good for all seasons with a more pragmatic algorithm producing the double-side limits of all possible “true” indexes aggregating the available data. It could be used to find tight “true” bounds of alternative values of real GDP and its implicit deflator, standard of living and the cost-of-living index, and other aggregate economic variables as it is becoming customary in national and international institutions. Point estimations, when needed, could however be constructed by taking the geometric average of the tight “true” bounds satisfying all Fisher’s tests, including transitivity.

13. An empirical illustration using Irving Fisher’s data

In his classical quest for the ideal index number formula, what results would Irving Fisher (1922) obtain with his data if he used our method outlined above? Would these data pass the LP-inequality test? Would his ranking order of index-number formulas need to be revised with reference to our results? We now try to answer the first two questions leaving the third question to be answered in another occasion for lack of space here.

The original data on consumer prices and quantities used by Fisher (1922) are reported in Tables 1 and 2, respectively. A brief description of the data is given by Fisher (1922, p. 14) himself. These data refer to 36 price and quantity movements in six consecutive years, from 1913 to 1918. All the empirical results presented in Fisher (1922) are originated from these data. They were the basic information for calculating alternative index numbers using 134 alternative formulas. These were a part of a much wider database collected by Wesley C. Mitchel for the US War Industry Board, for wholesale prices and quantities of 1474 commodities marketed in the United States.

The chief reason for Irving Fisher to employ these data is that they were, at the time, the only ones that included figures for quantities as well as prices for each traded commodities. The author emphasized that this little sample provided a good basis for comparison of the alternative formulas because the period covered was one of "extraordinary dispertion in the movements both of prices and quantities"[17] (p. 14). Using these data, he searched for the ideal formula which could minimize an important source of error: "Of the four sources of error, formula, assortment, number of commodities, and original data, the two first are usually most at fault. The error in the Sauerbeck-Statist index number today reaches over 35 per cent from the first source alone" (p. 349).

Our computations of the bilateral Laspeyres and Paasche matrices on the data used by Fisher (1922) are shown and compared in Table 3. We note that only one third of bilateral comparisons pass the LP-inequality test, revealing that these data are inconsistent with aggregation conditions. As shown above, under these circumstances, even Fisher "ideal" index number and his proposed ranking of alternative formulas would lack economic ground. To our knowledge, this important fact about Fisher's data and his computed index numbers has been overlooked until now.

The computations of the optimized chained indexes, the elements of the matrices M and H, have been made by means of our FORTRAN program previously written for the numerical illustrations in Afriat and Milana (2009, pp. 83-86). The inconsistency is reflected by the diagonal elements less than 1 resulting in the first round of powers. The computations are shown in Table 4, whereas the matrices M and H (with $H_{ij} = 1/M_{ji}$) resulting after correction for inefficiency are reported in Table 5. For comparison with Fisher "ideal" index number, also the geometric mean of the elements of M and H is reported. We may note that, since the inconsistency in transitivity of the elements of M and that of the corresponding elements of H partially offset each other.

[17] It turns out, however, that the wide dispersion in the movements of prices and quantities in the sample used is not sufficient to establish a conclusive ranking in performance of the alternative formulas. The direction of change in relative prices and quantities of the single commodities may also affect the ranking itself of the compared formulas.

The chain-consistent “true” bounds of the relative price levels are reported in Table 6. The cross ratios of these “true” bounds are shown in Table 7, where also the geometric average of these index numbers is reported for immediate comparison with the Fisher “ideal” index numbers.

It is easy to verify that, differently from the Fisher’s “ideal” index number, our proposed indices pass all Fisher’s tests, including transitivity. Moreover, these turn out to be invariant with respect to the base year, a requirement considered as early as Edgeworth (1896, p. 137, fn. 5), whereas it is known that Fisher “ideal” index generally fails this test. It is noteworthy how this index appears to vary as the base year changes also in the case of Fisher’s data (see our computations in Table 7). Also this fact seems to have been overlooked by Fisher.

Our proposed bounds are optimized chained indexes. Therefore, they can be more consistently compared with traditional chained index numbers rather than fixed-based indexes. Table 7 includes comparisons of the geometric mean of these bounds with chained “ideal” Fisher, chained Laspeyres, and chained Paasche as well as with fixed-based Laspeyres and Paasche index numbers using Irving Fisher’s data. Interesting points are worth noting: (*i*) chained “ideal” Fisher is outside the fixed-based LP-inequality in the last two compared years; (*ii*) the geometric mean of our proposed bounds are outside the fixed-based LP-inequality in the last compared year, thus confirming that this inequality cannot be considered the proper double-sided limits of the set of possible “true” index numbers when it does not have the proper algebraic sign; (*iii*) the geometric mean of our proposed bounds is systematically below the chained “ideal” Fisher and chained Laspeyres and Paasche (except a couple of years where the chained Laspeyres is lower than the former index by around 0.20 per cent).

More importantly, our proposed “true” bounds are fully transitive by construction. After a correction for inconsistency with the LP-inequality requirement, these indices have a full economic meaning. The relative difference between Fisher’s “ideal” index and the geometric mean of the two bounds are, in some cases, as high as more than half percentage point and shows a significant variation over the whole period.

14. Conclusion

An historical account of the economic index-number problem has brought us to the determination that a solution can be found which satisfies all the traditional requirements. This is, however, achieved at the cost of some compromises. It has been shown that, under easily testable conditions, the observed data could be rationalized by a family of well-behaved index numbers regardless the actual determinants that have generated them. This solution is achieved by maintaining a certain indeterminacy regarding the numerical values of “true” indexes, but this is restricted within tight bounds which can be considered themselves as “true” index numbers. However, in cases were a point estimation is needed for practical or institutional purposes, a geometric average of these bounds can always be defined and calculated in a way to respect all Fisher’s tests and all the other important requirements for a typical economic index number.

From the perspective of the history of economic thought, the accumulated knowledge in the field of index number theory suggests that the search for the ideal index-number formula has come to a dead end. It is well known that the index-number problem is nothing more than an aggregation problem and economic theory has given us a simple and powerful devise to find a solution: the conical or homothetic aggregating function. Our solution is constructed using an algorithm rather than a simple formula to recover the boundaries of the family of all possible aggregating functions if the conditions are satisfied with the available data. A new paradigm is making his way slowly but inexorably along the path open with Samuelson’s revealed-preference approach more than fifty years ago and its later reformulation by Afriat. We foresee that it will eventually replace the old formula-based paradigm completely and dominate the field for at least the whole new century.

Appendix: A diagrammatical representation

The solution of chain-consistent tight bounds appear to satisfy all Fisher's test. This result seems to contrast the conclusions derived from Frisch's "impossibility theorem" in index number theory, but is perfectly in line with Samuelson and Swamy (1974, p. 566), who have claimed. We could remark that, while Frisch (1930) was referring to the "impossibility to find well-behaved *formulae* our solution regards index *numbers*. His "impossibility theorem" is still valid because it reveals only that applying the same formula for comparisons back and forth between pairs of observation points is self-contradictory. This can be clearly shown also with the aid of a geometrical representation.

We cannot deal with a *single* measure of the "true" index number just because this remains unknown, but we can construct two bounds (the tight upper and lower bounds) of the closed set of possible numerical values of this index number, if the conditions of its existence are satisfied. Given that approximation is reliable only if changes are close to infinitesimal, The toll we pay for satisfying all Fisher's tests and overcome the "impossibility theorem" is to deal with two bound estimates rather than an "ideal" single measure, which ends unavoidably to fail to satisfy at least one of those requirements.

Afriat's original method is to find whether a well behaved utility function can be reconstructed that is consistent with the finite set of observed choices satisfying the axioms of revealed preference (Afriat, 1967, Varian, 1982) testing the demand data for consistency with a *multi-valued* (*piecewise linear*) utility function. This, however, is not unique. There are generally other utility functions and the *recoverability problem* becomes how to reconstruct the entire set of these utility functions that would fit the observed data simultaneously.

For any given q_0, there is the set of q's that are revealed preferred to q_0 $(RP(q_0))$ and set of q's that are revealed worse than q_0 $(RW(q_0))$. A simple example is given in Figure 1. The area corresponding to the set of possible utility functions which satisfy the revealed preference test is that which does not belong to $RP(q_0)$ and $RW(q_0)$.

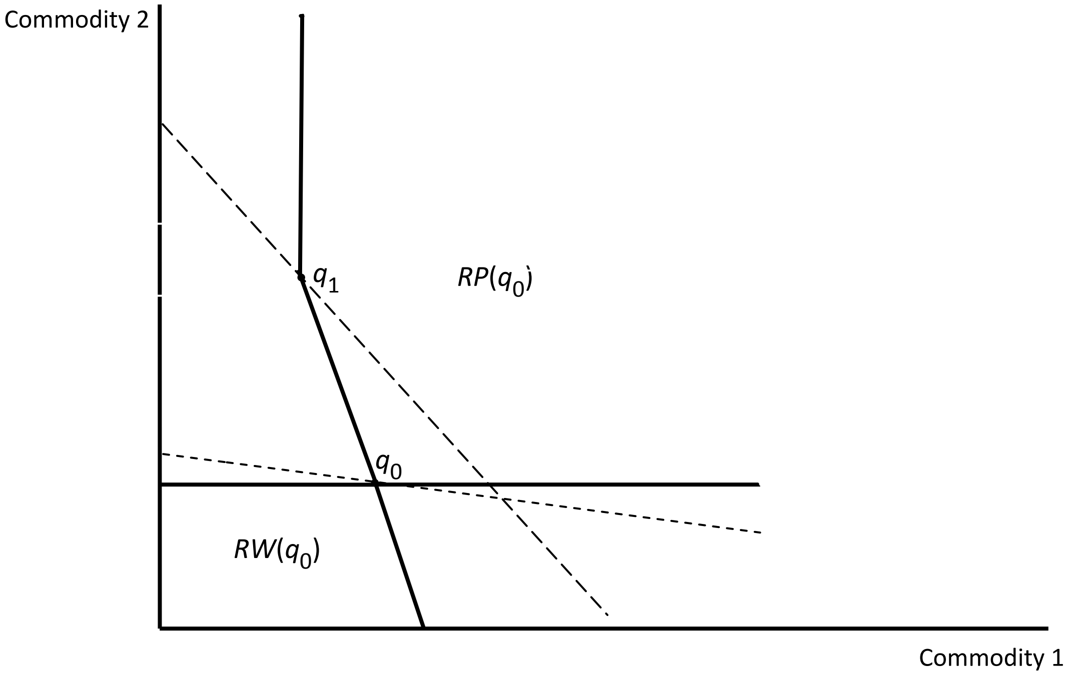
The necessary and sufficient condition for the existence of a price and quantity aggregate measure is that the observed data are consistent with homothetic preferences. Following Keynes'

(1930, pp. 105-106) “method of limits”, as re-exposed by Afriat (1977, pp. 108-115), we may ask whether it is possible to identify the area corresponding to the set of money metric utility functions passing through the reference point. The same observations considered in Figure 1 are shown in Figure 2, where this area is restricted between upper and lower bounds given by the expenditure function at the based period inflated by the Laspeyres and Paasche price indexes, respectively. We may note that these bounds are generally tighter than the limits represented with revealed-preference methods.

It is straightforward to see that, by shifting the piece-wise linear isoquant along the *P*-ray from crossing q_0 up to crossing q_1 in Figure 2, an upper bound (Laspeyres-type) isoquant can be as seen from q_1 as the base observation point. Similarly, by shifting the upper bound (Laspeyres-type) isoquant along the L-ray from that crossing q_0 up to that crossing q_1, we end up to a lower bound (Paasche-type) isoquant as seen from q_1 as the base observation point.

The inclusion of a third point of observation, as that between the two former points in Figure 3, permits us to track the isoquant or curve of indifference using hypothetical budget lines passing through the base points in an approximating path followed by the optimal chained Laspeyres indexes. These make up the upper bound which is tighter than the bilateral fixed-based Laspeyres index. The same applies to the approximation obtained by updating the value of the expenditure with the optimal Paasche indexes. Chaining these yields the lower bound that is generally tighter than the bilateral fixed-based Paasche indexes.

“Revealed-preference” method (as outlined by Varian, 2006)



gure 1: $RP(q_0)$ and $RW(q_0)$: simple case of “revealed preference test.

□

Keynes’ method of limits

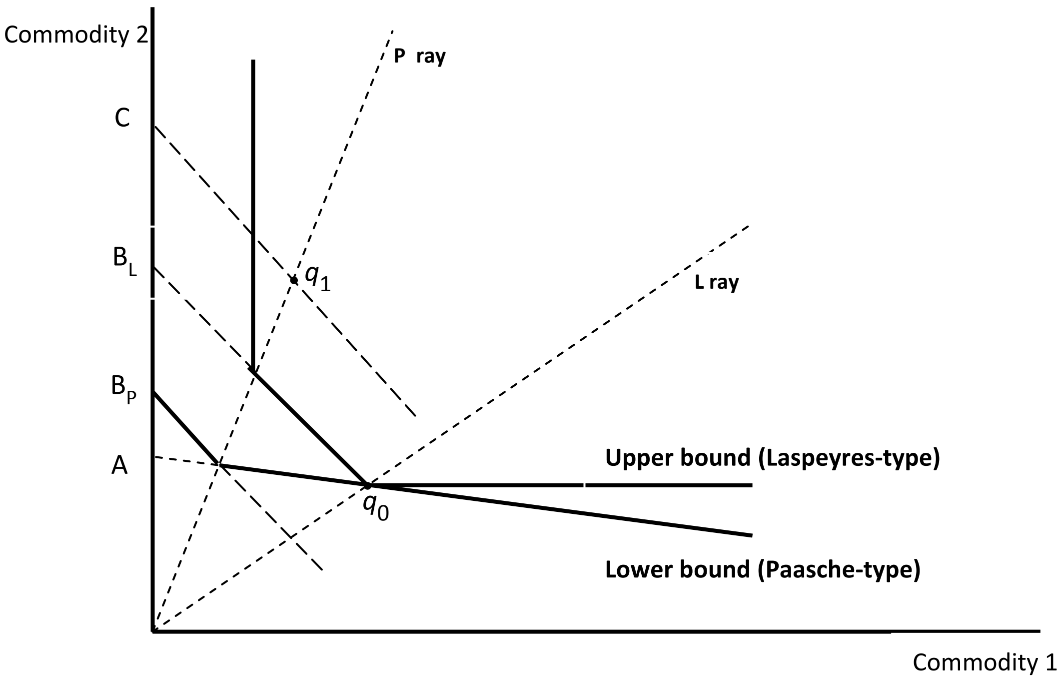


Figure 2: Laspeyres- and Paasche-type bounds

AC: Observed increase in nominal expenditure from L to P (at the relative Prices represented by the slope of AL and CP, respectively);

AB_P: Price component of the increase in nominal expenditure measured with the direct Paasche index number (implicit Laspyeres index number);

AB_L: Price component of the increase in nominal expenditure measured with the direct Laspeyres index number (implicit Paasche index number);

B_PC: Quantity component of the increase in nominal expenditure measured with the implicit Paasche index number (direct Laspeyres index number);

B_LC: Quantity component of the increase in nominal expenditure measured with the implicit Laspeyres index number (direct Paasche index number);

Samuelson-Afriat tight bounds

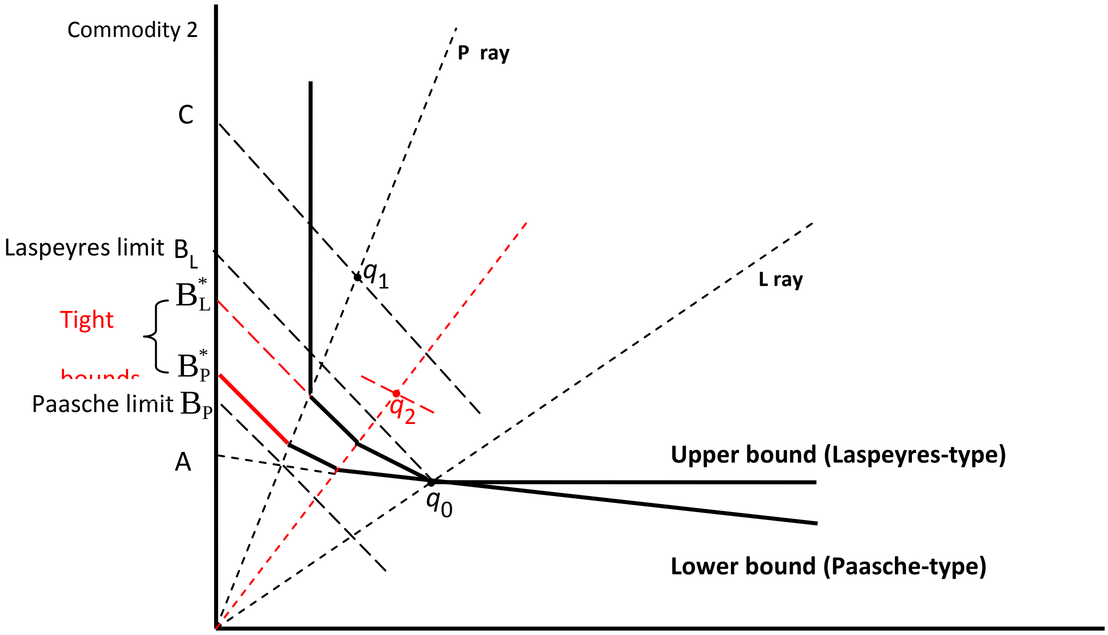


Figure 3: Tightening the bounds by adding a third observation point

A B^*_P : Price component of the increase in nominal expenditure measured with the chained Paasche index number (implicit chained Laspeyres index number);

A B^*_L : Price component of the increase in nominal expenditure measured with the chained Laspeyres index number (implicit chained Paasche index number).

References

Afriat, Sydney N. (1953), "Utility Separability and Budget Decentralization", Department of Applied Economics, Cambridge, UK.

Afriat, Sydney N. (1953-56). Theory of Economic Index Numbers. The Consistency Condition and Other Concepts in the Theory of Value and Demand. The Calculation of Index Numbers of the Standard and Cost of Living. On Index Numbers in the Theory of Value and Demand. *Research Reports* (mimeo.), Department of Applied Economics, Cambridge, UK.

Afriat, Sydney N. (1959). Value and Expenditure. *Research Memorandum* No. 7 (February), Econometric Research Program, Princeton University.

Afriat, Sydney N. (1967), "The Construction of Utility Functions from Expenditure Data", *International Economic Review* 8: 67-77.

Afriat, Sydney N. (1972), "The Theory of International Comparisons of Real Income and Prices", In In D. J. Daly (ed. by), *International Comparisons of Price and Output*, Studies in Income & Wealth, No.37, National Bureau of Economic Research, New York: Columbia University Press, pp. 208-216.

Afriat, Sydney N. (1975), "Index Numbers in Theory and Practice by R.G.D. Allen", *Canadian Journal of Economics* 11(2): 367-369.

Afriat, Sydney N. (1977a), "Slutsky and Frobenius", *Zeitschrift für Nationalökonomie* (*Journal of Economics*) 37(3-4): 1617-1634.

Afriat, Sydney N. (1977b), *The Price Index*, Cambridge, UK: Cambridge University Press.

Afriat, Sydney N. (1978), "Theory of the Price Index: Fisher's Test Approach and Generalizations. By Wolfgang Eichhorn and Joachin Voeller", *Journal of Economic Literature* 16(1): 129-131.

Afriat, Sydney N. (1979), "The Power Algorithm for Generalized Laspeyres and Paasche Indices". Athens Meeting of the Econometric Society. September.

Afriat, Sydney N. (1980a), *Demand Functions and the Slutsky Matrix*, Princeton, NJ, Princeton University Press.

Afriat, Sydney N. (1980b), "Matrix Powers: Classical and Variations" presented at the Matrix Theory Conference, Auburn, Alabama, 19-22 March.

Afriat, Sydney N. (1981), "On the Constructability of Consistent Price Indices Between Several Periods Simultaneously", in Angus Deaton (ed. by), *Essays in the Theory and Measurement of Consumer Behaviour: in Honour of Sir Richard Stone*, Cambridge, UK, Cambridge University Press, 133-161.

Afriat, Sydney N. (1982), “The Power Algorithm for Minimum Paths, and Price Indices”, presented at The Eleventh International Symposium on Mathematical Programming, University of Bonn, 23-27 August.

Afriat, Sydney N. (2005). *The Price Index and Its Extension. A Chapter in Economic Measurement*, London and New York, Routledge.

Afriat, Sydney N. (2008), “Computation of Consistent Price Indices”, presented at The 2008 European Meeting of the Econometric Society, Milan 27-21 August.

Afriat, Sydney N. and C. Milana (2009), *Economics and the Price Index*, New York and London, Routledge.

Aldrich, John (1992), “Probability and Depreciation: A History of the Stochastic Approach to Index Numbers”, *History of Political Economy* 24(3): 657-687.

Allen, R.G.D. (1932), “The Foundations of a Mathematical Theory of Exchange”, *Economica* 36: 197-226.

Allen, E.S. (1941), “The Scientific Work of Vito Volterra”, *American Mathematical Monthly* 48(8, Part I): 516-519.

Allen, R.G.D. (1949), “The Economic Theory of Index Numbers”, *Economica N.S*. 16(63): 197-203

Antonelli, Gian Battista (1882), “La funzione di Green” [“Green’s Function”], Thesis (manuscript).

Antonelli, Gian Battista (1886), “Sulla teoria matematica dell’economia pura. Capitolo Primo: Concetto di utilità”, Tipografia del Folchetto, Pisa. Reprinted in *Giornale degli Economisti e Annali di Economia,* Nuova Serie 10 (May-June 1951): 233-263 and in G.B. Antonelli (1952), *Sulla teoria matematica della economia politica,* Malfasi Editore, Milano, 1952, pp. 15-45. English translation by J.S. Chipman and A.P. Kirman, edited and annotated by J.S. Chipman with revisions supplied by W. Jaffé: “On the Mathematical Theory of Political Economy”, in J.S. Chipman, Leonid Hurwicz; Marcel K. Richter; and Hugo F. Sonneschein (eds.) *Preferences, Utility, and Demand. A Minnesota Symposium*, Harcourt Brace Jovanovich, Inc., New York, 1971, pp. 333-364 (with an Appendix by the author’s son Saverio Antonelli on biographical notes and bibliography of G.B. Antonelli).

Balk, Bert M. (1995), “Axiomatic Price Index Theory: A Survey”, *International Statistical Review,* 63(1): 69-93.

Balk, Bert M. (2008), *Price and Quantity Index Numbers. Models for Measuring Aggregate Change and Difference*, Cambridge University Press, Cambridge, UK.

Bainbridge, S. (1978), “Power Algorithm for Minimum Paths”, Private communication, department of Mathematics, University of Ottawa.

Banker, Rajiv D. and Ajay Maindiratta (1988), “Nonparametric Analysis of Technical and Allocative Efficiencies in Production”, *Econometrica* 56(6): 1315-1332.

Bennet, T. L. (1920), "The Theory of Measurement of Changes in Cost of Living", *Journal of the Royal Statistical Society* 83(3): 455-462.

Blundell, Richard W., Martin Browning and Ian A. Crawford (2003), "Nonparametric Engel Curves and Revealed Preferences", *Econometrica* 71, 208-240.

Boskin, Michael J.; Ellen R. Dulberger; Robert J. Gordon; Zvi Griliches; and Dale W. Jorgenson (1996), "Toward a More Accurate Measure of the Cost of Living", Final Report to the Senate Finance Committee, Advisory Commission to Study the Consumer Price Index", December 4 (http://www.ssa.gov/history/reports/boskinrpt.html)

Bowley, A. L. (1899), "Wages, Nominal and Real", in R.H.I. Palgrave (ed. by), *Dictionary of Political Economy*, Vol. 3, London, Macmillan.

Bowley, A. L. (1923), "Reply to 'Professor Bowley on Index Numbers' by Irving Fisher", *Economic Journal* 33(130): 251-52.

Bowley, A. L. (1928), "Notes on Index Numbers", *Economic Journal* 38(150): 216-237.

Carey, Malachy (1977), "Integrability and Mathematical Programming Models: A Survey and a Parametric Approach", *Econometrica* 45(8): 1957-1976.

Caves, D.W., L.R. Christensen, and W.E. Diewert (1982), "Multilateral Comparisons of Output, Input, and Productivity Using Superlative Index Numbers", *Economic Journal* 92: 73-86.

Chance, W. A. (1966), "A Note on the Origins of Index Numbers", *Review of Economics and Statistics* 48(1): 108-110.

Chavas, Jean-Paul and Thomas L. Cox (1990), "A Non-Parametric Analysis of Productivity: The Case of U.S. and Japanese Manufacturing", *American Economic Review* 80(3): 450-464.

Chavas, Jean-Paul and Thomas L. Cox (1997), "On Nonparametric Demand Analysis", *European Economic Review* 41: 75-95.

Chipman, John S. (1971), "Introduction to Part II", in Chipman, John S.; Leonid Hurwicz; Marcel K. Richter; and Hugo F. Sonnenschein (eds.) (1971), *Preferences, Utility, and Demand. A Minnesota Symposium*, Harcourt Brace Jovanovich, Inc., New York, pp. 321-332.

Court, Louis M. (1941), "Entrepreneurial and Consumer Theories for Commodity Spectra; Part I", *Econometrica* 9(2): 135-162.

Crawford, Ian A. and J. Peter Neary (2008), "Testing for a Reference Consumer in International Comparisons of Living Standards", *American Economic Review* 98(4): 1731-1732.

Deaton (1979), "The Distance Function in Consumer Behaviour with Applications to Index Numbers and Optimal Taxation", *Review of Economic Studies* 46(3): 391-405.

Debreu, G. (1972), "Smooth Preferences", *Econometrica* 40(4): 603-615.

Debreu, G. (1976), “Smooth Preferences: A Corrigendum”, Econometrica 44(4):831-832.

Denny, Michael (1972), “Trade and Production Sector: An Exploration of Models of Multi Product Technology”, Ph. D. Dissertation, University of California at Berkeley.

Denny, Michael (1974), "The Relationship between Functional Forms for the Production System", *Canadian Journal of Economics* 7: 21-31.

Derks, Scott V. (2004), *The Value of a Dollar, 1860-2004*, Third Edition, Grey House Publishing, Millerton, NY.

Diamond, Peter; Daniel McFadden and Miguel Rodriguez (1078), “Measurement of the Elasticities of Factor Substitution and Bias of Technical Change”, in Melvyn Fuss and Daniel McFadden (eds), *Production Economics: A Dual Approach to Theory and Application*, Vol. 2, Amsterdam, North-Holland Publ. Co., Ch. IV.

Diewert, W. Erwin (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4: 115-145.

Diewert, W. Erwin (1993), “The Early History of Price Index Research”, in W.E. Diewert and A.O. Nakamura (eds), *Essays in Index Number Theory*, Vol. I, Elsevier Science Publishers B.V., pp. 33-65.

Diewert, W. Erwin and Robert J. Hill (2009), “Comment on Different Approaches to Index Number Theory”, Discussion paper 09-05, Department of Economics, University of British Columbia, Vancouver, CA.

Divisia, François (1925), L’indice monétaire et la théorie de la monnaie”, *Revue d’Economie Politique* 39: 980-1008.

Dowrick, Steve and John Quiggin (1994), “International Comparisons of Living Standards and Tastes: A Revealed-Preference Analysis”, *American Economic Review* 84(1): 332-341.

Dowrick, Steve and John Quiggin (1997), “True Measures of GDP and Convergence”, *American Economic Review* 87(1): 41-64.

Edgeworth, F.Y. (1896), “A Defence of Index-Numbers”, *Economic Journal* 6(21): 132-142.

Edgeworth, F. Y. (1925), “The Plurality of Index-Numbers”, *Economic Journal* 35(139): 379-388.

Eltetö, O. and P. Köves (1964), “On a Problem of Index Number Computation Relating to International Comparison”, *Statisztikai Szemle* 42: 507-518.

Ferger, Wirth F. (1946), “Historical Note on the Purchasing Power Concept, and Index Numbers”, *Journal of the American Statistical Association* 41(233): 53-57.

Fisher, Irving (1911), *The Purchasing Power of Money. Its Determination to Credit, Interest and Crisis*, MacMillan, New York.

Fisher, Irving (1922), *The Making of Index Numbers. A Study of Their Varieties, Tests, and Reliability,* Houghton Mifflin Company, Boston and New York. (Third edition, 1927, reprint of 1967 by Augustus M. Kelley).

Frisch, Ragnar (1936), "Annual Survey of General Economic Theory: The Problem of Index Numbers", *Econometrica* 4: 1-39.

Gini, Corrado (1924), "Quelques considerations au sujet de la construction des nombres indices des prix et des questions analogues, *Metron* 4: 3-24.

Gini, Corrado (1931), "On the Circular Test of Index Numbers", *Metron* 2: 3-24.

Gershenkron, Alexander (1947), "The Soviet Indices of Industrial Production", *Review of Economic Studies* 29(4): 217-226.

Goldberger, A. S. and T. Gamaletsos (1970), "A Cross-Country Comparison of Consumer Expenditure Patterns", *European Economic Review* 1: 347-399.

Gorman, W.M. (1953), "Community Preference Fields", *Econometrica* 21: 63-80.

Gorman, W.M. (1961), "On a Class of Preference Fields", *Metroeconomica* 13: 53-56.

Hartman, P. (1970), "Frobenius Theorem under Charathéodory Type Conditions", *Journal of Differential Equations* 7: 307-333.

Hanoch, G. (1970), "Generation of New Production Functions through Duality", Discussion paper No. 118, Harvard Institute of Economic Research, Cambridge, MA (published in M. Fuss and D. McFadden, eds., Production Economics: A Dual Approach to Theory and Application, Amsterdam, North-Holland.)

Hicks, John R. (1940), "The Valuation of the Social Income", *Economica,* pp. 105-124.

Hicks, John R. (1942), "Consumer Surplus and Index Numbers", *Review of Economic Studies* 9(2): 126-137.

Hicks, John R. (1939), *Value and Capital. An Inquiry into Some Fundamental Principles of Economic Theory*, Oxford, Clarendon Press (2nd Ed., 1946).

Hicks, John R. (1956), *A Revision of Demand Theory*. Oxford: The Clarendon Press.

Hicks, John R. (1958), "The Measurement of Real Income", *Oxford Economic Papers* 10(2): 125-162.

Hill, Robert J. (1999), "Comparing Price Levels across Countries Using Minimum Spanning Trees", *Review of Economics and Statistics* 81: 135-142.

Hill, Robert J. (2000), "Constructing Bounds on Per Capita Income Differentials across Countries", *Scandinavian Journal of Economics* 102(2): 285-302.

Hill, Robert J. (2004), "Constructing Price Indexes across Space and Time: The Case of the European Union", *American Economic Review* 94(5): 1379-1410.

Hill, Robert J. (2006a), “Superlative Index Numbers: Not All of Them Are Super”, *Journal of Econometrics* 130: 25-43.

Hill, Robert J. (2006b), “When Does Chaining Reduce the Paasche-Laspeyres Spread? An Application to Scanner Data”, *Review of Income and Wealth* 52: 309-325.

Houthakker, H.S. (1952-1953), “Compensated Changes in Quantities and Qualities Consumed”, *Review of Economic Studies* 19(3): 155-164.

Hotelling, Harold (1932), “Edgeworth’s Taxation Paradox and the Nature of Demand and Supply Functions”, *Journal of Political Economy* 40(5): 577-616.

Hotelling, Harold (1935), “Demand Functions with Limited Budgets”, *Econometrica* 3: 66-78.

Hulten, Charles R. (1973), “Divisia Index Numbers”, *Econometrica* 41: 1017-1025.

Hurwicz, Leonid (1971), “On the Problem of Integrability of Demand Functions”, in Chipman, J.S.; L. Hurwicz; M.K. Richter; and H.F. Sonnenschein (eds.) (1971), *Preferences, Utility, and Demand*, New York, Harcourt Brace Jovanovich, Inc., pp. 174-214.

Hurwicz, Leonid and Hirofumi Uzawa (1971), “On the Integrability of Demand Functions”, in Chipman, J.S.; L. Hurwicz; M.K. Richter; and H.F. Sonnenschein (eds.) (1971), *Preferences, Utility, and Demand*, New York, Harcourt Brace Jovanovich, Inc., pp. 114-148.

ILO/IMF/OECD/UN/Eurostat/The World Bank (2004a), *Consumer Price Index Manual: Theory and Practice*, Geneva: International Labour Office.

IMF/ILO/OECD/UN/Eurostat/The World Bank (2004b), *Producer Price Index Manual: Theory and Practice*, Washington: International Monetary Fund.

IMF/ILO/OECD/UN/Eurostat/The World Bank (2008), *Export and Import Price Index Manual,* Washington: International Monetary Fund.

Kadiyala, K. Rao (1972), “Production Functions and the Elasticity of Substitution”, *Southern Economic Journal* 38: 281-284.

McCusker, John J. (2001), *How Much Is That in Real Money?: A Historical Price Index for Use as a Deflator of Money Values in the Economy of the United States*, 2nd ed., rev. and enl. Worcester, Ma., American Antiquarian Society.

Kendall, M.G. (1969), “Studies in the History of Probability and Statistics, XXI. The Early History of Index Numbers”, *Review of the International Statistical Institute* 37(1): 1-12.

Keynes, John Maynard (1909), “The Method of Index Numbers with Special Reference to the Measurement of General Exchange Value”, Adam Smith Prize Essay (Reprinted in *The Collected Writings of John Maynard Keynes*. Volume 11, London: Macmillan, Cambridge University Press, 1983, pp. 43-173).

Keynes, John Maynard (1930), *Treatise on Money, Vol. 1, The Pure Theory of Money*, Harcourt Brace, New York.

Konüs, Alexandr A. (1924), "Problema istinovo indeksa stoimosti zhizni", *Ekonomicheskii Byulleten Konyunkturnovo Instituta* 3: 64-71. English translation: "The Problem of the True Index of the Cost of Living", *Econometrica* 7 (Jan. 1939): 10-29.

Konüs, Alexandr A. and Sergei S. Byushgens (1926), "K probleme pokupatelnoi cili deneg" (English translation of Russian title: "On the Problem of the Purchasing Power of Money"), *Voprosi Konyunkturi* II(1) (supplement to the *Economic Bulletin of the Conjuncture Institute*), pp. 151-172.

Leontief, Wassily (1936), "Composite Commodities and the Problem of Index Number", *Econometrica* 4(1): 39-59.

Lloyd, P. J. (1975), "Substitution Effects and Biases in Nontrue Price Indices", *American Economic Review* 65(3): 301-313.

Marshall, Alfred (1887), "Remedies for Fluctuations of General Prices", *Contemporary Review* (Reprinted in H.C. Pigou, ed. by, *Memorials of Alfred Marshall*, London, Macmillan & Co., 1925).

McCarthy, M. D. (1967), "Approximation of the CES Production Function: A Comment", *International Economic Review* 8: 190-192.

Malmquist, S. (1953), "Index Numbers and Indifferent Surfaces", *Trabajos de Estadistica* 4: 209-242.

Manser, Marilyn E. and Richard J. McDonald (1988), "An Analysis of Substitution Bias in Measuring Inflation, 1959-85", *Econometrica* 56(4): 909-930.

Martina, Alan (2000), "Antonelli's Analytical Techniques: Their Exploitation to Derive Results in Duality Theory", *European Journal of Economic Thought* 7(3): 363-376.

Milana, Carlo (2005), "The Theory of Exact and Superlative Index Numbers Revisited", Working paper no. 3, EU KLEMS Project (http://www.euklems.net/project_site.html).

Nataf, A. (1953), "Sur des questions d'agregation an économétrie", *Publications de l'Institut de Statistique de l'Université de Paris*, 2, fasc. 4: 5-61.

Neary, J. Peter (2004), "Rationalizing the Penn World Table: True Multilateral Indices for International Comparisons of Real Income", *American Economic Review* 94: 1411-1428.

Officer, Lawrence H. and Samuel H. Williamson (2008), *Purchasing Power of Money in the United States from 1774 to 2007*, MeasuringWorth.

Oulton, Nicholas (2007), "Chain Indices of the Cost of Living and the Path-Dependence Problem: An Empirical Solution", CEP Discussion Paper No. 797, Centre for Economic Performance, London School of Economics, London.

Persky, Joseph (1998), “Retrospectives: Price Indexes and General Exchange Values”, *Journal of Economic Perspectives* 12(1): 197-205.

Pfouts, R. W. (1966), “An Axiomatic Approach to Index Numbers”, *Review of the International Statistical Institute* 34(2): 174-185.

Phlips, Louis (1983), *Applied Consumption Analysis*, Amsterdam, North-Holland Publ. Co.

Pigou, A.C. (1912), *Wealth and Welfare*, London, Macmillan.

Quiggin, John and Van Veelen, Matthijs (2007), “Multilateral Indices: Conflicting Approaches?”, *Review of Income and Wealth* 53(2): 372-378.

Reinsdorf, Marshall B. (2007), “Axiomatic Price Index Theory”, in Marcel Boumans (ed. By), *Measurement in Economics – A Handbook*, Academic Press, London, UK, Ch. 7, pp. 153-188.

Roy, René (1943), *De l’Utilité. Contributions à la théorie des choix,* Paris, Hermann.

Roy, René (1947), “La distribution du revenue entre les divers biens”, Econometrica 15(3): 205-225.

Roy, René (1949), “De la theorie des choix aux budgets de familles”, *Econometrica* 17 (Supplement: Report of the Washington Meeting (Jul. 1949)): 179-186.

Rothenberg, Winifred B. (1979), “A Price Index for Rural Massachusetts, 1750-1855”, *Journal of Economic History* 39(4): 975-1001.

Samuelson, Paul A. (1947), *Foundations of Economic Analysis*, Cambridge, Mass.: Harvard University Press.

Samuelson, Paul A. (1972), “Unification Theorem for the Two Basic Dualities of Homothetic Demand Theory”, *Procedings of the National Academy of Sciences of the USA* 69(9): 2673-2674.

Samuelson, Paul A. (1974), “Analytical Notes on International Real-Income Measures”, *Economic Journal* 84(335): 595-608.

Samuelson, Paul A. (1984), “Second Thoughts on Analytical Income Comparisons”. *Economic Journal* 94: 267-278.

Samuelson, Paul A. and Subramanian Swamy (1974), “Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis”, *American Economic Review* 64(4): 566-593.

Schreyer, Paul (2004), “Chain Index Number Formulae in the National Accounts”, 8th OECD – NBS Workshop on National Accounts, 6-10 December 2004, OECD Headquarters, Paris.

Shephard, Ronald W. (1953), *Cost and Production Functions*, Princeton, Princeton University Press.

Slutsky, E. (1915), “Sulla teoria del bilancio del consumatore”, *Giornale degli Economisti* 51: 1-26. English translation by Olga Ragusa: “On the Theory of the Budget of the Consumer”, in G.J.

Stigler and K.J. Boulding (eds.), *Readings in Price Theory*, Homewood, Ill., Richard D. Irwin, Inc., 1952, pp. 27-56.

Staehle, H. (1935), "A Development of the Economic Theory of Price Index Numbers", *Review of Economic Studies* 2(3): 163-188.

Stuvel, G. (1989), *The Index-Number Problem and Its Solution*, London, Macmillan.

Szulc, B. (1964), "Indices for Multiregional Comparisons", *Przeglad Statystyczny* 3: 239-254.

Swamy, Subramanian (1984), "Samuelson's Analytical Notes on International Real-Income Measures: A Comment", *Economic Journal* 94: 265-266.

Triplett, Jack E. (2006), "The Boskin Commission Report After a Decade", *International Productivity Monitor*, Spring, 12: 42-60.

Van Veelen, Matthijs (2002), "An Impossibility Theorem Concerning Multilateral International Comparison of Volumes", *Econometrica* 70: 369-375.

Van Veelen, Matthijs and Roy van der Weide (2008), "A Note on Different Approaches to Index Number Theory", *American Economic Review* 98(4): 1722-1730.

Varian, Hal R. (1982), "The Nonparametric Approach to Demand Analysis", *Econometrica* 50: 945-973.

Varian, Hal R. (2006), "Revealed Preference", in Michael Szenberg, Lall Ramrattan and Aron A. Gottesman (eds), *Samuelsonian Economics and the Twenty-First Century.* Oxford University Press.

Verblunsky, S. (1949), "On Green's Formula", *Journal of the London Mathematical Society* 24: 146-148.

Ville, J. (1946), "Sur les conditions d'existence d'une ophelimité totale et d'un indice du niveaux des prix", *Annales de l'Université de Lyon* 9: 32-39 [English translation by P.K. Newman: "The Existence Conditions of a Total Utility Function", Review of Economic Studies 19 (1951-52): 123-128.]

Vogt, Arthur and János Barta (1997), *The Making of Tests of Index Numbers*, Physica-Verlag, Heidelberg, Germany.

von der Lippe, Peter (2001), *Chain Indices. A Study in Price Index Theory*, Metzler-Poeschel, Stuttgart, Germany.

von der Lippe, Peter (2007), *Index Theory and Price Statistics*, Peter Lang, Frankfurt am Main, Germany.

Walsh, Correa Moylan (1901), *The Measurement of General Exchange-Value*, New York, Macmillan.

Walsh, Correa Moylan (1921), The Problem of Estimation: A Seventeenth-Century Controversy and Its Bearing on Modern Statistical Questions, Especially Index-Numbers, P. S. King And Son, First Edition. Wraps. edition

Wold, Herman (1944), "A Synthesis of Pure Demand Analysis, III", *Scandinavisk Aktuarietidskrift* 27: 69-120.

Wold, Herman (1953), *Demand Analysis: A Study in Econometrics*, New York, John Wiley & Sons.

Table 1. Prices of the 36 Commodities, 1913-1918

	1	2	3	4	5	6
	p0 1913	p1 1914	p2 1915	p3 1916	p4 1917	p5 1918
01 Bacon	0.1236	0.1295	0.1129	0.1462	0.2382	0.2612
02 Barley	0.6263	0.6204	0.7103	0.8750	1.3232	1.4611
03 Beef	0.1295	0.1364	0.1289	0.1382	0.1672	0.2213
04 Butter	0.2969	0.2731	0.2743	0.3179	0.4034	0.4857
05 Cattle	12.0396	11.9208	12.1354	12.4375	15.6354	18.8646
06 Cement	1.5800	1.5800	1.4525	1.6888	2.0942	2.6465
07 Coal, anth.	5.0636	5.0592	5.0464	5.2906	5.6218	6.5089
08 Coal, bit.	1.2700	1.1700	1.0400	2.0700	3.5800	2.4000
09 Coffe	0.1113	0.0816	0.0745	0.0924	0.0929	0.0935
10 Coke	3.0300	2.3200	2.4200	4.7800	10.6600	7.0000
11 Copper	0.1533	0.1318	0.1676	0.2651	0.2764	0.2468
12 Cotton	0.1279	0.1121	0.1015	0.1447	0.2350	0.3178
13 Eggs	0.2468	0.2660	0.2597	0.2945	0.4015	0.4827
14 Hay	11.2500	12.3182	11.6250	10.0625	17.6042	21.8958
15 Hides	0.1727	0.1842	0.2076	0.2391	0.2828	0.2144
16 Hogs	8.3654	8.3608	7.1313	9.6459	15.7047	17.5995
17 Iron bars	1.5100	1.2000	1.3700	2.5700	4.0600	3.5000
18 Iron, pig	14.9025	13.3900	13.5758	18.6708	38.8082	36.5340
19 Lead (white)	0.0676	0.0675	0.0698	0.0927	0.1121	0.1271
20 Lead	0.0437	0.0386	0.0467	0.0686	0.0879	0.0741
21 Lumber	90.3974	90.9904	90.5000	91.9000	105.0400	121.0455
22 Mutton	0.1025	0.1010	0.1073	0.1250	0.1664	0.1982
23 Petroleum	0.1233	0.1200	0.1208	0.1217	0.1242	0.1695
24 Pork	0.1486	0.1543	0.1429	0.1618	0.2435	0.2495
25 Rubber	0.8071	0.6158	0.5573	0.6694	0.6477	0.5490
26 Silk	3.9083	4.0573	3.6365	5.4458	5.9957	6.9770
27 Silver	0.5980	0.5481	0.4969	0.6566	0.8142	0.9676
28 Skins	2.5833	2.6250	2.7188	4.1729	5.5208	5.5625
29 Steel rails	28.0000	28.0000	28.0000	31.3333	38.0000	54.0000
30 Tin, pig	44.3200	35.7000	38.6600	43.4800	61.6500	87.1042
31 Tin plate	3.5583	3.3688	3.2417	5.1250	9.1250	7.7300
32 Wheat	0.9131	1.0412	1.3443	1.4165	2.3211	2.2352
33 Wool	0.5883	0.5975	0.7375	0.7900	1.2841	1.6600
34 Lime	1.2500	1.2500	1.2396	1.4050	1.7604	2.3000
35 Lard	0.1101	0.1037	0.0940	0.1347	0.2170	0.2603
36 Oats	0.3758	0.4191	0.4958	0.4552	0.6372	0.7747

Source: Irving Fisher, The Making of Index Numbers, Appendix VI, Table 63, p.489.

Table 2. Quantities Marketed of the 35 Commidities, 1913-1918 (millions of units)

	1 q0 1913	2 q1 1914	3 q2 1915	4 q3 1916	5 q4 1917	6 q5 1918
01 Bacon, lb.	1077.00	1069.00	1869.00	1481.00	1187.00	1408.00
02 Barley, bu.	178.20	195.00	228.90	182.30	209.00	256.40
03 Beef, lb.	6589.00	6522.00	6820.00	7134.00	8417.00	10244.00
04 Butter, lb.	1757.00	1780.00	1800.00	1820.00	1842.00	1916.00
05 Cattle, cwt.	69.80	67.60	71.50	83.10	103.50	118.30
06 Cement, bbl.	85.80	84.40	84.40	92.00	88.10	69.40
07 Coal, anth., ton.	6.90	6.86	6.78	6.75	7.83	7.69
08 Coal, bit., ton	477.00	424.00	443.00	502.00	552.00	583.00
09 Coffe, lb.	863.00	1002.00	1119.00	1201.00	1320.00	1144.00
10 Coke, short ton.	46.30	34.60	41.60	54.50	56.70	55.00
11 Copper, lb.	812.30	620.50	1043.50	1420.80	1316.50	1648.30
12 Cotton, lb.	2785.00	2820.00	2838.00	3235.00	3423.00	3298.00
13 Eggs, doz.	1722.00	1759.00	1791.00	1828.00	1882.00	1908.00
14 Hay, ton.	79.20	83.00	103.00	111.00	94.90	89.80
15 Hides, lb.	672.00	924.00	1227.00	1212.00	1113.00	663.00
16 Hogs, cwt.	68.40	65.10	76.80	86.20	67.80	82.40
17 Iron bars, cwt.	79.20	50.40	82.60	132.40	133.00	132.00
18 Iron, pig,, ton.	31.00	23.30	29.90	39.40	38.70	38.10
19 Lead (white), lb.	286.00	318.00	312.00	258.00	230.00	216.00
20 Lead, lb.	823.70	1025.60	1014.10	1104.50	1099.80	1083.00
21 Lumber, M bd, ft.	21.80	20.70	20.50	22.30	21.20	19.20
22 Mutton, lb.	732.00	734.00	629.00	618.00	474.00	513.00
23 Petroleum, gal.	10400.00	11200.00	11840.00	12640.00	14880.00	15680.00
24 Pork, lb.	9211.00	8871.00	9912.00	10524.00	8427.00	11426.00
25 Rubber, lb.	115.80	136.60	231.40	258.00	375.90	351.50
26 Silk, lb.	19.10	19.10	20.00	24.40	29.40	27.10
27 Silver, oz.	146.10	144.00	173.40	139.30	133.60	140.70
28 Skins, skin	6.70	5.90	4.30	5.60	2.70	0.70
29 Steel rails, ton.	3.50	1.95	2.20	2.86	2.94	2.37
30 Tin, pig, cwt.	1.04	0.95	1.16	1.43	1.56	1.59
31 Tin plate, cwt.	15.30	17.30	19.70	22.80	29.50	28.00
32 Wheat, bu.	555.00	654.00	588.00	642.00	605.00	562.00
33 Wool, lb.	448.00	550.00	699.00	737.00	707.00	752.00
34 Lime, bbl., 300 lb.	23.00	22.50	25.00	27.10	24.00	20.20
35 Lard, lb.	1100.00	955.00	1050.00	1141.00	927.00	1107.00
36 Oats, bu.	1122.00	1240.00	1360.00	1480.00	1587.00	1538.00

Source: Irving Fisher, The Making of Index Numbers, Appendix VI, Table 64, p.490.

Table 3. Laspeyres and Paasche matrices and LP-inequality

Laspeyres matrix (*)

	1	2	3	4	5	6
1	1.00000	0.99684	0.99901	0.87457	0.62093	0.56367
2	0.99931	1.00000	0.99987	0.87263	0.61694	0.56051
3	0.99672	1.00200	1.00000	0.87378	0.61900	0.56005
4	1.14081	1.13344	1.13673	1.00000	0.70872	0.64315
5	1.62067	1.59943	1.61430	1.42725	1.00000	0.90805
6	1.77865	1.77036	1.78189	1.56539	1.09922	1.00000

Paasche matrix (**)

	1	2	3	4	5	6
1	1.00000	1.00069	1.00329	0.87657	0.61703	0.56222
2	1.00317	1.00000	0.99801	0.88227	0.62522	0.56486
3	1.00099	1.00013	1.00000	0.87972	0.61946	0.56120
4	1.14342	1.14595	1.14445	1.00000	0.70065	0.63882
5	1.61050	1.62089	1.61550	1.41100	1.00000	0.90973
6	1.77410	1.78410	1.78555	1.55484	1.10126	1.00000

(*) $L_{i,j}$ where row i refers to the comparison year and column j refers to the base year

(**) $K_{i,j}$, where row i refers to the comparison year and column j refers to the base year

LP-inequality:
Laspeyres-Paasche difference

	1	2	3	4	5	6
1	0.00000	-0.00385	-0.00428	-0.00200	0.00390	0.00145
2	-0.00386	0.00000	0.00186	-0.00964	-0.00828	-0.00435
3	-0.00427	0.00187	0.00000	-0.00594	-0.00046	-0.00115
4	-0.00261	-0.01251	-0.00772	0.00000	0.00807	0.00433
5	0.01017	-0.02146	-0.00120	0.01625	0.00000	-0.00168
6	0.00455	-0.01374	-0.00366	0.01055	-0.00204	0.00000

Laspeyres/Paasche ratio

	1	2	3	4	5	6
1	1.00000	0.99615	0.99573	0.99772	1.00632	1.00258
2	0.99615	1.00000	1.00186	0.98907	0.98676	0.99230
3	0.99573	1.00187	1.00000	0.99325	0.99926	0.99795
4	0.99772	0.98908	0.99325	1.00000	1.01152	1.00678
5	1.00631	0.98676	0.99926	1.01152	1.00000	0.99815
6	1.00256	0.99230	0.99795	1.00679	0.99815	1.00000

Table 4. Steps of computations of matrix M

```
STARTING LASPEYRES MATRIX L

POWER  1

1.0000000 0.9968400 0.9990100 0.8745700 0.6209300 0.5636700
0.9993100 1.0000000 0.9998700 0.8726300 0.6169400 0.5605100
0.9967200 1.0020000 1.0000000 0.8737800 0.6190000 0.5600500
1.1408100 1.1334400 1.1367300 1.0000000 0.7087200 0.6431500
1.6206700 1.5994300 1.6143000 1.4272500 1.0000000 0.9080500
1.7786500 1.7703600 1.7818900 1.5653900 1.0992200 1.0000000


POWER  2

0.9957332 0.9912726 0.9941500 0.8698725 0.6149905 0.5587388
0.9955050 0.9867523 0.9919447 0.8726300 0.6161238 0.5599772
0.9961329 0.9900472 0.9932519 0.8717014 0.6156182 0.5600500
1.1326579 1.1334400 1.1332927 0.9890737 0.6992645 0.6353045
1.5983264 1.5994300 1.5992221 1.3957106 0.9867523 0.8964965
1.7691385 1.7581254 1.7701299 1.5448692 1.0922059 0.9923045


POWER  3

0.9905886 0.9836342 0.9888102 0.8650142 0.6115557 0.5556182
0.9860715 0.9854469 0.9866241 0.8610697 0.6087670 0.5530846
0.9893640 0.9846382 0.9899185 0.8639449 0.6107997 0.5549313
1.1283452 1.1184246 1.1243098 0.9890737 0.6983394 0.6347006
1.5922406 1.5782413 1.5865461 1.3957106 0.9854469 0.8956443
1.7569123 1.7469069 1.7560992 1.5341930 1.0846579 0.9854469


POWER  4

0.9829555 0.9781406 0.9832876 0.8583487 0.6068433 0.5513368
0.9823169 0.9736802 0.9788038 0.8599305 0.6079616 0.5523528
0.9839588 0.9769314 0.9820720 0.8592248 0.6074627 0.5518995
1.1176529 1.1169449 1.1182792 0.9759708 0.6900009 0.6268882
1.5771523 1.5761533 1.5780361 1.3772207 0.9736802 0.8846200
1.7457015 1.7348344 1.7439632 1.5244034 1.0777367 0.9791588


POWER  5

0.9774657 0.9706034 0.9757107 0.8535548 0.6034541 0.5482576
0.9730083 0.9723920 0.9735536 0.8496625 0.6007023 0.5457575
0.9762573 0.9715940 0.9767066 0.8524996 0.6027080 0.5475798
1.1133973 1.1036081 1.1094153 0.9746797 0.6890880 0.6260588
1.5711472 1.5573333 1.5655281 1.3753987 0.9723920 0.8834497
1.7336374 1.7237645 1.7328350 1.5138685 1.0702887 0.9723920


POWER  6

0.9699336 0.9651825 0.9702614 0.8469776 0.5988040 0.5440329
0.9693035 0.9607812 0.9658369 0.8485385 0.5999075 0.5450355
0.9709236 0.9639893 0.9690619 0.8478421 0.5994152 0.5445882
1.1028466 1.1021480 1.1034646 0.9630415 0.6808600 0.6185834
1.5562587 1.5552730 1.5571309 1.3589758 0.9607812 0.8729009
1.7225751 1.7118519 1.7208598 1.5042086 1.0634592 0.9661872
```

```
INCONSISTENCY CASE SINCE SOME DIAGONAL ELEMENTS < 1
DIAGONAL ELEMENTS < 1 (IN THIS CASE ALL)
ASSOCIATED COST EFFICIENCY e(i)

i       M(i,i)    d(I)   e(i)= M(i,i)^(1/d(i))

1     0.9699336     7         0.995648
2     0.9607812    12         0.996672
3     0.9690619    12         0.997385
4     0.9630415     8         0.995304
5     0.9607812     6         0.993354
6     0.9661872     6         0.994283
```

CRITICAL COST EFFICIENCY e* = MIN$_i$ e(i)   FOR i = 1,2, ... 6
= e(5) = 0.993354

USED TO DETERMINE THE ADJUSTED LASPEYRES MATRIX

L*(i,j) = L(i,j)/e*   FOR i ≠ j

A NEW POWER ITERATION PROCEDURE FOLLOWS:

```
=================================================================
STARTING LASPEYRES MATRIX L*

POWER  1

1.0000000 1.0035092 1.0056938 0.8804212 0.6250843 0.5674412
1.0059958 1.0000000 1.0065595 0.8784682 0.6210676 0.5642600
1.0033884 1.0087038 1.0000000 0.8796259 0.6231413 0.5637969
1.1484424 1.1410231 1.1443352 1.0000000 0.7134616 0.6474529
1.6315129 1.6101308 1.6251003 1.4367988 1.0000000 0.9141252
1.7905498 1.7822044 1.7938115 1.5758630 1.1065742 1.0000000


POWER  2

1.0000000 1.0035092 1.0056938 0.8804212 0.6232470 0.5662401
1.0059958 1.0000000 1.0052621 0.8784682 0.6210676 0.5642600
1.0033884 1.0033391 1.0000000 0.8796259 0.6231413 0.5637969
1.1478644 1.1410231 1.1443352 1.0000000 0.7086525 0.6438337
1.6197847 1.6101308 1.6206924 1.4144487 1.0000000 0.9085324
1.7905498 1.7817292 1.7938115 1.5656099 1.1065742 1.0000000


POWER  3

1.0000000 1.0035092 1.0056938 0.8804212 0.6232470 0.5662401
1.0059958 1.0000000 1.0052621 0.8784682 0.6210676 0.5642600
1.0033884 1.0033391 1.0000000 0.8796259 0.6231413 0.5637969
1.1478644 1.1410231 1.1443352 1.0000000 0.7086525 0.6438337
1.6197847 1.6101308 1.6186034 1.4144487 1.0000000 0.9085324
1.7905498 1.7817292 1.7915824 1.5651925 1.1065742 1.0000000


POWER  4

1.0000000 1.0035092 1.0056938 0.8804212 0.6232470 0.5662401
1.0059958 1.0000000 1.0052621 0.8784682 0.6210676 0.5642600
1.0033884 1.0033391 1.0000000 0.8796259 0.6231413 0.5637969
1.1478644 1.1410231 1.1443352 1.0000000 0.7086525 0.6438337
1.6197847 1.6101308 1.6186034 1.4144487 1.0000000 0.9085324
1.7905498 1.7817292 1.7911048 1.5651925 1.1065742 1.0000000


POWER  5
```

```
1.0000000 1.0035092 1.0056938 0.8804212 0.6232470 0.5662401
1.0059958 1.0000000 1.0052621 0.8784682 0.6210676 0.5642600
1.0033884 1.0033391 1.0000000 0.8796259 0.6231413 0.5637969
1.1478644 1.1410231 1.1443352 1.0000000 0.7086525 0.6438337
1.6197847 1.6101308 1.6186034 1.4144487 1.0000000 0.9085324
1.7905498 1.7817292 1.7911048 1.5651925 1.1065742 1.0000000

POWER  6

1.0000000 1.0035092 1.0056938 0.8804212 0.6232470 0.5662401
1.0059958 1.0000000 1.0052621 0.8784682 0.6210676 0.5642600
1.0033884 1.0033391 1.0000000 0.8796259 0.6231413 0.5637969
1.1478644 1.1410231 1.1443352 1.0000000 0.7086525 0.6438337
1.6197847 1.6101308 1.6186034 1.4144487 1.0000000 0.9085324
1.7905498 1.7817292 1.7911048 1.5651925 1.1065742 1.0000000
```

Table 5. Matrices M and H

Matrix M (*) adjusted for inefficiency

	1	2	3	4	5	6
1	1.00000	1.00351	1.00569	0.88042	0.62325	0.56624
2	1.00600	1.00000	1.00526	0.87847	0.62107	0.56426
3	1.00339	1.00334	1.00000	0.87963	0.62314	0.56380
4	1.14786	1.14102	1.14434	1.00000	0.70865	0.64383
5	1.61978	1.61013	1.61860	1.41445	1.00000	0.90853
6	1.79055	1.78173	1.79110	1.56519	1.10657	1.00000

Matrix H(**) adjusted for inefficiency

	1	2	3	4	5	6
1	1.00000	0.99404	0.99662	0.87118	0.61737	0.55849
2	0.99650	1.00000	0.99667	0.87641	0.62107	0.56125
3	0.99434	0.99477	1.00000	0.87387	0.61782	0.55831
4	1.13582	1.13835	1.13685	1.00000	0.70699	0.63890
5	1.60450	1.61013	1.60477	1.41113	1.00000	0.90369
6	1.76604	1.77223	1.77369	1.55320	1.10068	1.00000

(*) $M_{i,j}$, where row i refers to the comparison year and column j refers to the base year

(**) $H_{i,j}$, where row i refers to the comparison year and column j refers to the base year

Geometric mean of matrices M and H (only approximately transitive)

$$(M_{ij} \cdot H_{ij})^{1/2}$$

	1	2	3	4	5	6
1	1.00000	0.99876	1.00115	0.87579	0.62030	0.56235
2	1.00124	1.00000	1.00096	0.87744	0.62107	0.56275
3	0.99885	0.99904	1.00000	0.87674	0.62047	0.56105
4	1.14183	1.13968	1.14058	1.00000	0.70782	0.64136
5	1.61212	1.61013	1.61167	1.41279	1.00000	0.90611
6	1.77825	1.77697	1.78238	1.55918	1.10362	1.00000

Table 6. Price level computations: Upper and lower bounds

	Upper bounds $\widehat{p}_i$	Lower bounds $\breve{p}_i$
1	1,00000	1,00000
2	1,00600	0,99650
3	1,00935	0,99129
4	1,15504	1,12694
5	1,63375	1,59026
6	1,80786	1,75036

Table 7. Chain-consistent (transitive) bounds for the price index number

Ratios of upper level price indexes: $\widehat{P}_{ij} = \widehat{p}_i / \widehat{p}_j$

	1	2	3	4	5	6
1	1,00000	0,99404	0,99073	0,86577	0,61209	0,55314
2	1,00600	1,00000	0,99667	0,87096	0,61576	0,55646
3	1,00935	1,00334	1,00000	0,87387	0,61782	0,55831
4	1,15504	1,14816	1,14434	1,00000	0,70699	0,63890
5	1,63375	1,62401	1,61860	1,41445	1,00000	0,90369
6	1,80786	1,79709	1,79110	1,56519	1,10657	1,00000

Ratios of lower level price indexes: $\breve{P}_{ij} = \breve{p}_i / \breve{p}_j$

	1	2	3	4	5	6
1	1,00000	1,00351	1,00879	0,88736	0,62883	0,57131
2	0,99650	1,00000	1,00526	0,88425	0,62663	0,56931
3	0,99129	0,99477	1,00000	0,87963	0,62335	0,56633
4	1,12694	1,13090	1,13685	1,00000	0,70865	0,64383
5	1,59026	1,59584	1,60424	1,41113	1,00000	0,90853
6	1,75036	1,75650	1,76575	1,55320	1,10068	1,00000

Geometric mean of the ratios of upper and lower levels of prices: $(\widehat{P}_{ij} \cdot \breve{P}_{ij})^{1/2}$

	1	2	3	4	5	6
1	1,00000	0,99876	0,99972	0,87650	0,62040	0,56215
2	1,00124	1,00000	1,00096	0,87758	0,62117	0,56285
3	1,00028	0,99904	1,00000	0,87674	0,62058	0,56231
4	1,14090	1,13949	1,14058	1,00000	0,70782	0,64136
5	1,61186	1,60986	1,61140	1,41279	1,00000	0,90611
6	1,77888	1,77668	1,77838	1,55918	1,10362	1,00000

Fisher "ideal" index number no. 353 (Fisher, 1922, p. 493):
$(L_{i,j} \cdot K_{i.j})^{1/2}$ (intransitive)

	1	2	3	4	5	6
1	1.00000	0.99876	1.00115	0.87557	0.61897	0.56294
2	1.00124	1.00000	0.99894	0.87744	0.62107	0.56268
3	0.99885	1.00107	1.00000	0.87675	0.61923	0.56063
4	1.14212	1.13968	1.14058	1.00000	0.70467	0.64098
5	1.61558	1.61012	1.61490	1.41910	1.00000	0.90889
6	1.77637	1.77722	1.78372	1.56011	1.10024	1.00000

Relative price levels that are implicit in Fisher “ideal” index numbers

	1	2	3	4	5	6
1	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000
2	1,00124	1,00124	0,99779	1,00214	1,00339	0,99954
3	0,99885	1,00231	0,99885	1,00135	1,00042	0,99590
4	1,14212	1,14109	1,13927	1,14211	1,13846	1,13863
5	1,61558	1,61212	1,61304	1,62077	1,61559	1,61454
6	1,77637	1,77943	1,78167	1,78182	1,77753	1,77639

Percentage difference between Fisher “ideal” index numbers and the geometric mean of the upper and lower “true” bounds

	1	2	3	4	5	6
1	0,00	0,00	0,00	0,00	0,00	0,00
2	0,00	0,00	-0,34	0,09	0,22	-0,17
3	-0,14	0,20	-0,14	0,11	0,01	-0,44
4	0,11	0,02	-0,14	0,11	-0,21	-0,20
5	0,23	0,02	0,07	0,55	0,23	0,17
6	-0,14	0,03	0,16	0,17	-0,08	-0,14

Comparison of chained index numbers
with fixed-base Laspeyers and fixed-base Paasche

	Fixed L	Fixed P	$(\widehat{p}_i \cdot \breve{p}_i)^{1/2}$	Chained Fisher	Chained Laspeyres	Chained Paasche
1	1,00000	1,00000	**1,00000**	1,00000	1,00000	1,00000
2	0,99931	1,00317	**1,00124**	1,00124	0,99931	1,00317
3	0,99672	1,00099	**1,00028**	1,00231	1,00131	1,00330
4	1,14081	1,14342	**1,14090**	1,14322	1,13822	1,14823
5	1,62067	1,61050	**1,61186**	1,62234	1,62452	1,62015
6	1,77865	1,77410	**1,77888**	1,78496	1,78571	1,78420

Percentage difference with respect to the geometric mean of "true" bounds

	Fixed L	Fixed P	$(\widehat{p}_i \cdot \breve{p}_i)^{1/2}$	Chained Fisher	Chained Laspeyres	Chained Paasche
1	0,00	0,00	**0,00**	0,00	0,00	0,00
2	-0,19	0,19	**0,00**	0,00	-0,19	0,19
3	-0,36	0,07	**0,00**	0,20	0,10	0,30
4	-0,01	0,22	**0,00**	0,20	-0,24	0,64
5	0,55	-0,08	**0,00**	0,65	0,79	0,51
6	-0,01	-0,27	**0,00**	0,34	0,38	0,30