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THE CONCEPT OF THE STOCHASTIC EQUIVALENCE SCALES: THEORY AND APPLICATIONS.

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THE CONCEPT OF THE STOCHASTIC EQUIVALENCE SCALES: THEORY AND APPLICATIONS.

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1. Introduction

In the present paper there are depicted theoretical basis of the stochastic equivalence scales (*SES*) concept. Furthermore, estimation methods of such scales on the basis of statistical data are proposed. Theoretical considerations are illustrated with empirical examples.

Theoretical basis of *SES* is constituted by *holistic (stochastic)* research paradigm of welfare. The outline of this paradigm has been presented in the papers of Kot (2002, 2003, and 2004), whilst its further discussion is to be depicted in a separate paper.

The hitherto attitude towards the problem of equivalence scales is founded on individualistic welfare paradigm. The starting point here is an individual consumer provided with individualistic income utility function[1]. What is searched for here is the way of aggregating the welfare of the individuals in order to obtain single characteristics named social welfare function. Alas, the conclusion arising from the famous statement of Arrow (1951) is that such an aggregation is impossible. Within the frames of individualistic paradigm, there are attempts to omit the consequences of the above statement by the acceptance of additional and immensely controversial assumptions. This particularly concerns the interpersonal comparability assumption, which is unacceptable for many economists [c.f. Pollak, 1991].

Theoretical difficulties of individualistic paradigm apply also to the issues of equivalence scales, and that is indicated *inter alia* by the statement of Blundel and Lewbell (1991). It denotes the fact that equivalence scales applied in practice are arbitrary in such sense that they cannot be derived from the existing theory of consumer's behaviour. Theoretical welfare incomparability is transferred in an inevitable way onto the impossibility of the theoretical solu-

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[1] Utility function is treated here as a convenient mathematical representation of preference relation.

tion to the problem of equivalence scales. The concept of stochastic equivalence scales proposed in the present paper aims at overcoming this impossibility.

The remainder of the present paper is structured as follows. Section 2 provides an overview of the holistic paradigm of welfare research. Section 3 defines stochastic equivalence scales and the methods of their estimation on statistical data basis. Section 4 contains empirical examples of such scales, estimated on the basis of micro-data from 2000 Polish HBS. Finally, section 5 makes some concluding remarks and recommendations with regard to further research areas.

2. Holistic concept of welfare

Holistic research paradigm of welfare proposed by the authors constitutes an alternative for the hitherto valid individualistic paradigm. In holistic paradigm the existence of benefit function (*BF)* is postulated as a social instrument for the evaluation of income distribution. In other words, it is assumed that the society as a whole is provided with *BF*[2]. The essence of *BF* consists in the transformation of income distribution into *welfare distribution*.

Formal representation of this transformation is as follows. Let the positive random variable X with c.d.f. $F(x)$ describe the income distribution in the society (population) [3]. The author postulates the existence of *BF* in the form b: $R^+ \rightarrow R$, which transforms the random variable X into a new random variable W:

$$W = b(X) \tag{1}$$

with c.d.f $G(w)$. The values of *BF* will be named *welfare*. We will say that (1) defines *welfare distribution.*

Therefore, the basis of holistic welfare paradigm is constituted by the ordered triple $\langle X,b,W \rangle$, i.e. random variable X (income distribution), non-random *benefit function* $b(\cdot)$ and random variable W (welfare distribution). Each element of the above triple concerns *the population as a whole,* not *an individual* of this population. Holistic paradigm might also be defined as stochastic due to the here applied probabilistic (stochastic) mathematical apparatus describing the population as a whole.

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[2] Let us notice that Arrow's theorem does not exclude the existence of *BF*, being the mathematical representation of social preference relation. The above theorem only infers that it is not possible to obtain this social preference relation on the basis of individual preference relation.

[3] We use here interchangeably the commonly understood mental abbreviations in the form of "probability distribution", "random variable", always meaning a certain random variable, which distribution is described with c.d.f. Consequently, we will also apply the following symbolics: capital Latin letters will denote random variables (measurable functions), while small Latin letters will stand for the values of these functions.

Holistic (stochastic) paradigm proposes theoretical research perspective of welfare so to say from the opposite side than individualistic paradigm does. In individualistic paradigm the starting point are individual welfare, predicted by the ‘trajectories’ of individual consumers’ behaviours. On the basis of these individual trajectories it is attempted to obtain social ‘trajectory’ in the form of social welfare. However, in holistic paradigm it is proposed to start from the other side, i.e. from the decomposition of welfare for the society as a certain whole.

A similar transition from individualistic (deterministic) paradigm to holistic (stochastic) paradigm took place with regard to thermodynamics at one time. It will be illuminating to follow the problem and motives which inclined the physicians in the early XIX century to search for solutions out of the conventional at that time deterministic paradigm.

Thermodynamics of the early XIX century was faced with the problem of measuring the total kinetic energy of gas closed in a certain container[4]. Initially there were attempts to solve this problem within the frames of Newton’s mechanics, which had for the then physicians the value of universal theory. The solution appeared to be simple. The only thing to be done was to define the initial position and momentum of a single gas molecule, describe its trajectory by the known movement equations and predict the collision with the trajectory of another molecule, then with the trajectory of another molecule and so forth. However, this way of solving the problem ended as a failure[5].

The solution of the problem was found by Boltzman, who formulated the basis of stochastic gas theory. He proposed considering kinetic energy *probability distribution* in spite of the hitherto attempts of “aggregating” individual trajectories of gas molecules. By the way he discovered that temperature – which was at that time quite a mysterious physical quantity – is simply the average value of this kinetic energy distribution.

The analogy between the situation in thermodynamics described above and the present situation in welfare economics is very illuminating. Let us pay attention to at least two elements.

Firstly, in welfare research the independence of individual preferences of particular consumers is assumed. The inadequacy of this assumption was pointed out by many authors. One of the attempts to diminish this assumption lies in supplementing the utility function u(x)

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[4] We summarize here the example provided by Prigogine and Stengers (1997).

[5] The reason for this failure was the inadequacy of Newton’s theory for the considered issues. Newton’s equations constituted the idealization of a single particle (a material point) movement with the lack of external impacts, i.e. of other particles. Let us notice that the individualistic welfare paradigm also uses the preferences of individuals, isolated from the preferences of other persons.

with an additional argument in the form c.d.f *F(x)*, regarded as the income rank x in the distribution, i.e. the usage of functional *u(x,F(x))* [c. f. Lambert (2001) p. 123].

Secondly, in welfare economics it is searched for the total (or rather averaging) social welfare - an economic unobservable and very mysterious quantity. This quantity is measured by the mean value $\bar{u}$ of individual welfare, where averaging is performed with regard to income distribution *F(x)*, i.e.

$$\bar{u} = \int_0^\infty u(x)dF(x) \tag{2}$$

In holistic (stochastic) paradigm we employ *welfare distribution W*, which is a category not existing in individualistic paradigm. The mean value $E[W] = \mu_w$ in this distribution is equal to:

$$\mu_w = \int_D w dG(w) = \int_0^\infty b(x)dF(x) \tag{3}$$

where *D* is the relevant range of integration.

It is easy to notice that social welfare $\bar{u}$ in individualistic paradigm is nothing but the mean value in welfare distribution[6] in holistic paradigm. It seems to be obvious that describing the welfare distribution *W*, just like describing any other distribution, only with the mean value is insufficient. Nevertheless, nothing apart from the mean value is offered by individualistic paradigm. However, holistic paradigm allows describing the welfare distribution in a more complete way, e.g. with the use of standard descriptive statistics: position, variability, skewness, etc. One might also analyse inequalities in welfare distribution *W*.

The research perspective offered by holistic (stochastic) paradigm would be heuristically barren if we were not able to determine welfare distribution on the basis of empirical data. In the papers of Kot (2002, 2003, 2004) some theorems have been proved, which allow identifying the welfare distribution form and estimating its parameters, together with the parameters of *BF*, on the basis of empirically observed income distribution. More extensive and more general description of these methods will be presented in a separate paper.

3. Stochastic equivalence scales.

The need to compare the welfare of households with various needs underlies the concept of all equivalence scales. The differentiation of needs is usually associated with the differentiation of household demographic structure[7], for example the size of the household or the

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[6] Similarly, the temperature occurred to be the average value in the kinetic energy distribution of gas molecules.

[7] Obviously, the diversification of needs might also result from other reasons, e.g. disability.

size of the household and the age of its members (whether they are adults or children). At the same time a group of the reference households is established, e.g. single-person households. Equivalence scales are assumed to serve as a tool for converting the income of an analysed household into the income of a reference household in order to obtain the same welfare level of comparable households.

Within the frames of holistic (stochastic) paradigm we propose formulating the problem of establishing equivalence scales on the basis of welfare distribution of comparable groups of households. In practice, this will resolve to compare the *income distribution* in the analyzed group of households with the *income distribution* in the group of reference households.

For the formal problem expression we will divide the population of all households into H decomposable subgroups due to the chosen criterion differentiating the needs of those households, e.g. due to their demographic structure. Let random variables $X_1, X_2, \ldots, X_H$ represent the income distributions of the separated subgroups. For the accepted form *BF* $b(x)$, these income distributions will be matched with welfare distributions, i.e. random variables $W_1 = b(X_1), W_2 = b(X_2), \ldots, W_H = b(X_H)$.

Let there be given a certain function $q\colon R^+ \to R^+$, for which there exists differentiable[8] inverse function $q^{-1}(\cdot)$. Let us choose the group r of reference households with income distribution X_r, and let us mark with $Y = q(X_h)$ the income distribution X_h of the examined group h of households transformed with the function $q(\cdot).h, r = 1,2,\ldots,H, h \neq r$. The welfare distribution of the transformed income distribution Y will be described with the random variable $W_y = b(Y)$.

Definition. The function $q(x)$ will be called *stochastic equivalence scale (SES)* if and only if for each $h,r = 1,2,\ldots, H, h \neq r$:

$$W_y = W_r \tag{14b}$$

or equivalently:

$$Y = q(X_h) = X_r \tag{14a}$$

for the established *BF* $b(x)$, where index h denotes the analyzed group of households, while r denotes the group of reference households.

In other words, *SES* transforms the income distribution of the analyzed group of households (X_h) into the income distribution of the group of reference households (X_r). The equation (14b) shows that after the transformation $q(\cdot)$ the income distribution in the analyzed

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[8] Differentiability $g^{-1}(\cdot)$ is supposed to assure the uniqueness of transformations of random variables.

group of households is the same as the welfare distribution in the group of reference households.

Let us by the way notice that the above-mentioned definition does not specify any particular parametric or non-parametric form of *SES* $q(x)$. It indicates that every function $q(x)$ with the properties such as those specified by definition 1 might be recognized as *SES* [9].

The empirical verification of that whether the given function $q(x)$ might be recognized as *SES* is very simple and is based on statistical test of two distributions equality, e.g. of Kolomogorov-Smirnov *K-S*. In order to do that, we divide the whole sample of incomes of households $(x_1,\ldots,x_n)$ into H groups according to the accepted criterion, e.g. the size of the household. Let $(x_1^r,\ldots,x_k^r)$ denotes the k-element sample of reference households' incomes and $(x_1^h,\ldots,x_m^h)$ the m-element sample of the households' incomes from the examined/analysed group h, $r,h = 1,\ldots,H$, $r \neq h$. We transform now the values of the incomes of the analysed group of households h with the function $q_h(\cdot)$, i.e. we calculate $y_i = g_h(x_i^h)$, $i = 1,\ldots,m$. If $F_r(x)$ and $F_y(x)$ are cumulative distribution functions of, respectively, distribution X_r i Y, then we verify the hypothesis H_0: $F_r(x) = F_y(x)$ against the alternative hypothesis H_1: $F_r(x) \neq F_y(x)$. If the *K-S* test does not reject the null hypothesis H_0, then we can recognize the function to be *SES*. The null hypothesis H_0 testing is repeated for the consecutive groups of households h and the function $q_h(x)$, $h,r = 1,\ldots,H$, $h \neq r$.

The aforementioned procedure we may be enhanced. We extract, as previously, the observations of reference group incomes. The remaining observations are transformed with the function $g_h(x)$ for each h-group separately and the transformed values of incomes obtained in such a way are joined in one group. Now with the use of the *K-S* test we compare the income distribution of the reference group with the income distribution Y of this joint group. If the test does not reject the null hypothesis of distributions equality in both groups, then the family $\{q_h(x)\}_{h=1,\ldots,H}$ of transforming functions might be recognised as *SES*.

Transforming functions $q_h(x)$, which as a result of testing have been recognized as *SES* do not have to be of parametric form. These might be for example certain constants, let us say g_h, treated as 'deflators' of h-group incomes. The sequence $(g_1,\ldots,g_H)$ of such deflators might be then approximated with the selected function with a certain number of parameters, which arguments might be variables, serving as a division criterion of households into H groups, e.g. the number of members, the number of adults and children, etc. In the next section, empirical examples illustrating the discussed method of finding *SES* will be presented.

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[9] Particularly the role of *SES* might be performed by the Ebert and Moyes (1999) transforming function, only if it has differentiable reverse function.

4. Empirical examples.

4.1. The example of non-parametric equivalence scale

Let us consider a certain, very simple and easy to implement *SES*, which can be obtained in the following way. As previously, we divide the households into H decomposable groups and let the indicator *r* relates to the group of reference households, and *h* to the group of examined households. Let us mark with X_r and X_h the income distributions of these groups. Let the mean values in the welfare distributions W_r and W_h corresponding to these two income distributions be respectively equal to: μ_r and μ_h. Let us introduce the following transformations of the analysed income distributions:

$$Z_r = \frac{X_r}{\mu_r}, \quad Z_h = \frac{X_h}{\mu_h} \tag{15}$$

Let us observe that new random variables Z_r and Z_h no more depend on the average level of welfare in their groups[10]. If random variables Z_r and Z_h have the same distribution, i.e. if the equality holds:

$$Z_r = Z_h \tag{16}$$

then from (15) and (16) the equality follows:

$$X_r = \frac{X_h}{\mu_h / \mu_r} \tag{17}$$

The function $q(\cdot)$ in the form (17) might be such a candidate for *SES* which ensures the equality of the mean values in welfare distribution.

For the purpose of estimating the deflator μ_h/μ_r let us call the above-analysed parallel between the average value μ_w in the welfare distribution (2) and the utilitarian social welfare $\bar{u}$ (3). The value $\bar{u}$ tends to be described within the individualistic paradigm by means of the *Abbreviated Social Welfare Function* (*ASWF).* If by μ we mark the average income and by *G* Gini coefficient, then the following *ASWF* $\bar{u}$ might be accepted as the approximation μ_w:

$$\bar{u} = \mu(1-G) \tag{18}$$

[c.f. Sen (1973), p. 33],

$$\bar{u} = \frac{\mu}{1+G} \tag{19}$$

[c.f. Kwakani (1986), p. 200]. As the approximation of μ_w, also the equally distributed equivalent income μ_α might be accepted [c.f. Atkinson (1970)].

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[10] In a similar way we compare the variability in distributions differentiating in the average level, when we use variation coefficients $V = D(X)/E[X]$.

If we decide to accept *ASWF* in the form of (18), the deflator estimator μ_h/μ_r will have the following form:

$$\frac{\mu_h}{\mu_r} = \frac{\bar{x}_h(1-G_h)}{\bar{x}_r(1-G_r)} \tag{20}$$

where $\bar{x}_h$ and G_h together with $\bar{x}_r$ and G_r are the average value and the Gini coefficient respectively in the group *r* and *h* of households.

In order to verify whether the transformation (17) in the form:

$$Y = q(X_h) = X_h \Big/ \frac{\bar{x}_h(1-G_h)}{\bar{x}_r(1-G_r)} \tag{21}$$

might be recognized as *SES*, each income value of the group *h* households is divided by the deflator (20) and then the hypothesis about the equality of the obtained in such a way distribution *Y* and the distribution X_r in the group of reference households *r* is tested. If the test (e.g. *K-S*) does not reject our hypothesis, the function (21) might be recognized as *SES.* The above procedure is repeated for all *H* groups of households (apart from the group *r*, of course).

Example 1. Let us assume that the income *X* is formulated by means of households expenditures. On the basis of Polish HBS 2000, nine groups of households have been distinguished due to the number of members. Let the reference group be single-person households. In table 1 there are the estimates of mean income, Gini coefficient and deflator value (20) presented. There are also placed the values of Kolmogorov-Smirnov (*K-S)* test *Z* and of asymptotic significance (two-tailed p-value).

Table 1. The results of calculating *SES* (equation 21) and *K-S* test for the groups of households with different number of persons

Group ***h***	**No. of persons**	**Mean** $\bar{x}$	**Gini** ***G***	***ASWF*** $\bar{x}$ ***(1-G)***	**Deflator Eq. (20)**	**Sample size**	***K-S Z***	**Asym.Sign. (2-tailed)**
1	1	990.13	0.30814	685.034	1.000000	5098	-	-
2	2	1644.02	0.30561	1141.591	1.666475	9123	0.565	0.907
3	3	1963.22	0.30679	1360.922	1.986650	7673	0.781	0.576
4	4	2125.06	0.29504	1498.077	2.186867	7784	0.553	0.920
5	5	2144.15	0.28244	1538.553	2.245953	3792	1.022	0.248
6	6	2152.47	0.26563	1580.711	2.307495	1661	1.113	0.168
7	7	2294.65	0.28107	1649.695	2.408196	571	0.765	0.602
8	8	2239.32	0.27697	1619.086	2.363514	285	0.582	0.888
9*)	9.69	2514.84	0.30630	1744.549	2.546660	172	0.935	0.347

*) 9 or more persons

Source: Author's calculations based on micro-data from Polish HBS 2000.

The results of *K-S* test presented in table 1 prove that the function (21) might be recognized as *SES*. The application of the deflator (20) has guaranteed that the transformed income distributions of each group did not differ statistically significantly from the income distribution of the reference households group. This can be confirmed by the level of p-value, which has exceeded by far the standard significance level 0.05 for each of the analysed groups of households.

Furthermore, the income distributions of reference households have been compared with the joint distribution of the transformed income of all the groups (31061 observations). The value of *K-S* test amounted to 0.523, and p-value was as high as 0.947, which indicates that the income distributions of both groups did not differ statistically significantly.

Figures 1 and 2 display the concept of stochastic equivalence scales. In fig.1, the theoretical (Dagum) density functions of the distribution of expenditures per household are plotted for five groups of households. Fig.2 depicts the distributions of expenditures per equivalent unit, where the deflator (20) was applied, for the same groups of households.

Fig.1. The distribution of expenditures/household
for various household size.
(Dagum density functions, Polish HBS, 2000)

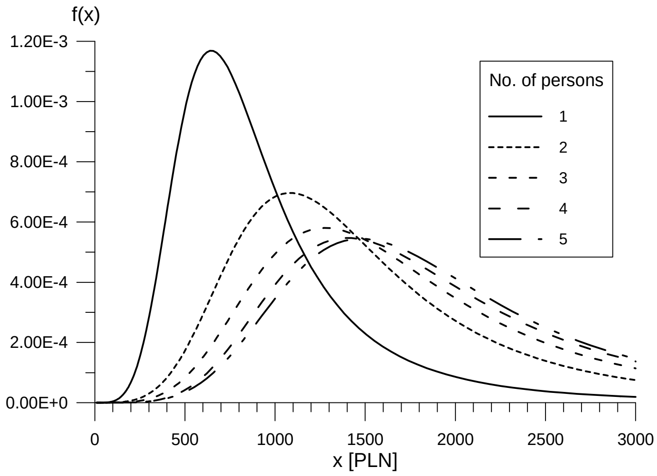
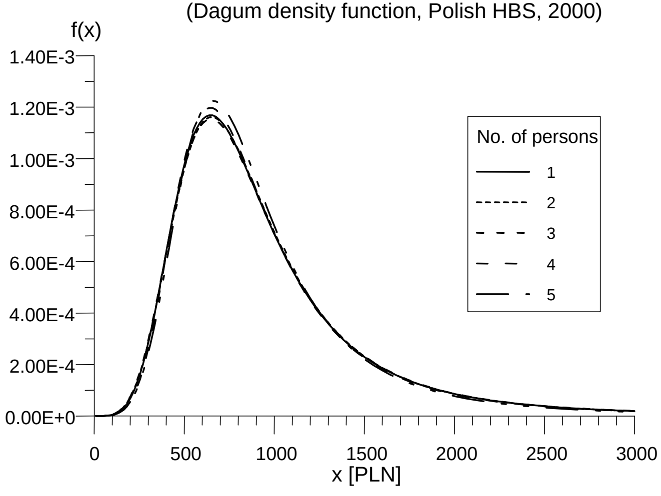


Fig.2. The distribution of expenditures/equiv.unit
(deflator (20)) for various household size.
(Dagum density function, Polish HBS, 2000)



The above figures infer that the non-parametric stochastic equivalence scale with the deflator (20) functions perfectly: the resulting distributions of expenditures (after adjustment) are almost undistinguishable from the distribution of the reference (single-member) households.

Example 2. In this example the whole population of households was divided into 25 groups with regard to the number of adults and children aged less than 18. Single-person households were adopted as the reference group. The evaluation results of the deflator (20) and the values of *K-S* test are presented in table 2.

Table 2. The results of calculating *SES* (equation 21) and the *K-S* test for the groups of households with different number of adults and children

Group h	Group code	No. of adults	No. of children	Mean $\bar{x}$	Gini G	*ASWF* $\bar{x}(1-G)$	Deflator Eq. (20)	Sample size	*K-S* Z	Asym.Sign. (2-tailed)
1	10	1	0	990.13	0.30814	685.034	1.000000	5098	-	-
2	11	1	1	1416.34	0.30811	979.982	1.430560	596	1.034	0.236
3	12	1	2	1377.19	0.27528	998.0778	1.456977	310	0.596	0.936
4	13	1	3.53	1432.32	0.24423	1082.502	1.580218	177	0.861	0.449
5	20	2	0	1659.94	0.30483	1153.938	1.684499	8527	0.519	0.950
6	21	2	1	1971.88	0.31494	1350.855	1.971955	3884	0.790	0.560
7	22	2	2	2083.63	0.29874	1461.163	2.132980	4005	0.692	0.724
8	23	2	3	1963.20	0.28008	1413.343	2.063174	1476	1.217	0.103
9	24	2	4	1847.57	0.24748	1390.332	2.029583	433	1.440	0.032*)
10	25	2	5.69	1812.60	0.25770	1345.489	1.964121	226	0.912	0.376
11	30	3	0	2005.77	0.29527	1413.529	2.063445	3479	0.836	0.487
12	31	3	1	2162.58	0.28758	1540.663	2.249033	2136	0.574	0.897
13	32	3	2	2112.47	0.27180	1538.301	2.245585	1049	1.169	0.130
14	33	3	3	2019.74	0.24318	1528.580	2.231395	350	1.025	0.244
15	34	3	4.71	1986.38	0.24472	1500.276	2.190077	180	1.076	0.197
16	40	4	0	2243.52	0.28979	1593.373	2.325978	1516	0.769	0.596
17	41	4	1	2356.47	0.27816	1700.993	2.483080	957	0.679	0.745
18	42	4	2	2301.93	0.26526	1691.323	2.468964	550	0.767	0.599
19	43	4	3.5	2352.96	0.26855	1721.073	2.512393	237	1.064	0.208
20	50	5	0	2556.62	0.27884	1843.733	2.691449	281	0.828	0.499
21	51	5	1	2491.00	0.27018	1817.981	2.653857	251	0.603	0.860
22	52	5	2.53	2511.80	0.23567	1919.847	2.802559	191	0.933	0.349
23	60	6	0	2380.07	0.28582	1699.796	2.481333	68	0.735	0.653
24	61	6	1.95	2979.59	0.32889	1999.635	2.919032	131	0.854	0.459
25	77	7.18	1.98	3278.90	0.31079	2259.849	3.298888	51	0.449	0.988

*) The differences between the compared distributions are significant statistically (at 0.05 significance level)

Source: Author's calculations based on micro-data from Polish HBS 2000.

The results of the *K-S* test presented in table 2 prove that the deflator (20) meets all the requirements of *SES* also in the case of dividing households into groups on the basis of more complex criterion than the previous division. All income distributions transformed with the deflator (20) did not differ statistically significantly from the income distribution of the reference households. The only exception was in the case of group 9 households (2 adults, 4 children).

In addition to the comparison in pairs between the transformed income distribution of each group and the income distribution of the reference households, the latter distribution was compared with the distribution of all the transformed income together. The value *Z* of the statistics *K-S* was equal to 0.704, while the p-value equal to 0.705 exceeded the critical significance level 0.05. This indicates that the distribution of income transformed by means of the deflator (20) did not differ statistically significantly from the income distribution of the reference households.

The non-parametric *SES* applied here is particularly simple and easy to employ. To convert the income of the examined household group to the income of the reference households, it is sufficient to use the deflator in the form of the quotient of the examined households *ASWF* and the reference households *ASWF* .

4.2. The examples of the parametric equivalence scales

The stochastic equivalence scales can also be expressed in the parametric form. Furthermore, the parameter estimation of such scales is also possible.

The practice uses various parametric deflators. If as the criterion of households division only the number of members h is chosen, then the income of the examined household X_h is converted to the income of reference household X_r according to the following equivalence scale:

$$X_r = \frac{X_h}{h^{\varepsilon}}, \qquad 0 \leq \varepsilon \leq 1 \tag{22}$$

[c.f. Buhmann et al., (1988)]. The parameter ε of the scale (22) is determined in an arbitrary way.

Using the *SES* concept, the parameter ε occurring in the deflator form h^{ε} can be very easily estimated. If with d_h we denote the non-parametric evaluation of the deflator (20) for an h-person household, then having the sequence of these evaluations for $h=1,2,\ldots,H$ the parameter ε can be estimated by means of the following model of non-linear regression:

$$D = h^{\varepsilon} + U \tag{23}$$

where U is the random term with the null mathematical expectation and the variance σ_u^2.

Using the deflator values presented in table 1, ε was estimated accepting square loss function. The evaluation $\varepsilon = 0.451619$ was obtained with the standard deviation equal to 0.020678. The 95% confidence interval for this parameter was equal to (0.403936, 0.499302). Residual sum of squares obtained the value of 0.43951029. The participation of explained variance was equal to 0.76148 ($R = 0.87263$).

In fig. 3 there is depicted the graph of d_h values, observed and expected by the model (23). Figure 3 shows that the power equivalence scale does not very well fit the empirical values of the deflator. It definitely overestimates these values for very big households and underestimates for the remaining households.

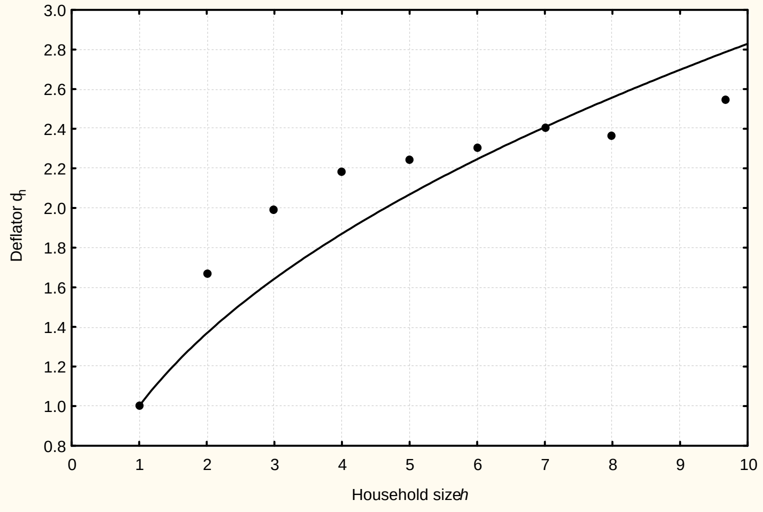


Fig.3. The deflator of the power equivalence scale
$d_h = h^{0.451619}$

Fortunately, we are in such a convenient situation that we may experiment with any parametric forms of deflator and choose the best one from the point of view of the goodness of fit criterion. In table 3 there are compared three additional deflator forms and the evaluations of their parameters. Moreover, there are also presented the elasticities of equivalence scales in relation to the number of persons in a household[11].

Table 3. Parametric relative equivalence scales and their estimates obtained with Nonlinear LSM

No.	The form of the deflator d_h	Elasticity of the scale	Estimates	Standard error	95% Confidence interval		Residual Sum of Squares	Fraction of explained variance (R^2)
					Lower limit	Upper limit		
1	h^{ε}	$-\varepsilon$	0.451619	0.020678	0.403936	0.499302	0.439510	0.76148
2	$1+\varepsilon \cdot log(h)$	$-\frac{\varepsilon}{1+\varepsilon\, log(h)}$	0.733215	0.026931	0.671113	0.795318	0.131573	0.928597
3	$1+(h-1)^{\varepsilon}$	$-\frac{\varepsilon}{(h-1)^{1-\varepsilon}+h-1}$	0.175697	0.032158	0.122294	0.229101	0.144160	0.921767
4	$1+\gamma(h-1)^{\varepsilon}$	$-\frac{\varepsilon\gamma}{(h-1)^{1-\varepsilon}+\gamma(h-1)}$	γ= 0.767107	0.042451	0.666725	0.867488	0.028451	0.984560
			ε = 0.327258	0.033055	0.249096	0.405420		

Source: Author's calculations based on the data from Table 2

The power equivalence scale with the deflator h^{ε} (no 1 in table 3) has constant elasticity in relation to the size of the household *h*. The remaining equivalence scales (with deflators no 2, 3 and 4 in table 3) are characterized by variable elasticity.

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[11] The elasticity of relative equivalence scale $y = x_h/d_h$, where d_h is the deflator, was calculated in accordance with the formula: $(\partial y/\partial h) : y/h$.

Table 3 infers that the equivalence scales, alternative for (22), provide a much better consistency with empirical values. Special attention should be paid to the equivalence scale no 4 with the two-parameter deflator in the form of $d_h=1+\gamma\cdot(h-1)^{\varepsilon}$. The parameter γ of this scale can be interpreted as the weight assigned to the first additional person in a household. The adjustment of this deflator to the empirical data is illustrated by fig. 4.

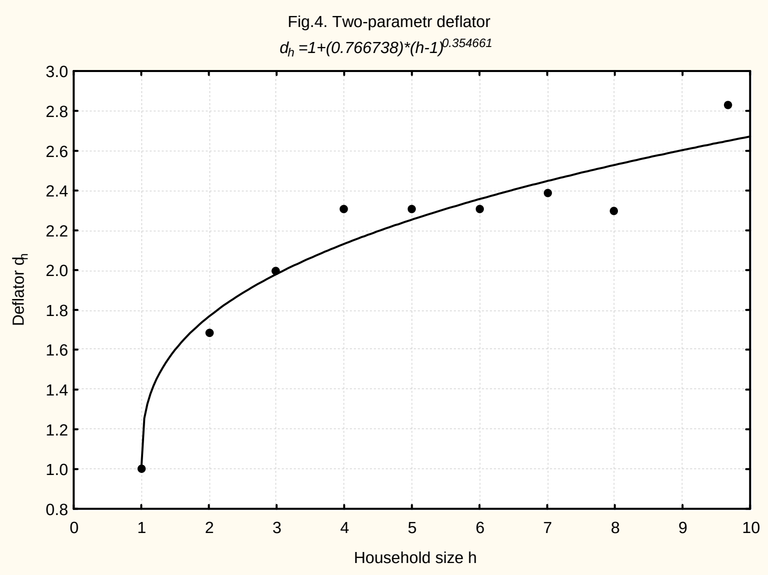

Fig.4. Two-parametr deflator

d_h =1+(0.766738)*(h-1)$^{0.354661}$

It is visible in fig. 4 that the deflator of the two-parameter equivalence scale approximates the empirical values much better than we observed in the case of the power equivalence scale in fig. 1.The fact that the two-parameter equivalence scale has variable elasticity with regard to the size *h* of the household does not constitute any hindrance as this elasticity might be easily calculated. This is illustrated by fig. 5.

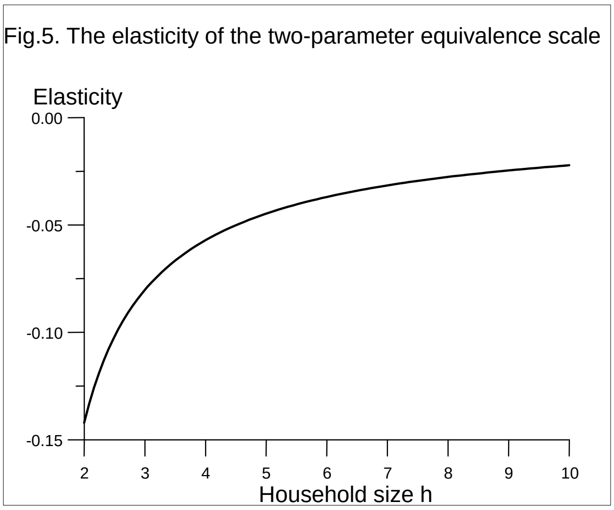

Fig.5. The elasticity of the two-parameter equivalence scale

The following example will illustrate the application of the parametric *SES* in the case when the diversification of the household needs is expressed by the number of adults *a* and the number of children *k*. In practice, the following equivalence scales are applied:

Coutler and Katz scale (1992), (abbr. *CK*):

$$X_r = \frac{X_h}{(a + \delta \cdot k)^{\varepsilon}} \tag{24}$$

and the so-called OECD scale:

$$X_r = \frac{X_h}{1 + \gamma(a-1) + \delta \cdot k} \tag{25}$$

The parameters of these scales are set arbitrarily.

Being in possession of the deflator (20) evaluations presented in table 1, we can estimate the unknown values of these scales parameters. The results of the estimation are presented in table 4.

Table 4. The estimates of the Cutler-Katz, OECD, and the combined OECD-Coutler-Katz equivalence scales obtained by Nonlinear LSM

Equivalence scale	The form of the deflator d_h	Estimate	Standard error	95% Confidence interval		Residual Sum of Squares	Fraction of explained variance (R^2)
				Lower limit	Upper limit		
Coutler-Katz	$(a+\delta \cdot k)^{\varepsilon}$	δ=0.347059	0.059814	0.223324	0.470794	0.777064	0.876602
		ε=0.580134	0.014336	0.550478	0.609790		
OECD	$1+\gamma(a-1)+\delta \cdot k$	γ=0.367624	0.021745	0.322641	0.412607	1.524916	0.757879
		δ=0.152376	0.025134	0.100382	0.204369		
Combined OECD-Coutler-Katz	$1+[\gamma(a-1)+\delta \cdot k]^{\varepsilon}$	γ=0.639066	0.067678	0.498710	0.779422	0.334143	0.946946
		δ=0.107752	0.025016	0.055871	0.159632		
		ε=0.513963	0.048793	0.412773	0.615154		

Source: Author’s calculations based on the data from Table 2

The parameter evaluations of the *C-K* scale indicate that the “cost” of a child constitutes around 35% of the expenditure of an adult. The elasticity ε of this scale with regard to the “effective household size” is equal to 0.58; therefore, it is greater than the value 0.45, which was obtained previously for the power scale (22), thus in the case when the cost of a child is considered equal to the cost of an adult.

The obtained parameter evaluations of the OECD scale differ significantly from those commonly applied in practice. Let us remind that for this scale the arbitrary values $\gamma = 0.7$ and $\delta = 0.5$ are accepted, while in the case of the so-called *augmented OECD* scale, γ is set at the level of 0.5 and δ at the level of 0.3. The results in table 4 also indicate that the *C-K* equivalence scale much better approximates the empirical data than the OECD scale does.

An attempt was made to create a scale “combined” from the OECD scale and the *C-K* scale (abbr. *OECD-C-K*), i.e. the equivalence scale with the deflator form: $d=1+[\gamma(a-1)+\delta \cdot k]^{\varepsilon}$. This new, three-parametric equivalence scale provided much better fit to the empirical data than the both scales separately. In the *OECD-C-K* scale the parameter γ is the scale assigned to each additional adult person (as in the OECD scale), whereas the parameter δ is the weight assigned to a child and represents the cost of a child as the fraction of the expenditure of an adult. In this new scale the effective size of the household is depicted in the form *1* with the surplus of an additional adult and a child added. The parameter ε reflects here the scale elasticity in relation to this "surplus" effective size of the household.

The evaluation of the parameter γ in the new *OECD-C-K* scale equals 0.64 and is much greater than the one obtained for the OECD scale. On the other hand, the expenditure of a child constitutes here about 11% of the expenditure of an additional adult person. The evaluation of the parameter ε of the *OECD-C-K* scale was equal to 0.51; therefore, it was smaller than the value of 0.58 in the *C-K* scale.

4.3. The influence of the equivalence scales on the income distribution

Let us finally examine the problem of how selecting the form of equivalence scale influences the expenditures distribution parameters. The evaluations of the basic statistics for all the equivalence scales discussed above are displayed in table 5. In addition to the equivalence scales based on the *SES* concept, in table 5 there are also presented the values of statistics in the distribution of expenditures per person.

Table 5. Descriptive statistics for the distribution of expenditures adjusted by non-parametric and parametric *SES*

No.	**Deflator**	**Description**	**Mean**	**V**	**Skew-ness**	**Kur-tosis**	**Gini**	***ASWF*= $\bar{X}$ *(1-G)***	
								Value	**% dev.3)**
1	h	*Per capita*	574.80	.81180	6.61	117.67	.34103	378.77	44.71
2	***Eq. 20*1)**	***Nonparametric***	**971.12**	**.68953**	**5.58**	**69.53**	**.29467**	**684.96**	**0.01**
3	***Eq. 20*2)**	***Nonparametric***	**965.96**	**.68345**	**5.73**	**74.20**	**.29109**	**684.78**	**0.04**
4	h^{ε}	*Power*	1105.08	.70085	5.60	68.95	.29872	774.97	13.13
5	$1+\varepsilon \cdot log(h)$	*Logarithmic*	1032.39	.69501	5.60	69.18	.29634	726.46	6.05
6	$1+(h-1)^{\varepsilon}$	*Modified power*	926.34	.68864	5.48	68.55	.29592	652.22	4.79
7	$\boldsymbol{1+\gamma(h-1)^{\varepsilon}}$	***Modified power***	**978.41**	**.68895**	**5.51**	**68.06**	**.29507**	**689.72**	**0.68**
8	$(a+\delta \cdot k)^{\varepsilon}$	*Coulter-Katz (C-K)*	1059.89	.69716	5.77	74.70	.29566	746.52	8.98
9	$1+0.7(a-1)+0.5k$	*OECD*	760.76	.74883	6.15	90.43	.31602	520.34	24.04
10	$1+0.5(a-1)+0.3k$	*OECD augmented*	933.84	.72175	5.92	79.91	.30518	648.85	5.28
11	$1+\gamma(a-1)+\delta \cdot k$	*OECD estimated*	1121.14	.70391	5.78	74.58	.29841	786.58	14.82
12	$\boldsymbol{1+[\gamma(a-1)+\delta \cdot k]^{\varepsilon}}$	***Combined C-K and OECD***	**978.41**	**.68763**	**5.70**	**74.28**	**.29300**	**691.74**	**0.98**
Reference households (1 person)			**990.13**	**.78201**	**8.56**	**161.61**	**.30814**	**685.03**	**0.0**

1) Household selection due to household size *h* (as in Table 1)

2) Household selection due to the number of adults *a* and children *k* (as in Table 2)

3) Relative deviation [%] from 685.03 (*ASWF* for reference households), signs omitted.

Source: Author's calculations based on the micro-data from Polish HBS 2000.

Let us remark that we accepted the Sen *ASWF*, calculated as $\bar{x}\,(1\text{-}G)$ as the compatibility criterion of the distribution transformed by means of *SES* with the distribution in the group of reference households. In the last column of table 5 there are displayed the values of percentage deviation of the calculated value of the *ASWF* for the given equivalence scale from the value 685.03 for the distribution of expenses in reference households (single person). For the simplification, the sign of this deviation was omitted. The last column of table 5 infers that two non-parametric *SES*s provide the greatest consistency of *ASWF* with the value 685.03 of the reference distribution, which is obviously the consequence of these scales definitions. Among the parametric equivalence scales the best ones – from the accepted criterion point of view – occurred to be the modified two-parameter power scale (no 7 in table 5) and the combination of Coulter-Katz scale with the OECD scale (no 12 in table 5).

This observation seems to be significant at least for the reason that the power scale distinguished here (no 7 in table 5) is based only on the size of the household *h*, while for the estimation of the combined scale *C-K-OECD* (no 12), two characteristics of a household are used: the number of adults *a* and the number of children *k*. This implies that the additional information about the age structure of a household does not contribute in a significant way to the construction of equivalence scales. However, it is obvious that the combined equivalence scale might be useful in other analyses regarding for example the costs of a child in a family.

Against the background of the two discussed here equivalence scales, the scales commonly applied in practice do not perform well: the power scale either with the parameter $\varepsilon = 1$ (per capita) or the estimated value ε, and also the OECD and the Coulter-Katz scales. To simplify further comparisons, we accept the statistics in the distribution obtained by means of the modified two-parameter power scale (no 7 in Table 5).

As we can infer from table 5, in practice the most popular scale – the *per capita (PC)* scale (no 1) – very much underrates the evaluation of the average income, while, on the other hand, significantly overrates the inequality evaluation (Gini). The composition of these two quantities leads to the very considerable underrating of the average welfare, measured here with Sen *ASWF*. The same applies to the OECD scale. The equivalence scale called the augmented OECD scale occurs to be much better than the two previously mentioned scales.

The influence of the equivalence scales on the poverty measures is depicted in table 6. As the reference point, like previously, the evaluations of the poverty measures obtained by means of the modified two-parameter power scale (no 7 in table 6) will be accepted.

Table 6. Poverty measures for the distribution of expenditures adjusted by non-parametric and parametric Stochastic Equivalence Scales. Poverty line = 315 [PLN] (subsistence level)

No.	Deflator	Description	Head-count	Mean among poor	Poverty gap	Per-capita poverty gap	P_2
1	h	*Per capita*	0.24771	234.52	0.25550	0.06329	0.02352
2	*Eq. 20*1)	*Nonparametric*	0.02195	261.47	0.16993	0.00373	0.00106
3	*Eq. 20*2)	*Nonparametric*	0.02133	263.18	0.16452	0.00351	0.00099
4	h^{ε}	*Power*	0.01215	262.30	0.16732	0.00203	0.00059
5	$1+\varepsilon \cdot log(h)$	*Logarithmic*	0.01616	261.35	0.17033	0.00275	0.00079
6	$1+(h-1)^{\varepsilon}$	*Modified power*	0.02878	259.21	0.17712	0.00510	0.00149
7	$1+\gamma(h-1)^{\varepsilon}$	*Modified power*	0.02168	261.24	0.17067	0.00370	0.00106
8	$(a+\delta \cdot k)^{\varepsilon}$	*Coulter-Katz*	0.01416	264.18	0.16135	0.00228	0.00065
9	$1+0.7(a-1)+0.5k$	*OECD*	0.08495	254.08	0.19339	0.01643	0.00506
10	$1+0.5(a-1)+0.3k$	*OECD augmented*	0.03068	261.64	0.16938	0.00520	0.00148
11	$1+\gamma(a-1)+\delta \cdot k$	*OECD estimated*	0.01097	263.54	0.16338	0.00179	0.00052
12	$1+[\gamma(a-1)+\delta \cdot k]^{\varepsilon}$	*C-K-OECD*	0.02105	262.86	0.16553	0.00348	0.00099
Reference households (1 person)			**0.02668**	**258.41**	**0.17966**	**0.00479**	**0.00150**

1) Household selection due to the household size *h* (as in Table 1)

2) Household selection due to the number of adults *a* and children *k* (as in Table 2)

Source: Author's calculations based on micro-data from Polish HBS 2000.

The *PC* scale (no 1 in table 6) drastically overrates the evaluations of all the relative poverty indices. Head-count Ratio calculated by means of the *PC* scale is more than eleven times greater than the value calculated on the basis of the reference scale 7. Overrating of the P_2 by means of the *PC* scale is even greater (more than twenty-two times). In a similar way, although in a lesser extent, the OECD scale functions. The *Augmented OECD* equivalence scale appears to be much more precise than the two previously mentioned scales.

It is obvious that the above observations cannot be, at least now, generalised. The comparative analysis of equivalence scales depicted here concerned only the income distribution (measured in expenditures) in Poland in 2000. To draw conclusions of a general nature, it is required to conduct the comparative studies of income distribution of many countries and for many years.

5. Final conclusions.

The holistic paradigm of welfare economics offers new research possibilities, unreachable in the individualistic paradigm. The concept of the stochastic equivalence scales is one, although not the only one, of such possibilities. "Axiomatic" formulation of *SES* is very general. It does not specify one definite form of such scale, but defines the properties, which should be possessed by a certain function *q(x),* in order to be recognized as *SES.* To solve this problem, it is sufficient to apply the statistical test of two distributions equality. It should be emphasised that *SES* are not arbitrary in such sense that they have theoretical bases in the stochastic paradigm proposed by the authors.

The application of *SES* is in practice very simple. What might appear to be particularly useful are the deflators created by means of *ASWF*s , e.g. the deflator of (20) type, based on Sen *ASWF*. For this purpose, households have to be divided into groups using the criterion of need diversification chosen by the researcher, the group of reference households has to be selected, the value of *ASWF* for each group has to be calculated, and then deflator has to be calculated by dividing these values by the value of *ASWF* of the reference group. Next we divide the incomes of households in each of the separated group by the deflator calculated for this group.

Let us notice that many criteria for diversifying needs of a household are possible. We may divide households into homogeneous groups taking into account for example the number of members, their age (including several age groups of children, elderly persons, etc.), sex, or socio-occupational characteristics of the household head. The territorial and temporal diversification of households is also possible. The aim of the research always determines the choice of the particular criterion.

Non-parametric estimation of the values of deflators might give grounds to parametric modelling of equivalence scales. The empirical examples presented in the present paper show the possibilities of applying many parametric forms of such scales.

In the present paper we did not *explicite* use the welfare distribution. We only used the mean value approximation in this distribution by means of *ASWF*. In the papers of Kot (2002, 2003, 2004) we obtained the evaluations of parameters of power equivalence scales and the Coulter-Katz scale on the basis of evaluations of welfare distribution parameters.

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