Session Number: 6C
Session Title: Productivity II
Session Chair: Harry Wu, Hong Kong Polytechnic University, Hong Kong, China

Paper Prepared for the 29th General Conference of
The International Association for Research in Income and Wealth

Joensuu, Finland, August 20 – 26, 2006

Measurement Problems with Non-Invariant Economic Index Numbers of Outputs, Inputs, and Productivity: The Case of Italy

Carlo Milana

For additional information please contact:

Author Name(s) : Carlo Milana
Author Address(es) : *Istituto di Studie Analisi Economica*, Piazza dell'Indipendenza, no.4, I-00185 Rome, Italy
Author E-Mail(s) : *c.milana@isae.it*
Author FAX(es) : +39-06-4448-2249
Author Telephone(s) : +39-06-4448-2300 (office), +39-347-8000984 (mobile)

Measurement Problems with Non-Invariant Economic Index Numbers of Outputs, Inputs, and Productivity: The Case of Italy

Carlo Milana*

"*The fundamental and well-known theorem for the existence of a price index that is invariant under change in level of living is that each dollar of income be spent in the same way by rich or poor, with all income elasticities exactly unity (the homothetic case). Otherwise, a price change in luxuries could affect only the price index of the rich while leaving that of the poor relatively unchanged. This basic theorem was well known already in the 1930's, but is often forgotten and is repeatedly being rediscovered*".

"[…] *Although most attention in the literature is devoted to price indexes, when you analize the use to which price indexes are generally put, you realize that quantity indexes are actually most important. Once somehow estimated, price indexes are in fact used, if at all, primarily to 'deflate' nominal or monetary totals in order to arrive at estimates of underlying 'real magnitudes' (which is to say,* quantity *indexes!)*".

"[…] *The fundamental point about an economic quantity index, which is too little stressed by writers, Leontief and Afriat being exceptions, is that it must itself be a cardinal indicator of ordinal utility*".

P.A. Samuelson and S. Swamy (1974, pp. 567-568)

"[…] all *of the traditional test criteria of Fisher (1911) for an index number are satisfied by the canonical pair* $[p(P^1, P^0), q(P^1, P^0)]$ *in the homothetic case*".

P.A. Samuelson and S. Swamy (1974, p. 571)

---

* *Istituto di Studi e Analisi Economica*, Piazza dell'Indipendenza, no. 4, I-00185 Rome, Italy. Phone (office): +39-06-4448-2300; Mobile: +39-347-8000984; FAX: +39-06-4448-2249; E-mail: *c.milana@isae.it*; Web site: http://www.isae.it; personal Web site: http://users.libero.it/carlo.milana/.

This paper has been prepared for presentation at the 29th General IARIW Conference to be held on 20-26 August, 2006 at the University of Joensuu, Finland. It is part of the results of the Specific Targeted Research Project "EUKLEMS-2003. Productivity in the European Union: A Comparative Industry Approach" supported by the European Commission within the Sixth Framework Programme with Contract No. 502049 (SCS8).

Abstract

This paper is focused on measurement problems with composite index numbers of outputs, inputs, and productivity when input and output aggregation conditions are not satisfied. The discussion is aimed at devising a general framework where non-separable outputs and inputs can be aggregated together and an invariant measure of technical change can be obtained. In economic theory, it is well-known that price and quantity aggregates of inputs and outputs exist if and only if the underlying functions are homothetically separable. Under this condition a composite quantity index can be constructed with the required property of linear homogeneity (a scalar change in all elementary quantities changes the index by the same proportion), while its "dual" composite price index is independent from reference variables. In production activities, constant returns to scale and Hicks-neutral technical progress leave the price indexes invariant with respect to the output levels and technical change. By contrast, in the general non-homothetic case, any attempt to define composite indexes of outputs, inputs and total factor productivity ends up to path-dependent magnitudes that are not robust to measurement choices. In the current state of the art of index numbers, this difficulty is assumed to be circumvented by adopting implicitly "approximating formulas". In this paper, we show that this is not correct and an alternative method based on the use of profit functions is introduced. This does not separate *a priori* inputs and outputs and permits us to construct *net* aggregates that are always linearly homogeneous even in the non-homothetic case. This method is applied empirically to the data of the Italian industries using the newly built database of the EUKLEMS project. Homotheticity seems to be the exception rather than the rule during the period 1970-2003 and the results obtained are contrasted with those of traditional approaches that assume homothetic input-output separability. Although these alternative measures are not fully comparable, we conclude that the TFP decline recently reported in Italy is not confirmed in size and direction by our findings on technical change.

1. Introduction

Recent papers have attempted to propose methods for decomposing the standard total factor productivity (TFP) index (defined as the ratio of aggregate index of outputs to the aggregate index of inputs) in a number of components, among which non-constant returns to scale and technical progress are seen as the most important while the cyclical capacity utilisation and changes in market structure and imperfect competition are often considered as additional elements. The practice of decomposing the TFP index is, however, not new (see for example Young, 1928, Griliches, 1963, and Westfield, 1966), but it is with the recent contributions that it has become important to assess which are the most relevant drivers of a sustained growth. Under the hypothesis of constant returns to scale, Takayama (1974) had found empirical evidence of a biased technical change in the US. Other famous empirical findings of biased technical changes in the US industries has been those obtained by Jorgenson and Fraumeni (1981) and Jorgenson, Gollop, and Fraumeni (1989).

Under the influence of the "new growth theory" stressing the importance of scale economies in explaining the high growth registered in newly industrialized countries and also in some economically advanced regions, a number of studies have been recently devoted to find new empirical evidence. Park and Known (1995), for example, have found that the traditional TFP estimates for the Korean case have been distorted because they have not taken into account short-run equilibrium choices and increasing returns to scale. In fact, these have seemed to play the major role in explaining the observed rapid economic growth, suggesting that this would be possible even with a negative technical change. Basu and Fernald (1997) have found that, in the US, increasing returns are present in many firms although many others seem to operate under constant returns to scale. Diewert and Fox (2004) and Fox (2005), in examining the New Zealand case, have found a little contribution to TFP growth from technical progress, whereas increasing returns to scale may have played a substantial role also there. These results were also in line with those obtained by Nakajma et al. (2002) for Japan before and after the bursting of the financial bubble.

All these studies have reached their conclusions by means of parametric (econometric) methods, which, by their nature, impose *a priori* hypotheses both on the deterministic and stochastic parts of the models used. Severe problems of multicollinearity arise usually in the econometric estimations of flexible functional forms because outputs and inputs have strong trends over time. An attempt in the direction of using index numbers recently done by Diewert and Fox (2005) have used "technically" (not economically) defined distance functions which are consistent with any degree of returns to scale. This extends an innovative study previously made by Caves, Christensen, and Diewert (1982), which did not allow the identification of the contribution of increasing returns to scale,. They still need, however, some "external" econometric estimation of the rate of (Hicks-neutral) technical change and the degree of returns to scale over the period. A more relevant drawback of this approach is that, as we shall see later, with non-homothetic effects of non-constant returns to scale, the advantage of constructing an appropriate linearly homogeneous quantity index is accompanied by the disadvantage of having its implicit "dual" price index that, in general, fails to meet the requirements of the linear homogeneity property. On the other hand, as noted by Samuelson and Swamy (1974, p. 576), in the general non-homothetic case, if one constructs a price index number independently from the quantity index so that they both satisfy the homogeneity requirements, then these two index numbers are inconsistent with the nominal value given by the sum of elementary prices times the respective quantities.

In this paper, we follow a different route by remaining in the field of economic index numbers. In the vein of Samuelson (1947, pp.146-163)(1950)(1953), Samuelson and Swamy (1974), and Swamy (1985), we construct indicators of "netput" price and quantity changes both satisfying the required properties for aggregation even in the presence of non-constant returns to scale. This can be done by defining indexes in the spaces of input-output quantity and price transformation functions which are always homogeneous by construction with respect to their arguments. The resulting normalized aggregate index is net of the effects of returns to scale and represents technical change rather than TFP. The returns-to-scale effects are, instead, incorporated into the general price-induced substitutions and are taken into account implicitly by the economic index number formula.

The economic theory of the producer price and quantity index numbers is largely isomorphic to the economic theory of cost-of-living and consumer welfare indexes. The concepts of "true" index numbers measuring the cost of achieving a certain standard of living and consumer's real income and economic welfare can be adapted in the context of changes in producer's outputs, inputs and productivity. A major difficulty in constructing these index numbers may arise with production activities involving more than one output or one input. Finding an index number that reduces the observed changes in the elements of one vector to a single scalar while satisfying simultaneously several desirable properties may reveal to be a problem that cannot be solved completely in the general case. This impossibility theorem has not fully discouraged a never-ending search of the best class of possible formulas that approximate the theoretical concept of "true" index numbers. In the spirit of this search, this paper starts with a collection of alternative possible formulas based on profit, revenue, and cost functions and tries to clarify the pitfalls of using index numbers derived from these functions under conditions that rule out aggregation.

With non-homothetic changes in production technology, the implicit quantity index that is obtained by deflating nominal or money values by means of an economic price index number may fail to satisfy the one-degree homogeneity property (that is, if all the elementary items change proportionally, also the aggregating index number should result multiplied by the same factor of proportionality). This property is very important for the index to have an economic meaning and is strictly related to the property of the deflator of being invariant with respect to relevant reference variables. Some authors remain within the economic-theoretic approach and prefer to define cost- or profit-based index numbers of prices, while giving up the linear homogeneity property of the implicit quantity indexes. On the other hand, if one prefers to define linearly homogeneous quantity index numbers (based, for example, on technically optimised distance functions), he has to face implicit price index numbers that may fail, in the non-homothetic case, to preserve the desired property in question. However, it is by now a standard procedure to take the geometric average of two economic-theoretic price (or technically optimised quantity) index numbers, defined, respectively, with reference to the the base and current period, in order to mitigate the lack of “invariance" property. Another solution is obtained by using specific weights that permit the implicit index number of quantities to respect at least *locally* the linear homogeneity property.

However, all the solutions that can be found are not completely satisfactory in the general non-homothetic case as they do not lead us to "true" index numbers of quantities (or prices) that are *globally* invariant along the whole range of possible reference vectors of prices (or quantities). Because non-homotheticity does not justify separability and aggregation of the outputs or inputs within the internal functional structure of production, the lack of invariance of the resulting composite index numbers can be, at best, reduced only partially. It will be shown that the so-called Diewert’s (1976) “superlative” index numbers, which correspond to an average between the values taken by the economic indexes in the two compared situations, may not offer a satisfactory solution to this problem . In the general non-homothetic case, it is not at all guaranteed that they “average out” the measurement distortions significantly. As a consequence, even with superlative index numbers their dual counterparts may not respect fundamental properties.

The rest of the paper proceeds as follows. In section 2, separability conditions for outputs, inputs, and technical change are defined within a general description of the production technology. In section 3, the dual profit, revenue, and cost functions are formulated under various hypotheses of separability conditions. In section 4, with reference to Appendix A, a full collection of economic index numbers based on revenue and cost functions is presented and studied at the light of the theory of bounds and the economic theory of production, focusing the attention on the degree-one homogeneity property required for aggregation. In section 5, the superlative index numbers of outputs, inputs and productivity that are constructed in the general non-homothetic case are critically discussed and new generalized theorems are referred to with indication of exact identities between these index numbers and different functional forms. In section 6, more general decomposition procedures based on the profit function are proposed in order to overcome the non-separability problem between outputs and inputs in constructing indicators of technical change and net output deflators. Section 7 presents an application to the Italian case and compares the results obtained with alternative approaches. Section 8 concludes with final remarks and suggestions.

2. General conditions of the production technology

> *"The neoclassical production function f(**x**) is an inconvenient functional representation, because it is not definable for the general case".*
>
> Shephard (1974, pp. 200-201)

Basic hypotheses concerning the production activities are made with the most general description of technological conditions. The social, economic and institutional environment, the constraints to which producers are subject, the firms' objectives, the entrepreneurs' organizational and production behaviour are also important for describing the producers' behavioural choices, but for simplicity and without any loss for further generalizations we concentrate here only on the production technology.

2.1 The production possibility set

The *production possibility set* at time S^t contains all feasible combinations of nonnegative M outputs and nonnegative N inputs of the production activity that uses the technology available at time t and is defined as:

$$S^t \equiv \{(\mathbf{y},\mathbf{x}) : T^t(\mathbf{y},\mathbf{x}) \leq 0\} \tag{2.1}$$

where $\mathbf{y} = [y_1, y_2, \ldots, y_M]'$ is a nonnegative M dimensional (column) vector of outputs, $\mathbf{x} = [x_1, x_2, \ldots, x_N]'$ is a nonnegative N dimensional (column) vector of inputs, $T^t(\mathbf{y},\mathbf{x})$ is a function that, when set equal to zero, represents the *producer's indifference curve* or *transformation function* in the space of output and input quantities at time *t*. It is well-defined and, in particular, it is single valued for all outputs and inputs, continuous and twice differentiable; it implies no outputs with no inputs and no decreases in outputs with increases in inputs. Moreover, it may vary over time in all its parameters or even in its functional form. It is worth noting that using the general transformation function $T^t(\mathbf{y},\mathbf{x})=0$, rather than to the usual production function, places us in a position to unveil the difficulties that are present when a homothetic production function cannot be defined and to find (partial) solutions to the consequent measurement problems.

We do not introduce, at this stage, any assumption regarding the degree of returns to scale and the effects on input-output combinations of economies of scale and technical change. These can be of any type, bringing about homothetic (proportional) or non-homothetic (non-proportional) changes in optimal input and output quantities or real-prices. These assumptions are more general than the usual restrictive hypotheses made on the technology of production using input-output separable functional relationships, often described by means of production functions. As we shall see, most of the problems encountered in measuring production or consumption aggregates stem from imposing the existence of a production function that separates outputs from inputs.

The resulting set S^t is non-empty, closed and bounded for any feasible combinations of $\mathbf{y}$ and $\mathbf{x}$. It is also non-empty, so that free disposability (waste) of outputs and inputs in production processes is possible. The subset

(2.2) $$E^t \equiv \{(\mathbf{y},\mathbf{x}): T^t(\mathbf{y},\mathbf{x})=0\}$$

is the contour of set S^t and represents the so called *production possibility frontier*. It is the set of all the technically efficient combinations of outputs and inputs with the technology available at time *t*. All combinations $(\mathbf{y},\mathbf{x})$ that belong to S^t, but not to E^t, are defined to be technically inefficient in the sense of Debreu (1951) and Farrell (1957)[1].

[1] Since the analysis that follows does not concern technically inefficient solutions, but deals with only "restricted" or "unrestricted" solutions among all the *technically optimal* (non-wasteful) alternatives, we could disregard free disposability and technical inefficiency and, therefore, work only using the frontier subset E^t. However, we prefer to maintain the description of the full production possibility set S^t and remain in the tradition of most of the production-theoretical literature.

In non-joint production activity, every output is produced in its own process and is not a function of other outputs and inputs. However, we assume that, in general, joint production can take place, so that the resulting production possibility frontier represents an optimal technical relationship between many outputs and many inputs.

Technical change can be biased or unbiased (neutral) in many respects. It is Hicks-neutral when the functional form of the input-output functional relationship *remains the same*, except for the fact that its parameters change in such a way that all quantities consequently change proportionally in the output (input) space for given quantity levels of inputs (outputs). In this case, the transformation function shifts *homothetically* in the output and input spaces. Economies of scale may produce effects on the changes in the levels of outputs (inputs) that are similar to those of technical change, in the sense that the scale of production may affect the input-output coefficients in various possible directions. The scale effects, too, may cause proportional ("homothetic changes") or non-proportional ("non-homothetic changes") in outputs and inputs.

Moreover, the effects of technical change and economies of scale on output-input ratios are the two components of total factor productivity change, which is defined as the change in the ratio of an aggregate of outputs to an aggregate of inputs. The existence of such aggregates is a precondition for any productivity measure to have an economic sense.

2.2 Separability conditions of the production possibility frontier

The notion of separability has been independently introduced, essentially in the same terms, by Leontief (1947a, 1947b) and Sono (1945, 1961), respectively, in the field of the producer and consumer theory to analyse aggregation conditions[2]. They showed that two inputs x_i and x_j are separable from a third input x_k if and only if their marginal rates of substitutions are independent from the third input, that is, in our context,

[2] Their definition refers to local separability (requiring differentiability of functions), but can be potentially extended to a concept of global separability. Stigum (1967), Gorman (1968), and Bliss (1975) have given other definitions of separability, which have been used extensively also by Blackorby, Primont and Russell (1978) in consumer theory.

(2.3) $$\frac{\partial}{\partial x_k}\left(\frac{T_i^t}{T_j^t}\right)=0$$

where $T_i^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial x_i]|_{T(\mathbf{y},\mathbf{x})=0}$ and $T_j^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial x_j]|_{T(\mathbf{y},\mathbf{x})=0}$. T_i^t and T_j^t are, respectively, the marginal productivity of inputs x_i and x_j, evaluated on the production efficiency frontier. This condition can be extended to the output space. Inputs are separable from outputs and a function $\mathrm{f}^t(\mathbf{x})$ exists, if and only if (2.3) applies to the derivatives of all ratios between marginal input productivities with respect to the single outputs (with x_k being replaced by y_k). Similarly, outputs are separable from inputs and a function $g^t(\mathbf{y})$ exists if and only if (2.3) applies to the derivatives of all ratios between marginal contributions of outputs with respect to the single inputs (with $T_i^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial y_i]|_{T(\mathbf{y},\mathbf{x})=0}$ and $T_j^t \equiv [\partial T^t(\mathbf{y},\mathbf{x})/\partial y_j]|_{T(\mathbf{y},\mathbf{x})=0}$). Both outputs and inputs are mutually separable and the functions $f^t(\mathbf{x})$ and $g^t(\mathbf{y})$ exist within the transformation function if (2.3) applies, where x_k is replaced by any of the two functions $Y^t(\mathbf{y})$ and $X^t(\mathbf{x})$, while T_i^t and T_j^t are accordingly redefined with respect to the single inputs or outputs, respectively.

The technology of production is *input separable* when the separability conditions hold globally for all inputs and the transformation function can be written as

(2.4) $$T^t(\mathbf{y},\mathbf{x}) \equiv T^t[\mathbf{y}, f^t(\mathbf{x})]$$

where the function $f^t(\mathbf{x})$ is a degree-one homogenous function (that is, if all the elements of **x** are multiplied by a scalar λ, then also the numerical value of the function $f^t(\mathbf{x})$ turns out to be scaled by λ: $f(\lambda\mathbf{x}) = \lambda f(\mathbf{x})$).

Similarly, the technology of production is *output separable* when the separability conditions hold globally for all the outputs, in which case the transformation function can be defined as

(2.5) $$T^t(\mathbf{y},\mathbf{x}) \equiv T^t[g^t(\mathbf{y}),\mathbf{x}]$$

where $g^t(\mathbf{y})$ is a degree-one homogenous function.

We must note, at this point, that separability is a necessary but not a sufficient condition for constructing aggregates of inputs or outputs on the basis of the transformation function (2.4) or (2.5). Strotz (1959) and Gorman (1959) have shown that an aggregate of input quantities exists if, not only these input quantities are separable, but also the resulting quantity index must be a function homogeneous of degree one in its inputs (that is, if all these inputs change proportionally, also the quantity index changes by the same factor of proportionality). Green (1964, p. 25) has called these conditions "homogeneous functional separability".

The technology of production is *input-output separable* when the separability conditions hold globally and simultaneously for all the outputs and the inputs so that the transformation function can be defined as

(2.6) $$T^t(\mathbf{y},\mathbf{x}) \equiv T^t[g^t(\mathbf{y}), f^t(\mathbf{x})]$$

where the internal structure of $f^t(\mathbf{x})$ is independent from that of $g^t(\mathbf{y})$. In this case, the vectors **y** and **x** are said to be mutually *weakly separable* within the internal structure of the transformation function.

A special case of the input-output separable technology implied by (2.6) is the *homothetically separable* technology that was defined by Shephard (1953, p. 43), which leads us, in our context, to the following general form of the transformation function:

(2.7) $$T^t(\mathbf{y},\mathbf{x}) \equiv g^t(\mathbf{y}) - F^t[f^t(\mathbf{x})]$$

$$= {F^t}^{-1}[g^t(\mathbf{y})] - f^t(\mathbf{x}) \qquad \text{(since } T^t(\mathbf{y},\mathbf{x}) = 0\text{)}$$

The transformation function $T^t(\mathbf{y},\mathbf{x}) = 0$ defined by (2.7) is a special case of (2.6), since it is based on the additional hypothesis that $f^t(\mathbf{x})$ is linearly homogeneous. It is worth noting, here, that, although $f^t(\mathbf{x})$ is an order-one homogeneous function, in general the *homothetic function* $F^t[f^t(\mathbf{x})]$ may fall into one of the broader classes of homogeneous and non-homogeneous functions[3]. Furthermore, for the output-input relation to be in "additive" form, the aggregates $g^t(\mathbf{y})$ and $f^t(\mathbf{x})]$ must be either mutually perfectly substitutable or not at all substitutable. In this case, the technology is *additively separable* or *strongly separable* and the transformation function can be written as follows:

(2.8) $$T^t(\mathbf{y},\mathbf{x}) \equiv F^t[g(\mathbf{y}) - f(\mathbf{x})]$$

implying, by the implicit function theorem, $g(\mathbf{y}) = A^t[f(\mathbf{x})]$, which is the form that is usually assumed in the index number approach to productivity measurement. This relation requires, therefore, that the returns to scale are constant. If the separability holds globally as in (2.8), the vectors **y** and **x** are said to be mutually *strongly separable* within the internal structure of the transformation function.

Separability is a necessary but not a sufficient condition for constructing aggregates of outputs or inputs on the basis of the transformation function (2.6), since (non-linear) interaction effects between $f^t(\mathbf{x})$ and $g^t(\mathbf{y})$ could not be decomposable. As we shall see later, changes in output and input aggregates can be clearly measured only if they are strongly separable, as in (2.8).

Following Samuelson (1950, p. 23) and Debreu (1959, p. 38), we can interpret some of the inputs as negative outputs and include them in **y** rather than in **x**. The output aggregating function $g^t(\mathbf{y})$, if it exists, can be interpreted as a net-quantity aggregator. If all the intermediate inputs used in production are considered as negative outputs (so that the vector **x** represents only the inputs of primary factor services), then the net output quantity aggregator $g^t(\mathbf{y})$ has the meaning of a *real value-added function*. A number of contributions have questioned how consistent with the rather stringent conditions for separability and aggregation, given by a global

3 Shephard (1953, pp. 41-50(1970, pp. 30-36) has introduced the concept of the homothetic production function $F^t[(f^t(\mathbf{x})]$, which he defined to be a continuous, positive, monotone, increasing function of a homogeneous function of degree one. A homothetic production function is *non-homogeneous* if it changes by a factor that depends non-linearly on the scale of production when all inputs are multiplied by a positive scalar value λ, *i.e.* $F^t[(f^t(\lambda \cdot \mathbf{x})] = F^t[(\lambda \cdot f^t(\mathbf{x})] = \lambda^{\varphi(\mathbf{y})} \cdot F^t[(f^t(\mathbf{x})]$, whereas it is *homogeneous of degree r* if $F^t[(f^t(\lambda \cdot \mathbf{x})] = F^t[(\lambda \cdot f^t(\mathbf{x})] = \lambda^r \cdot F^t[(f^t(\mathbf{x})]$, with r = 1 meaning that it is *linearly homogeneous* and reflects constant returns to scale.

version of (2.3), are the usual practices of construction of value-added at constant prices as a measure of aggregate net output in national accounts and industrial statistics.

Thus far, we have concentrated our attention on input-output separability. We should note, however, that technical change often affects the whole internal structure of the functional relationship. This might have strong consequences on the measurement, not only of technical change itself, but also of the input- and output-quantity and price indexes. In fact, as we shall see, non-neutral technical change might create measurement problems that are difficult to solve. In the general case, technical change may bring about a change in the functional form of the production transformation function. When this happens, no index number can be constructed unless we can estimate both the old and new functions. The induced changes have to be identified eiher directly or indirectly. If technical change causes non-homothetic shifts in the space of the input quantities for given output levels, then the transformation function would be indexed to the technology in the parameters involved (or in its functional form) and the effects of this change could not, in general, be isolated. These difficulties should be contrasted with the case of Hicks-neutral technical change, where the function $F^t[f^t(\mathbf{x})]$ can be indexed to technical change and can be written as $A^t \cdot F[f^t(\mathbf{x})]$, where A^t is a separable technological variable and the function F(·) is not subject to change.

From the description of the technology given thus far, it is evident that only in some special cases is it possible to construct the following derived index numbers:

(2.9) $$TFP(\mathbf{y}^0,\mathbf{x}^0,\mathbf{y}^1,\mathbf{x}^1) \equiv \frac{g(\mathbf{y}^1)}{f(\mathbf{x}^1)} \Big/ \frac{g(\mathbf{y}^0)}{f(\mathbf{x}^0)} \text{ (if the aggregating functions } g(\mathbf{y}) \text{ and } f(\mathbf{x}) \text{ exist)}$$

$$= TC(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1) \cdot RS(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1)$$

where $TFP(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1)$ is the index number of (average) total factor productivity defined as the ratio between the aggregate of outputs to the aggregate of inputs; $TC(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1)$ and $RS(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1)$ are, respectively, the technical change component and the returns to scale component. In a technology with strong input-output separability and constant economies of scale, $TFP(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1) = TC(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1)$. The returns to scale component could be, in turn, decomposed as follows:

(2.10) $$RS(\mathbf{y}^0,\mathbf{y}^1,\mathbf{x}^0,\mathbf{x}^1) = ESC \cdot IOS$$

where *ESC* represents the effects on TFP of the changes in the degree of economies of scale and *IOS* is a component that captures effects of input-output substitution between $g(\mathbf{y})$ and $f(\mathbf{x})$ *on* the transformation curve.

It is, therefore, evident that the productivity index number can have a real meaning only in the particular cases where outputs and inputs are separable. If they are weakly separable and there are non-constant returns to scale, then a problem emerges when it is of interest to isolate input-output substitution on the transformation curve from economies of scale and technical changes. Input-output substitutions taking place on the transformation frontier are generally induced by changes in relative prices. As it can be seen in the following sections, these effects can be more directly measured, along with other productivity components, using the dual cost, revenue and profit functions.

3. Cost, revenue, and profit functions under different behavioral and technological conditions

> "*As a working hypothesis, it is not obvious that maximal* [*net*] *profits are more meaningful than maximal revenue or minimal cost. If a producer chooses to use an input vector **x**, whatever his reasons under whatever restrictions, it is reasonable to assume as a working hypothesis that he will select for given price vector **p** the output vector **y** which maximizes his revenue, similar to what one does for the input vector **x** in using a minimal cost function for given output vector (scalar) **y**, in making econometric studies. Nothing in either case need be assumed about constant, increasing or decreasing returns to scale*".
>
> Shephard (1974, pp. 204-205)

> "*There are as many alternative approaches to duality theory as there are individuals working in the field of duality theory*".
>
> Lau (1974, p. 178)

3.1 The producer's objectives and behavioral choices

The following analysis will consider a production unit operating as a price taker in the markets of its outputs and inputs. This is not without consequences on the method used because the assumption of price-taking behaviour leads us to the definition of value (profit, revenue, or cost) functions that are always homogeneously linear in prices by construction. Although the case considered here is an extreme stylized example, the results that are obtained could constitute a useful starting point for generalizations that include a variety of other producers' objectives and behaviour in many other economic contexts. We shall follow an approach that is based on the duality between the production possibility frontier and cost, revenue, and profit functions along the line of the pioneering works of Uzawa (1964), McFadden (1966) and Diewert (1971, 1973)[4,5].

The value function of the (static) profit maximization problem for a production unit operating in the long-run equilibrium is given by:

$$\Pi^t(\mathbf{p},\mathbf{w}) \equiv \max_{\mathbf{y},\mathbf{x}} \left\{ \mathbf{p}\cdot\mathbf{y} - \mathbf{w}\cdot\mathbf{x} : (\mathbf{x},\mathbf{y}) \in S^t \right\} \tag{3.1}$$

where $\mathbf{p} = [p_1, p_2, \ldots p_M]$ is a row vector of M output prices, $\mathbf{w} = [p_1, \mathrm{p}_2, \ldots p_N]$ is a row vector of N input prices, and $\Pi^t(\mathbf{p},\mathbf{w})$ is the long-run equilibrium profit function at time t. With constant returns to scale, an *indeterminacy problem* arises in the (long-run) partial equilibrium context. In this case, profit maximization may turn out to be a circular problem, since the optimal scale of

---

4 Lau (1974, pp. 178-80), for example, has classified a variety of different approaches to duality theory into three broad groups. The first group is based on the "conjugacy correspondence" developed by Fenchel (1949, 1953) and extended by Rockafellar (1970), which includes the pioneering work of Hotelling (1932) on the normalized profit function implicitly based on the classical Legendre transformation (this approach was also followed by Samuelson, 1953 and Lau, 1969). The second group of approaches is based on symmetric duality between gauge functions, or distance functions, or polar cones of convex sets (Shephard, 1953 provided a pioneering contribution employing the concept of distance functions, followed by similar approaches proposed by Gorman, 1968, McFadden, 1973, 1978a, 1978b, and Jacobsen, 1970, 1972). The third group of approaches is based on the duality between the set of production possibilities and its support function (the precursors of this line of analysis in production and consumption economics have been the cited works of Uzawa, 1964, McFadden, 1966, 1978a, 1978b and Diewert, 1971, 1973).

5 We deliberately shall not make explicit use of the concept of distance functions since we want to remain in the field of economically defined optimal cost, revenue and profit functions (these, however, can be interpreted as distance functions themselves in the space of their argument variables).

production and that of inputs could be at any positive level. This particular case, which - as we shall see - is important for the construction of canonical index numbers, can find a solution only in the general equilibrium of the economy. At any rate, there is no loss of generalization if the levels of all output and input quantities are normalized by one elementary quantity, say the i^{th} output, which represents, in this way, the scale of the whole production activity. All the solutions for quantities would then be in relative terms rather than in absolute units of measure, while we can assume that general-equilibrium market forces determine the scale of the production activity.

If the producer using a constant-returns-to-scale technology supplies outputs in perfectly competitive markets, then there are no profits, that is $\Pi^t(\mathbf{p},\mathbf{w}) = 0$. There is a dual relationship between the profit function $\Pi^t(\mathbf{p},\mathbf{w})$ and the transformation function $\mathrm{T}^t(\mathbf{y},\mathbf{x})=0$, which is the contour of the production possibility set S^t, since

(3.2) $$E^t \equiv \{(\mathbf{y},\mathbf{x}) : T^t(\mathbf{y},\mathbf{x}) = 0\}$$

$$\equiv \{(\mathbf{y},\mathbf{x}) : \mathbf{p}\cdot\mathbf{y} - \mathbf{w}\cdot\mathbf{x} = \Pi^t(\mathbf{p},\mathbf{w}), \forall\mathbf{p}, \forall\mathbf{w}\}$$ (using (2.2)

with $\Pi^t(\mathbf{p},\mathbf{w})$ defined by (3.1).

The profit function completely characterizes the technology, in the sense that it contains all the information needed to describe the production possibility frontier $T^t(\mathbf{y},\mathbf{x}) = 0$. As McFadden (1978, p. 92) has clarified, at its given value, the profit function is itself a *price possibility frontier* or transformation function defined in the space of producer's output and input prices. Moreover, both $T^t(\mathbf{y},\mathbf{x})$ and $\Pi^t(\mathbf{p},\mathbf{w})$ are linearly homogeneous in their arguments. They can be considered as aggregation functions of outputs, inputs, and technical progress effects taken together in the quantity and price spaces, respectively.

The maximization problem that defines the profit function in (3.1) can be decomposed in different stages of optimization. Let us consider the following optimized conditional revenue (or benefit) and cost functions:

(3.3) $$\mathrm{C}^t(\mathbf{w},\mathbf{y}) \equiv \min_{\mathbf{x}} \{\mathbf{w}\cdot\mathbf{x} : (\mathbf{y},\mathbf{x}) \in E^t\}$$

$$\text{(3.4)} \qquad \mathrm{R}^{t}(\mathbf{p},\mathbf{x}) \equiv \max_{\mathbf{y}} \left\{\mathbf{p}\cdot\mathbf{y} : (\mathbf{y},\mathbf{x}) \in E^{t}\right\}$$

These can be viewed as two different or separated optimization problems.

A simultaneous optimal solution leads us to the long-run equilibrium profit function defined by (3.1):

$$\text{(3.5)} \qquad \begin{aligned} \Pi^{t}(\mathbf{p},\mathbf{w}) &\equiv \max_{\mathbf{y},\mathbf{x}} \left\{\mathbf{p}\cdot\mathbf{y} - \mathbf{w}\cdot\mathbf{x} : (\mathbf{x},\mathbf{y}) \in E^{t}\right\} \\ &= \max_{\mathbf{y}} \left\{\mathbf{p}\cdot\mathbf{y} - \mathrm{C}^{t}(\mathbf{w},\mathbf{y})\right\} && \text{using (2.10)} \\ &= \max_{\mathbf{x},} \left\{\mathrm{R}(\mathbf{p},\mathbf{x}) - \mathbf{w}\cdot\mathbf{x}\right\} && \text{using (2.11)} \end{aligned}$$

In long-run equilibrium we have

$$\text{(3.6)} \qquad \Pi^{t}(\mathbf{p},\mathbf{w}) = \mathrm{R}(\mathbf{p},\mathbf{x}) - \mathrm{C}(\mathbf{w},\mathbf{y}) \qquad \text{using (2.13)}$$

since the levels of $\mathbf{x}$ and $\mathbf{y}$ are both consistent with the optimization problem (3.5). The long-run equilibrium profit function can, therefore, be seen as a revenue-cost or benefit-cost function.

Whether the cost-minimizing, revenue-maximizing, or "long-run equilibrium" profit maximizing solutions are to be considered as the closest to the producer's behavior at the particular time t depends on the specific conditions of the examined case. In general, the (conditional) short-run revenue and cost functions exhibit, in the short run, *decreasing* returns to scale in the *variable* outputs and/or inputs at given *fixed* reference levels of other outputs and/or inputs, also in cases where the economies of scale are constant or even increasing when all outputs and inputs are allowed to vary.

To encompass all these solutions in a unified approach, McFadden (1978, p. 66) has proposed the following *general restricted profit function* that includes, as particular cases, the cost (with negative sign), revenue and profit functions:

$$\text{(3.7)} \qquad \Pi^{McF^{t}}(\mathbf{q},\mathbf{k}) \equiv \max_{\mathbf{z}} \left\{\mathbf{q}\cdot\mathbf{z} : (\mathbf{z},\mathbf{k}) \in S^{t}\right\}$$

where $\mathbf{q}$ is a (row) vector of prices of *variable* outputs and/or (negative) variable inputs, the quantities of these outputs and (negative) inputs are the elements of a (column) vector $\mathbf{z}$, $\mathbf{k}$ is a (column) vector of quantities of *fixed* outputs and/or (negative) inputs. McFadden (1978, p. 61) called a vector of (positive) outputs and (negative) inputs a vector of *net outputs* or *netputs.*

By appropriately defining the *variable* output and/or input quantities as components of $\mathbf{z}$, the *conditional* output and/or input quantities as components of $\mathbf{k}$, and the prices of the elements of $\mathbf{z}$ as components of $\mathbf{q}$, the cost, revenue and long-run equilibrium profit functions can be obtained from McFadden's restricted profit function defined by (3.7) as follows

(*I*) if $\mathbf{z} \equiv -\mathbf{x}$, $\mathbf{q} \equiv \mathbf{w}$, and $\mathbf{k} \equiv \mathbf{y}$, then McFadden general restricted profit function represents a (negative) minimum cost function: $\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv -\mathrm{C}^t(\mathbf{w},\mathbf{y})$ (using (3.3) and noting that the definition of the cost function assigns a positive algebraic sign to inputs and a negative sign to the outputs that are treated as inputs);

(*II*) if $\mathbf{z} \equiv [\mathbf{y}\ -\mathbf{x}_1]'$, $\mathbf{q} \equiv [\mathbf{p}\ \mathbf{w}_1]$, and $\mathbf{k} \equiv -\bar{\mathbf{x}}_2$, with the input quantities being partitioned into *variable* inputs $\mathbf{x}_1$ and *fixed* inputs $\bar{\mathbf{x}}_2$, and their prices $\mathbf{w}$ being partitioned accordingly, then McFadden's general restricted profit function represents a revenue function: $\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \mathrm{R}^t([\mathbf{p} \vdots \mathbf{w}_1], \bar{\mathbf{x}}_2)$ (using (3.4));

(*III*) if $\mathbf{z} \equiv [\mathbf{y}\ (-\mathbf{x})]'$ and $\mathbf{k} = \mathbf{0}$, then McFadden's general restricted profit function *ceases to be restricted* and represents a long-run equilibrium profit function: $\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \Pi^t(\mathbf{p},\mathbf{w})$ (using (3.5)).

Duality between the producer's transformation function in the space of quantities and the cost, revenue and profit functions in the space of prices can be studied in a compact form using McFadden's general restricted profit function.

Note that McFadden's profit function collapses to particular cases that, in a sense, are not symmetric: this function actually refers to *net* profits only in the unrestricted case obtained with the translations (*III*). In the other cases, this form defines optimal values that are *not* net profits: the translations (*I*) and (*II*) lead us to cost and gross revenue functions, respectively.

3.2 The general conditional net-profit function

We can separate the long-run equilibrium profit maximization problem in a two- stage profit maximization procedures. To this end, we could consider at least two different views that have been historically developed in the economic theory of production. The first corresponds to the Marshallian view of the producer's partial-equilibrium problem, which consists in maximizing net profits by choosing the levels of the outputs, with no explicit attention for cost-minimizing adjustments in the techniques of production. In this context, the (conditional) profit function corresponds is

$$\Pi^{M\,t}(\mathbf{p},\mathbf{w};\mathbf{x}) = \mathrm{R}^t(\mathbf{p},\mathbf{x}) - \mathbf{w}\cdot\mathbf{x} \tag{3.8}$$

where $\Pi^{M\,t}(\mathbf{p},\mathbf{w};\mathbf{x})$ can be defined as the partial-equilibrium "Marshallian" net profit function, which is conditional to the vector of inputs $\mathbf{x}$.

The "Marshallian" net profit function (3.8) may have a value between those of the short-run "Marshallian" net-profit function"[6] (where the inputs are fixed at a given level $\bar{\mathbf{x}}$) and the long-run equilibrium profit function, that is

$$\Pi^{M\,t}(\mathbf{p},\mathbf{w};\bar{\mathbf{x}}) \le \Pi^{M\,t}(\mathbf{p},\mathbf{w};\mathbf{x}) \le \Pi^t(\mathbf{p},\mathbf{w}) \tag{3.9}$$

The second view stems from the Paretian theory of the firm, which is described to maximize profits through cost-minimizing adjustments in input combinations at given levels of outputs. Output levels, in turn, are viewed as determined by the general equilibrium of the economy through the interaction of all markets. In this context, the (conditional) profit function is

[6] The "short-run (or restricted) net profit function" defined here has the meaning of a value function referring to profits *net* of total costs. It should be contrasted with the widely used "short-run (or restricted) (gross) profit function", which is instead *gross* of the costs of fixed inputs and corresponds more closely to the concept of revenue function.

(3.10) $$\Pi^{P^t}(\mathbf{p},\mathbf{w};\mathbf{y}) = \mathbf{p}\cdot\mathbf{y} - C^t(\mathbf{w},\mathbf{y})$$

where we define $\Pi^{P^t}(\mathbf{p},\mathbf{w};\mathbf{y})$ as the partial-equilibrium "Paretian" net-profit function, which is conditional to the level of the outputs $\mathbf{y}$.

The "Paretian" net-profit function (3.10) may have a value between those of the short-run "Paretian" net profit function" (where the outputs are fixed at a given level $\overline{\mathbf{y}}$) and the long-run equilibrium profit function, that is

(3.11) $$\Pi^{P^t}(\mathbf{p},\mathbf{w};\overline{\mathbf{y}}) \le \Pi^{P^t}(\mathbf{p},\mathbf{w};\mathbf{y}) \le \Pi^t(\mathbf{p},\mathbf{w})$$

The two concepts of "Marshallian" and "Paretian" net profit functions that we have just defined do not appear to belong to the general concept of McFadden's (1978) "restricted profit function" represented by (3.7), but are, instead, more general cases.

3.3 The envelope theorem

The relationship between the general conditional net-profit function with profit, revenue, and cost functions in the short-run disequilibrium is described by the following theorem:

THEOREM 3.1. Envelope theorem. *The long-run equilibrium profit function* $\Pi^t(\mathbf{q},\mathbf{p}_k)$ *is the higher envelope of the general conditional net-profit function* $\Pi^{McF^t}(\mathbf{q},\overline{\mathbf{k}})$ *in the space of output and input prices, with some points in common, that is*

(3.12) $$\Pi^{McF^t}(\mathbf{q},\overline{\mathbf{k}}) \le \Pi^t(\mathbf{q},\mathbf{p}_k)$$

where $\overline{\mathbf{k}} = \mathbf{k}^t(\overline{\mathbf{q}},\overline{\mathbf{p}}_k)$, and the prices $(\overline{\mathbf{q}},\overline{\mathbf{p}}_k)$ are those that make compatible the levels of $\overline{\mathbf{k}}$ with the long-run equilibrium demand or supply functions.

In terms of the short-run equilibrium Marshallian and Paretian net-profit functions using the respective translations given by (*I*), (*II*), or (*III*), the results of Theorem (3.1) are the following:

(3.13)
$$\Pi^{M^t}(\overline{\mathbf{p}},\overline{\mathbf{w}};\overline{\mathbf{x}})) = \Pi^t(\overline{\mathbf{p}},\overline{\mathbf{w}})$$

$$\mathrm{R}^t(\overline{\mathbf{p}},\overline{\mathbf{x}}) - \overline{\mathbf{w}}\cdot\overline{\mathbf{x}} = \mathrm{R}^t(\overline{\mathbf{p}},\overline{\mathbf{x}}) - \mathrm{C}^t(\overline{\mathbf{w}},\overline{\mathbf{y}})$$ (using (3.6), (3.8), and Hotelling's lemma)

(3.14)
$$\Pi^{P^t}(\overline{\mathbf{p}},\overline{\mathbf{w}};\overline{\mathbf{y}})) = \Pi^t(\overline{\mathbf{p}},\overline{\mathbf{w}})$$

$$\overline{\mathbf{p}}\cdot\overline{\mathbf{y}} - \mathrm{C}^t(\overline{\mathbf{w}},\overline{\mathbf{y}}) = \mathrm{R}^t(\overline{\mathbf{p}},\overline{\mathbf{x}}) - \mathrm{C}^t(\overline{\mathbf{w}},\overline{\mathbf{y}})$$ (using (3.6), (3.10), and Hotelling's

where $\overline{\mathbf{x}} = \mathbf{x}(\overline{\mathbf{p}},\overline{\mathbf{w}})$ and $\overline{\mathbf{y}} = \mathbf{y}(\overline{\mathbf{p}},\overline{\mathbf{w}})$, and

(3.15)
$$\Pi^{M^t}(\mathbf{p},\mathbf{w};\overline{\mathbf{x}}) \le \Pi^t(\mathbf{p},\mathbf{w})$$

that is
$$R^t(\mathbf{p},\overline{\mathbf{x}}) - \mathbf{w}\cdot\overline{\mathbf{x}} \le R^t(\mathbf{p},\mathbf{x}(\mathbf{p},\mathbf{w})) - C^t(\mathbf{w},\mathbf{y}(\mathbf{p},\mathbf{w}))$$

(3.16)
$$\Pi^{P^t}(\mathbf{p},\mathbf{w};\overline{\mathbf{y}}) \le \Pi^t(\mathbf{p},\mathbf{w})$$

that is
$$\mathbf{p}\cdot\overline{\mathbf{y}} - C^t(\mathbf{w},\overline{\mathbf{y}}) \le R^t(\mathbf{p},\mathbf{x}(\mathbf{p},\mathbf{w})) - C^t(\mathbf{w},\mathbf{y}(\mathbf{p},\mathbf{w}))$$

The relations (3.15) and (3.16) reveal that $\Pi^t(\mathbf{p},\mathbf{w})$ is the higher envelope of $\Pi^{M^t}(\mathbf{p},\mathbf{w};\overline{\mathbf{x}})$ and $\Pi^{P^t}(\mathbf{p},\mathbf{w};\overline{\mathbf{y}})$, respectively, in the space of output and input prices. Disequations (3.15) and (3.16) imply that $\Pi^t(\mathbf{p},\mathbf{w})$ is an upper bound. Equation (3.13) and (3.14) reveal that the long-run equiliprium profit function and the conditional Marshallian and Paretian profit functions share some points in common. These common points are in correspondence with those input and output bundles that maximize profits in the long-run equilibrium.

The envelope relationship between the Marshallian and Paretian net profit functions, on one part, and the full long-run equilibrium profit function, on the other, holds since the expressions (3.13) and (3.14) imply that each of the two first net profit functions never intersect the long-run equilibrium profit function.

3.4 Separability conditions of the cost, revenue, and profit functions

The separability conditions of outputs and inputs on the transformation function in the space of quantities can be translated in terms of separability conditions on cost, revenue, and net-profit functions. Berndt and Christensen (1973) have shown that the Leontief (1947) - Sono (1961) condition given by (2.3) can be translated in terms of the Allen-Uzawa partial elesticities of substitution. If (2.3) is true, then

$$\frac{\partial}{\partial w_k}\left(\frac{C_i^t}{C_j^t}\right) = 0 \tag{3.17}$$

where $C_i^t \equiv \partial C^t(\mathbf{w},\mathbf{y})/\partial w_i$, $C_j^t \equiv \partial C^t(\mathbf{w},\mathbf{y})/\partial w_j$, and w_k is the price of input x_k. By Shephard's lemma, $C_i^t = x_i$ and $C_j^t = x_j$. Therefore, (2.49) is equivalent to

$$\frac{\partial}{\partial w_k}\left(\frac{x_i}{x_j}\right) = 0 \tag{3.18}$$

which means that the price of the k^{th} does not affect the ratio between the two inputs x_i and x_j.

The condition (3.17) implies

$$C_{ik}^t C_j^t = C_{jk}^t C_i^t \tag{3.19}$$

where $C_{ik}^t \equiv \partial C_i^t(\mathbf{w},\mathbf{y})/\partial w_k$. Multiplying both sides of (3.16) by $C^t/C_k^t \cdot C_i^t \cdot C_j^t$ yields

$$\frac{C^t C_{ik}^t}{C_i^t C_k^t} = \frac{C^t C_{jk}^t}{C_j^t C_k^t} \tag{3.20}$$

where $C^t \equiv C^t(\mathbf{w},\mathbf{y})$. In other terms, the functional separability of inputs x_i and x_j from the third input x_k implies $\sigma_{ik} = \sigma_{jk}$, where $\sigma_{ik} \equiv C^t C_{ik}^t / C_i^t C_k^t$ and $\sigma_{jk} \equiv C^t C_{jk}^t / C_j^t C_k^t$ are the

Allen-Uzawa partial elasticities of substitution between the pairs of inputs x_i and x_k and between the pairs of inputs x_j and x_k.

Consider the following input and/or output separability cases:

3.4.1 Input separability

In the case the separability condition is referred to a single pair of inputs x_i and x_j with respect to a single output y_k, (3.17) is replaced by

(3.21) $$\frac{\partial}{\partial p_k}\left(\frac{C_i^t}{C_j^t}\right)=0$$

which is equivalent to

(3.22) $$\frac{\partial}{\partial p_k}\left(\frac{x_i}{x_j}\right)=0$$

Similar conditions can be derived for the conditional revenue functions and the Marshallian and Paretian conditional net-profit functions. In terms of the general conditional net-profit function, the separability condition corresponding to (3.18) becomes:

(3.23) $$\frac{\partial}{\partial q_k}\left(\frac{\Pi_i^{McF\,t}}{\Pi_j^{McF\,t}}\right)=0$$

where $\Pi_i^{McF\,t} \equiv \partial \Pi^{McF\,t}(\mathbf{q},\mathbf{k})/\partial q_i$ and $\Pi_J^{McF\,t} \equiv \partial \Pi^{McF\,t}(\mathbf{q},\mathbf{k})/\partial q_j$.

If the technology is globally input separable as in (2.4), so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x})=0$ can be written as $T^t[\mathbf{y}, f^t(\mathbf{x})]=0$, where the separability conditions stated above are valid for any pair of inputs in **x** with respect to every single output in **y**, then the dual profit, cost, and revenue functions can be written as follows:

(3.24) $$\Pi^{McF\,t}(\mathbf{q},\mathbf{k}) \equiv \Pi^{McF\,t}(\mathbf{q},K(\mathbf{k}))$$

(3.25) $$\mathit{\Pi}^t(\mathbf{p},\mathbf{w}) \equiv \mathit{\Pi}^t[\mathbf{p},\omega(\mathbf{w})]$$

(3.26) $$C^t(\mathbf{w},\mathbf{y}) \equiv C^t[\omega(\mathbf{w}),\mathbf{y}]$$

(3.27) $$R^t(\mathbf{p},\mathbf{x}) \equiv R^t(\mathbf{p}, f(\mathbf{x}))$$

3.4.2 Output separability

If the technology is globally output separable as in (2.5), so that so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x}) = 0$ can be rewritten as $T^t[g^t(\mathbf{y}),\mathbf{x}] = 0,$ then the dual profit, cost, and revenue functions can be written as (see also Lau, 1978, p. 175):

(3.28) $$\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \Pi^{McF^t}[\phi(\mathbf{q}),\mathbf{k}]$$

(3.29) $$\Pi^t(\mathbf{p},\mathbf{w}) \equiv \Pi^t[\varphi(\mathbf{p}),\mathbf{w}]$$

(3.30) $$C^t(\mathbf{w},\mathbf{y}) \equiv C^t[\mathbf{w}, g(\mathbf{y})]$$

(3.31) $$R^t(\mathbf{p},\mathbf{x}) \equiv R^t[\varphi(\mathbf{p}),\mathbf{x}]$$

3.4.3 Input-output (weak) separability

If the technology is globally input-output separable as in (2.6), so that so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x}) = 0$ can be rewritten as $T^t[g^t(\mathbf{y}),\mathbf{x}] = 0,$ then the dual profit, cost, and revenue functions can be rewritten as (see also McFadden, 1978, p. 58):

(3.32) $$\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \Pi^{McF^t}[\phi(\mathbf{q}), K(\mathbf{k})]$$

(3.33) $$\Pi^t(\mathbf{p},\mathbf{w}) \equiv \Pi^t[\varphi(\mathbf{p}),\omega(\mathbf{w})]$$

(3.34) $$C^t(\mathbf{w},\mathbf{y}) \equiv C^t[\omega(\mathbf{w}), g(\mathbf{y})]$$

(3.35) $$R^t(\mathbf{p},\mathbf{x}) \equiv R^t[\varphi(\mathbf{p}), f(\mathbf{x})]$$

3.4.4 Homothetic input-output separability

If the technology is homothetically separable as in (2.7), so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x})=0$ can be rewritten as $T^t(\mathbf{y},\mathbf{x}) \equiv g^t(\mathbf{y}) - F^t[f^t(\mathbf{x})] = 0$ or, equivalently, $T^t(\mathbf{y},\mathbf{x}) \equiv {F^t}^{-1}[g^t(\mathbf{y})] - f^t(\mathbf{x}) = 0$ where f^t being linearly homogenous functions and f^t being a homothetic function, then the dual profit, profit, and revenue functions can be rewritten as follows (see Shephard, 1953, 1970 for the case of the cost function, and McFadden, 1978, p. 58, Denny and Pinto, 1978, p. 253, and Lau, 1978, pp. 159-160):

(3.36) $$\Pi^{McF^t}(\mathbf{q},\mathbf{k}) \equiv \phi^t(\mathbf{q}) \cdot F^t[K^t(\mathbf{k})]$$

(3.37) $$\Pi^t(\mathbf{p},\mathbf{w}) \equiv [\pi^t(\mathbf{p}) - \pi^t(\mathbf{w})]$$

(3.38) $$C^t(\mathbf{w},\mathbf{y}) \equiv \omega^t(\mathbf{w}) \cdot {F^t}^{-1}[g(\mathbf{y})]$$

(3.39) $$R^t(\mathbf{p},\mathbf{x}) \equiv \varphi^t(\mathbf{p}) \cdot F^t[f(\mathbf{x})]$$

where the function $F^t(\cdot)$ was defined in (2.7). Note that $f(\mathbf{x}) = {F^t}^{-1}[g^t(\mathbf{y})]$ and $g^t(\mathbf{y}) = F^t[f^t(\mathbf{x})]$. Therefore, as it is expected, $C^t(\mathbf{w},\mathbf{y}) = \omega^t(\mathbf{w}) \cdot f^t(\mathbf{x})$ and $R^t(\mathbf{p},\mathbf{x}) = \varphi^t(\mathbf{p}) \cdot g^t(\mathbf{y})$.

In the homothetic case, not only the internal structure, but also the levels of $\varphi^t(\mathbf{p})$ and $f^t(\mathbf{x})$ are mutually independent, as well as those of $\omega^t(\mathbf{w})$ and $g^t(\mathbf{y})$. Homothetic separability can be considered as a special case of input-output separability.

If the technology is characterized by additive (strong) input-output separability as in (2.8), so that the input-output transformation function $T^t(\mathbf{y},\mathbf{x})=0$ can be rewritten as $T^t(\mathbf{y},\mathbf{x})\equiv F^t[g(\mathbf{y})-f(\mathbf{x})]=0$ (implying, by the implicit function theorem, $g(\mathbf{y})=A^t[f(\mathbf{x})]$, then the dual profit, cost, and revenue functions can be written in a way directly derivable from the homothetic separability case, where the function $F^t[f^t(\mathbf{x})]$ reduces to $f^t(\mathbf{x})$, and $F^{t^{-1}}[g(\mathbf{y})]$ to $g(\mathbf{y})$.

3.4.5 Separability of technical change

Separability of technical change from outputs and/or inputs can be studied in a way analogous to that for input and output separability. If technical change of the transformation function is Hicks-neutral, then it is homothetically separable from input and output changes. The dual cost and revenue can be written as follows:

(3.40) $$C^t(\mathbf{w},\mathbf{y})\equiv\omega(\mathbf{w})\cdot A^{t^{-1}}\cdot F^{-1}[g^t(\mathbf{y})]$$

(3.41) $$R^t(\mathbf{p},\mathbf{x})\equiv\varphi(\mathbf{p})\cdot A^t\cdot F[f(\mathbf{x})]$$

Note that $f(\mathbf{x})=A^{t^{-1}}\cdot F^{-1}[g(\mathbf{y})]$ and $g(\mathbf{y})=A^t\cdot F[f(\mathbf{x})]$. Therefore, $C^t(\mathbf{w},\mathbf{y}))$ $=\omega(\mathbf{w})\cdot A^{t^{-1}}\cdot F^{-1}[g(\mathbf{y})]=\omega(\mathbf{w})\cdot f(\mathbf{x})$ and $R^t(\mathbf{p},\mathbf{x})=\varphi(\mathbf{p})\cdot A^t\cdot F[f(\mathbf{x})=\varphi(\mathbf{p})\cdot g(\mathbf{y})$. In the case of constant returns to scale and perfect competition, the long-run equilibrium yields $\frac{R^t(\mathbf{p},\mathbf{x})}{C^t(\mathbf{w},\mathbf{y})}=\frac{\varphi(\mathbf{p})\cdot g(\mathbf{y})}{\omega(\mathbf{w})\cdot f(\mathbf{x})}=1$ or, equivalently, $\frac{g(\mathbf{y})}{f(\mathbf{x})}=\frac{\omega(\mathbf{w})}{\varphi(\mathbf{p})}=TFP.$

4. Production-theoretic index numbers

One strand of the index number theory is the economic approach that Frisch (1936) defines as "functional approach" and distinguishes from the "statistical approach", while Allen (1949), Samuelson and Swami (1974, p. 573) and Diewert (1981) identify it as the "economic theory of index numbers". The economic approach to index numbers has highlighted the interconnection between quantity and price index numbers and the economic theory of consumer preferences and the economic theory of production.

The purpose of index numbers is to convert univocally the observed changes in the elements of a vector into changes in a single scalar. This mathematical operation is possible only under special conditions. Additional information concerning weights of aggregation may be useful, and in fact, index numbers have been traditionally defined mostly as "weighted" formulas. The economic theory of production provides us with indications on how to take into account of a number of additional informative elements, such as the general characteristics of the technology in use, market conditions and determination of prices and quantities, the producers' optimizing behavior, which may be useful in interpreting the alternative available index number formulas. If, in addition, the functional forms of output supply and input demand functions are “known” and their parameters are estimated, then producers’ actual or theoretical economic choices will be simulated and economic index numbers could be constructed explicitly. In the following discussion, we do not enter the territory of specification and estimation of behavioral functions and remain confined only in the discussion of traditional index numbers that can be related implicitly to economic index numbers. Therefore, we shall start from the conceptualization of the latter and discuss the limits and the conditions under which we could proceed in applying index number formulas.

In extreme synthesis, the economic index number problem consists in asking how can we find functional forms that aggregate price and quantity changes in outputs and inputs while remaining *invariant* with respect to the weights used. In other words, how can we find functional forms for index numbers that take into account the producer's indifferent curve and do not vary if the producer makes choices that bring about changes while remaining on this curve. This index number problem becomes even more complicated if the producer's indifference curve changes simultaneously along with changes on the curve. In the general case, possible non-homothetic (non-proportional or biased) changes in outputs (inputs) for given inputs (outputs) add the complication up to the level that the index number problem may be unresolvable.

In order to analyse these problems in a most complete way, let us examine all the production-theoretic index numbers of outputs and inputs prices and the derived formulas concerning productivity, economies of scale, and technical changes which may be constructed using revenue, cost, and profit functions. As we have extensively seen above, these functions imply that the producer maximizes profits by choosing output and/or input quantities for given prices. These functions, therefore, incorporate possible price-induced substitution or transformation effects between outputs and/or inputs, given a certain technology. Technical inefficiencies are ruled out in the present analytical context by assuming that the producer's choices are always technically optimized (he always operates on his indifference curve). However, allocative inefficiencies can arise by using input/output combinations *on* the producer's indifference curve that may result to be non-optimal from the economic point of view *at certain relative prices*. Allocative inefficiencies are to be interpreted as price components of the value function at hand (cost, revenue, or profit function) since, when remaining on his indifference curve, the producer do not perceive any effect from such allocative inefficiencies on the quantity side.

Appendix A offers a list of candidate economic index numbers of quantities and prices for outputs and inputs that may be constructed using the revenue and cost functions. A separate discussion will follow, which concerns the index numbers that can be constructed using profit functions. Candidate economic index numbers are also derived for productivity, scale economies, and technical changes. All these economic index numbers are constructed either directly or implicitly as ratios based on revenue or cost functions. To our knowledge, many of the candidate index numbers listed in Appendix A are new or have been used very seldom in the economic literature. We briefly concentrate on those that are more widely used or particularly relevant for our discussion and leave the interpretation of the others to the reader.

4.1 Economic index numbers of output quantities and prices

4.1.1. The Fisher-Shell (direct) output-price index number

Fisher and Shell (1972, Essay II, pp.50-59) examined the *revenue-based* output-price indexes defined by (A2) in our Appendix A, where the reference variables are those observed at the base period (with $r = 0$), or at the current period (with $r = t$). They recommended the Laspeyres-weighted revenue-based index number as the most appropriate output-price index, corresponding to our formula (A2) with $r = 0$, with the reference input quantities and the reference technology being set equal to those prevailing during the base period. Diewert (1983, p. 1056) called this particular direct output-price index the *Laspeyres Fisher-Shell output price* *index*, whereas he called the Paasche-weighted index, corresponding to our formula A2 with $r = t$, the *Paasche Fisher-Shell output price index*.

4.1.2. The Fisher-Shell (implicit) output-quantity index number

The output quantity index number has been defined implicitly by Fisher and Shell (1972, Essay II, p. 58) . They have recommended the use of the implicit *revenue-based* index number formula (A5) reported in our Appendix A by setting $r = 0$. Specifically, they obtain implicitly the output-quantity index as the nominal-revenue index deflated by the Laspeyres-weighted Fisher-Shell direct output-price index P_{FS}. Therefore, the resultin implicit output-quantity index is Paasche-weighted. It is worth noting the preference of these authors for this particular output-price index number and, for this reason, we call it *Fisher-Shell implicit output-quantity index* and denote it as Y_{FS}[7]

4.1.3. The Samuelson-Swamy-Sato (direct) output-quantity index number

Samuelson and Swamy (1974, p. 588) and Sato (1976, p. 438) have defined the *revenue-based* output (or input) quantity index numbers, in a form that can be viewed as a special case of (A4) (or A8). No reference is made to an indexed technology so that the input-quantity index number "coincides" with the output-quantity index number. Diewert (1983, p. 1063) has added an index for the technology, thus obtaining a more general form corresponding to (A4) in our Appendix A. He called the resulting index number the *Samuelson-Swamy-Sato output index*, where the effects of technical change combine with those of input changes to produce an output-change index[8].

[7] Diewert (1983, p. 1063) generalized the Fisher-Shell (1972, Essay II, pp.50-59) definition of the implicit output-quantity index to include that obtained using an output deflator referenced to an undefined period *r*. He called the general formula (A5) of our Appendix A the *Fisher-Shell output index.* However, Diewert (1983, p. 1105, fn. 8) recognizes that "[o]ur definition is perhaps slightly different from the Fisher-Shell definition, but it captures the same idea". In this view, it is possible to term "Paasche-type Fisher-Shell (implicit) output-quantity index number" that is obtained by deflating the nominal revenue index using a Laspeyers-type *revenue-based* output-price index and "Laspeyres-type Fisher-Shell (implicit) output-quantity index number" that is obtained by deflating the nominal revenue index by means of a Paasche-type *revenue-based* output-price index.

[8] Diewert (1983, p. 1105, fn. 9) recognizes: "Our definition is somewhat more general than that of Samuelson and Swamy who assume only one input and no technological change. Sato (1976, p. 438) has the many input definition without technical change. Both sets of authors noted the analogy of the output

4.1.4 Alternative economic output-quantity index numbers

Direct and implicit output-quantity index numbers can also be defined using the cost function, as in the candidate formulas (A19) and (A20). For given reference input prices and technology, candidates for the output quantity index numbers can be derived directly from the total cost function, under conditions of strong separability.

4.2 Economic index numbers of input quantities and prices

4.2.1 The Fisher-Shell (direct) input-price index number

In the context of consumer theory, Fisher and Shell (1972, Essay I) have recommended the use of a direct input-price index, which is constructed as a Paasche-weighted *cost-based* index number, conditional to reference variables observed at the current period.

4.2.2 The Fisher-Shell (implicit) input-quantity index number

The recommended input-quantity index number corresponds to our formula (A24) with $r = 1$, and is obtained implictly by deflating the nominal-cost index number by means of the direct Paasche-weighted *cost-based* input-price index. The resulting implicit input-quantity index is, therefore, Laspeyres-weighted.

4.2.3 The Samuelson-Swamy-Sato-type (direct) input-quantity index number

By analogy with the output-quantity index number defined by Samuelson and Swamy (1974, p. 588) and Sato (1976, p. 438), we could call the *revenue-based* formula given by (A8) the *Samuelson-Swamy-Sato-type direct input-quantity index number*.

---

index defined by (23) [our (A4)] to the Allen (1949) utility or real income index in the context of consumer theory".

4.3 Economic index numbers of productivity, scale economies, and technical change

Index numbers of productivity, scale economies, and technical change could be constructed using the output- and input-quantity index numbers or the input- and output-price index numbers. Appendix A contains the revenue- and cost-based formulas that could be considered as alternative candidate index numbers.

4.4 The theory of bounds in the economic approach

We concentrate on the cost function and leave to the reader a parallel symmetric reasoning on the revenue function. The theory of the input price and quantity indexes in production economics is largely isomorphic to the more widely studied theory of the cost-of-living, where the so-called Konüs "true" index number of cost of living and the Allen "true" index number of aggregate real inputs are constructed by using expenditure or cost functions[9]. Similarly, in the context of the production activity, the index numbers of aggregate input prices are theoretically based on the use of the cost function. As the theory of the cost of living has clearly shown, there is no unique way of accounting for the intertemporal or interspacial cost changes. Alternative decomposition procedures are equally possible, among which are the following[10].

(4.1) $$C^1(\mathbf{w}^1,\mathbf{y}^1)/C^0(\mathbf{w}^0,\mathbf{y}^0) = W_{L-K} \cdot X_{P-A}$$

(4.2) $$C^1(\mathbf{w}^1,\mathbf{y}^1)/C^0(\mathbf{w}^0,\mathbf{y}^0) = W_{P-K} \cdot X_{L-A}$$

---

[9] The basic contributions to the theory of aggregate input-price and input-quantity indexes in the context of production activities were given by Muellbauer (1972), Blackorby, Schwarm, and Fisher (1986), Diewert (1987), Fisher (1988, 1995), and Fisher and Shell (1998).

[10] See, for example, Diewert (1981, pp. 170-174) for the definitions of these index numbers.

(4.3) $$C^1(\mathbf{w}^1,\mathbf{y}^1)/C^0(\mathbf{w}^0,\mathbf{y}^0) = W_{F-K} \cdot X_{F-A}$$

where:

$W_{L-K} \equiv C^0(\mathbf{w}^1,\mathbf{y}^0)/C^0(\mathbf{w}^0,\mathbf{y}^0)$ is the Laspeyres-Konüs-type index number of input prices;

$X_{P-A} \equiv C^1(\mathbf{w}^1,\mathbf{y}^1)/C^0(\mathbf{w}^1,\mathbf{y}^0)$ is the Paasche-Allen-type index number of input quantities;

$W_{P-K} \equiv C^1(\mathbf{w}^1,\mathbf{y}^1)/C^1(\mathbf{w}^0,\mathbf{y}^1)$ is the Paasche-Konüs-type index number of input prices;

$X_{L-A} \equiv C^1(\mathbf{w}^0,\mathbf{y}^1)/C^0(\mathbf{w}^0,\mathbf{y}^0)$ is the Laspeyres-Allen-type index number of input quantities;

$W_{F-K} \equiv \left[W_{L-K} \cdot W_{P-K}\right]^{1/2}$ is the Fisher-Konüs-type index number of input prices;

$X_{F-A} \equiv \left[Q_{L-A} \cdot Q_{P-A}\right]^{1/2}$ is the Fisher- Allen-type index number of input quantities.

Under the assumption of cost minimization, the economic theory of index numbers implies that

(4.4) $$C^0(\mathbf{w}^0,\mathbf{y}^0) = \mathbf{w}^0 \cdot \mathbf{x}^0$$

(4.5) $$C^1(\mathbf{w}^1,\mathbf{y}^1) = \mathbf{w}^1 \cdot \mathbf{x}^1$$

(4.6) $$C^0(\mathbf{w}^1,\mathbf{y}^0) \le \mathbf{w}^1 \cdot \mathbf{x}^0$$

(4.7) $$C^1(\mathbf{w}^0,\mathbf{y}^1) \le \mathbf{w}^0 \cdot \mathbf{x}^1$$

Similar relations can be established for the revenue function, but with inverted inequality sign. The four relations (4.4)-(4.7) lead us to the following bounds for the input-price indexes:

(4.8) $$W_{L-K} \equiv \frac{C^0(\mathbf{w}^1,\mathbf{y}^0)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} = \frac{C^0(\mathbf{w}^1,\mathbf{y}^0)}{\mathbf{w}^0 \cdot \mathbf{x}^0} \le \frac{\mathbf{w}^1 \cdot \mathbf{x}^0}{\mathbf{w}^0 \cdot \mathbf{x}^0} \equiv W_L$$

(4.9) $$W_{P-K} \equiv \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^1(\mathbf{w}^0,\mathbf{y}^1)} = \frac{\mathbf{w}^1 \cdot \mathbf{x}^1}{C^1(\mathbf{w}^0,\mathbf{y}^1)} \geq \frac{\mathbf{w}^1 \cdot \mathbf{x}^1}{\mathbf{w}^0 \cdot \mathbf{x}^1} \equiv W_P$$

where W_L and W_P are, respectively, the Laspeyres and Paasche index numbers of input prices.

The relations (4.4)-(4.7) lead us also to the following bounds for the input-quantity indexes:

(4.10) $$X_{L-K} \equiv \frac{C^1(\mathbf{w}^0,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} = \frac{C^1(\mathbf{w}^0,\mathbf{y}^1)}{\mathbf{w}^0 \cdot \mathbf{x}^0} \leq \frac{\mathbf{w}^0 \cdot \mathbf{x}^1}{\mathbf{w}^0 \cdot \mathbf{x}^0} \equiv X_L$$

(4.11) $$X_{P-K} \equiv \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^1,\mathbf{y}^0)} = \frac{\mathbf{w}^1 \cdot \mathbf{x}^1}{C^1(\mathbf{w}^0,\mathbf{y}^1)} \geq \frac{\mathbf{w}^1 \cdot \mathbf{x}^1}{\mathbf{w}^1 \cdot \mathbf{x}^0} \equiv X_P$$

where X_L and X_P are, respectively, the Laspeyres and Paasche index numbers of input quantities.

When the two cost functions under comparison are homothetic to each other, the Laspeyres input-price index is the upper bound and the Paasche input-price index is the lower bound of the interval of possible values of the Konüs-type index number of input prices. In the general non-homothetic case we may have alternative one-sided bounds, (4.8) *or* (4.9) for input-price indexes, and (4.10) *or* (4.11) for input quantity indexes:

(4.12) $$W_{L-K} \equiv \frac{C^0(\mathbf{w}^1,\mathbf{y}^0)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} = \frac{\omega(\mathbf{w}^1)}{\omega(\mathbf{w}^0)} = \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^1(\mathbf{w}^0,\mathbf{y}^1)} \equiv W_{P-K} \qquad \text{(homothetic case)}$$

using (3.38), and

(4.13) $$X_{L-K} \equiv \frac{C^1(\mathbf{w}^0,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} = \frac{A^{1^{-1}} \cdot F^{-1}[g(\mathbf{y}^1)]}{A^{0^{-1}} \cdot F^{-1}[g(\mathbf{y}^0)]} = \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^1,\mathbf{y}^0)} \equiv X_{P-K} \quad \text{(homothetic case)}$$

Therefore,

(4.14) $$W_P \le W_{P-K} = W_{L-K} \le W_L$$ (homothetic case)

(4.15) $$X_P \le X_{P-K} = X_{L-K} \le X_L$$ (homothetic case)

By contrast, in the general non-homothetic case, as shown by Frisch (1936), it is invalid to combine those pairs of bounds in one single expression. In fact, only in the homothetic case is it possible to have the inequalities given above. When the cost functions $C^0(\mathbf{w}, \bar{\mathbf{y}}^0)$ and $C^1(\mathbf{w}, \bar{\mathbf{y}}^1)$ are non-homothetic to each other, both cases where $W_L \ge W_P$ and $W_L < W_P$ are possible. When non-homothetic changes bring about $W_{L-K} \le W_L < W_P \le W_{P-K}$, an average (even an arithmetic or geometric average) of the weighted "true" indexes W_{L-K} and W_{P-K} may lay outside the $[W_L, W_P]$ interval if they are positioned in a very asymmetric way with respect to their respective Laspeyres or Paasche bounds.

Caves, Christensen, and Diewert (1982) have, in fact, noted that, when $C^0(\mathbf{w}, \bar{\mathbf{y}}^0)$ and $C^1(\mathbf{w}, \bar{\mathbf{y}}^1)$ are two translog functions differing in parameters of their zero- and first-order terms, the geometric average $(W_{L-K} \cdot W_{P-K})^{\frac{1}{2}}$ is equal to a Törnqvist index number. (The converse is not true, however, since in the non-homothetic the case, this index number is also exact for other functions, including the linear ones.) Milana (2005) has found that, when $C^0(\mathbf{w}, \bar{\mathbf{y}}^0)$ and $C^1(\mathbf{w}, \bar{\mathbf{y}}^1)$ are two translog functions which may differ in *all* parameters (including those of the second-order terms), a weighted geometric average $(W_{L-K}^{1-\lambda} \cdot W_{P-K}^{\lambda})$ for some value of λ is exactly equal to a Törnqvist index number, and when $C^0(\mathbf{w}, \bar{\mathbf{y}}^0)$ and $C^1(\mathbf{w}, \bar{\mathbf{y}}^1)$ have other quadratic functional forms that may differ in all parameters (including those of the second-order terms), the weighted geometric average $(W_{L-K}^{1-\lambda} \cdot W_{P-K}^{\lambda})$ for each value of λ is exactly equal to specific index formulas belonging to the class of Diewert's (1976) "superlative index numbers".

It is easy to see that, in the non-homothetic case, when a situation is such that $W_{L-K} \le W_L < W_P \le W_{P-K}$ and the two weighted "true" economic indexes are positioned in a very asymmetric way with respect to their respective bounds, it may happen that, even when $\lambda = 1/2$, we may find the (weighted) average $(W_{L-K}^{1-\lambda} \cdot W_{P-K}^{\lambda})$ outside the numerical interval between W_L

and W_P, thus differing substantially from the Ideal Fisher's index number $(W_L \cdot W_P)^{\frac{1}{2}}$, which, instead, is always found within the two indexes W_L and W_P by construction (see, Samuelson and Swamy, 1974, p. 585). This remark warns us against the indiscriminate use of Fisher's ideal index formula as a good approximation to "true" economic index numbers or their average in non-homothetic cases.

4.5 The properties of economic index numbers

We may want to find the best index numbers among the alternative formulas presented thus far. One first question is how much the index numbers based on revenue functions differ from those based on cost functions. The answer can be found in the theory that we have presented in section 2, where we have seen that these two conditional optimized functions may not be consistent with each other and with long-run equilibrium optimal solutions. When the reference inputs that are conditional for the revenue function are not consistent with the reference outputs in the cost function, the revenue-based index numbers generally differ in value from the cost-based index numbers.

As anticipated during the description of cost-based output quantity indexes, the separability conditions described in section 2 are of great importance for the resulting index numbers to correctly represent the changes that they are intended to measure. We emphasize here that, if strong input-output separability (constant returns to scale) is not present, unless a more general formulation is found, it could not be possible to find a univocally determined index number. In case of input-output weak separability, in fact, the construction of index numbers can be possible, but under additional arbitrary hypotheses concerning the degree of input-output substitutions.

More specifically, under constant returns to scale, economic index numbers are invariant with respect to the reference variables. This means that price indexes are functions only of elementary prices, whereas quantity indexes are functions only of elementary quantities. Therefore, the candidate index numbers presented in Table 1, should be rewritten in canonical form with no reference variables. Samuelson and Swamy (1974, pp. 571-572) have established the "completeness theorem" for economic canonical index numbers for the homothetic case, stating that these index numbers (independently from their functional form) satisfy all the test criteria of Fisher (1911) appropriate to the primitive one-good case. These test criteria are the following:

- *(i)* *General mean of price relatives (or linear homogeneity of the price index) test* (if all the elementary prices are multiplied by a λ, also the resulting price index number is multiplied by a λ);
- *(ii)* *time-reversal test* (the index becomes the inverse of itself if the time order is reversed);
- *(iii)* *circular-reversal (or transitivity) test* (the index number comparing two situations does not change if it is constructed transitively by chaining index numbers referring to a third observation point);
- *(iv)* *dimensional invariancy test* (the index is invariant with respect to the dimensional change of the variables);
- *(v)* *factor-reversal test* (the price index multiplied by the quantity index equals the total nominal-value index). This test is called "strong factor-reversal test" if the price and quantity index numbers have the same functional form and "weak factor-reversal test" if the two index numbers have different functional forms.

Samuelson and Swamy (1974, p. 575), in the context of economic index numbers, drop the strong factor-reversal test in favor of the weak factor-factor reversal test. They wittingly state (with notation adjusted): "We must stress again that the factor-reversal test offers no stumbling block for our definitions of $P(\mathbf{p}^0, \mathbf{p}^1; \overline{\mathbf{q}}^r)$ and $Q(\mathbf{q}^0, \mathbf{q}^1; \overline{\mathbf{p}}^r)$ if, as we should do logically, we drop the *strong* requirement that the *same* formula should apply to $Q(\mathbf{q})$ as to $P(\mathbf{p})$. A man and wife should be properly matched; but that does not mean I should marry my identical twin!"

As for circularity test, rather surprisingly, even in the non-homothetic case, the economic index numbers do not fail to satisfy it. This is due to the fact that, differently from the traditional index numbers that have to rely on the information limited only to prices and quantities, economic index numbers can be constructed using *also* an explicit (known) functional form of the underlying aggregator function. This incorporates the additional information concerning technology-related behavioral choices. At least in principle, we could calculate the economic index numbers by simulation of the value function at hand at the given prices *and* reference variables and, therefore compare the results obtained (which are conditional to the *same* reference variables) transitively between any different situations whatsoever. With this in mind, Samuelson and Swamy (1974, p. 575) see Fisher's problem from the "external" economic point of view with subtle irony: "Where most of the older writers balk, however, is at the circular test that free us from one base year. Indeed, so enamoured did Fisher become with his so-called Ideal index […] that, when he discovered it failed the circular test, he had the hubris to declare '…, therefore a *perfect* fulfillment of this so-called circular test should really be taken as proof that the formula which fulfils it is erroneous' (1922, p. 271). Alas Homer has nodded; or, more accurately, a great scholar has been detoured on a trip whose purpose was obscure from the beginning".

Most important, both the price and quantity index number satisfy the linear homogeneity test (*i*) in the homothetic case. Samuelson and Swamy (1974, pp. 576-577) adjoined to this test the requirement that both price and quantity index numbers are homogeneous of degree zero with respect to the weights (that is, they are not affected by the scale of the weights). In other words, this widened test, which they called "widened (*i*)", requires that the economic price and quantity indexes are to be homogeneous of degree one in the elementary prices and quantities, respectively, and homogeneous of degree zero in their respective weights. This widened test is satisfied in the homothetic case, but fails in the non-homothetic case while satisfying all the other tests (*ii*)-(*v*). It is consistent with the homothetic separability requirements on the underlying economic function for aggregation. In particular, in the general non-homothetic case, if an economic index of prices (quantities) always fulfils, by construction, the requirements of the linear homogeneity test, its dual quantity (price) index number that is constructed implicitly by deflating the underlying nominal value function by the primal price (quantity) economic index fails to pass this test. Moreover, in the non-homothetic case, the direct economic index fails to be homogeneous of degree zero with respect to the variables taken as weights, while the implicit dual index is always homogeneous of degree zero but depends also on the reference variables.

As a consequence of these theoretical results, if the price *and* quantity index numbers are constructed independently so that they both satisfy the linear homogeneity test (*i*) (this is the procedure proposed by Pollak, 1971 and Diewert, 1983), then the factor-reversal test will fail in the non-homothetic case. More specifically, in the non-homothetic case, the economic price (quantity) index number can still satisfy the linear homogeneity property (*i*), but will fail the Samuelson-Swamy adjoined requirement that they are to be also invariant with respect to the reference quantities (prices). Quoting their words (and using our notation), "[i]f Fisher had adjoined to (i*) the requirement that the quantity index is never to be affected by scale changes in $\mathbf{p}^1$ and $\mathbf{p}^0$ (which leave their 'weightings' unchanged), we'd learn in the nonhomothetic case that both indexes must fail this widened (*i**) test" (p. 577).

The test for zero-degree homogeneity (invariance) of the price index with respect to the *current-period* quantity weights has been attributed to Vogt (1980, p. 70) by Diewert (1992, p. 217), who, in turn, proposed the test for the zero-degree homogeneity (invariance) of the price index with respect to the *base-period* quantity weights. Diewert (1992, p. 217, fn. 9) himself, however, reminds us that Irving Fisher (1911, pp. 400-406), in his "almost forgotten (but nonetheless brilliant work)" had actually considered the linear homogeneity requirements of the *proportionality test* together with the requirements of the zero-degree homogeneity of the price indexes with respect to the current-period quantity weights (which we may note turns to be the counterpart of the widened (*i**) test devised by Samuelson-Swamy for economic index numbers[11]). He considered this test as the most important among the eight tests that he had devised for price indexes because it might indicate what type of quantity weights was required. However, the later Fisher (1922, pp. 420-421) no longer seemed to consider this test was important and reduced the relevant tests to the five referred to above. It is remarkable that Samuelson and Swamy (1974, pp. 576-577) have, instead, considered this test among the most important and critical tests that should be satisfied by an economic index number.

4.6 The importance of the homogeneity property

> *"If, like Pollak, one employs a quantity definition that satisfies Fisher's (i*) [homogeneity or proportionality test], then one of the other tests, such as (v*) [weak factor-reversal test], will fail in the nonhomothetic case".*
>
> P.A. Samuelson and S. Swamy (1974, p. 576)

[11] We must, however, consider the difference between the index number formulas and the economic index numbers. Fisher (1911)(1922) considered the base- or current-period quantities or prices as reference variables in his formulas, whereas Samuelson-Swamy (1974, pp. 567-68) considered a *variable* optimal basket of price-dependent quantities required to attain a reference level of living or output in defining their economic price index and a given reference price situation in defining their economic quantity index. (In fact, they defined this economic quantity index as that obtained implicitly by deflating the nominal expenditure by means of the economic price index.)

Because of the duality relationship between the price and quantity indexes, the degree-one (linear) homogeneity of both these two aggregating indexes of economic variables is a sufficient and necessary condition for their dual counterparts to be of degree-zero homogeneous in the weights of aggregation. At the same time, the degree-zero homogeneity in the weights of aggregation of both economic indexes is a sufficient and necessary condition for their dual counterparts to be degree-one homogeneous in their economic variables. If an index number fails to be zero-degree homogeneous in the weights, then it is non-invariant with respect to the reference variables. These reference variables are consequently non-separable from the variables to be aggregated. Under this condition, aggregation is not possible and the index number itself loses its economic meaning[12].

In the economic approach, economic index numbers are usually based on cost and revenue functions, which are always linearly homogeneous in prices. This means that, in this approach, the linear homogeneity property of quantity indexes and the invariance of price index numbers are put in question.

Archibald (1977, p.70), showed that, while the revenue function and the output-price indexes are always homogeneous of degree one in the output prices (by construction), the output quantity index obtained implicitly by deflating the nominal revenue by means of the revenue-based economic price index does not satisfy the requirements of the linear homogeneity test in the general non-homothetic case.

Similarly, Fisher (1988) noted that, while the cost function and the Konüs-type input-price indexes are always homogeneous of degree one in the input prices (by construction), the Paasche-Allen-type index number of real aggregate inputs given by X_{P-A} is not, in the general non-homothetic case, homogeneous of degree one in the input quantities. That is, if $\mathbf{x}^1 = \lambda \cdot \mathbf{x}^0$, where λ is a positive scalar, then $[C^1(\mathbf{w}^1,\mathbf{y}^1)/C^0(\mathbf{w}^1,\mathbf{y}^0)] \geq \mathbf{w}^1 \cdot \lambda \cdot \mathbf{x}^0 / \mathbf{w}^1 \cdot \mathbf{x}^0 = \lambda,$ in view of (4.6). We can also show that the Laspeyres-Allen-type index number of real aggregate inputs given by X_{L-A} is not, in the general non-homothetic case, homogeneous of degree one in the input quantities. That is, if $\mathbf{x}^1 = \lambda \cdot \mathbf{x}^0,$ than $[C^1(\mathbf{w}^0,\mathbf{y}^1)/C^0(\mathbf{w}^0,\mathbf{y}^0)] \quad \leq \mathbf{w}^0 \cdot \lambda \cdot \mathbf{x}^0 / \mathbf{w}^0 \cdot \mathbf{x}^0 = \lambda,$ in view of (4.7).

---

[12] This point, highlighted by Samuelson and Swamy (1974, pp. 576-577), is a reformulation of the theory of separability and aggregation set up by Strotz (1959) and Gorman (1959).

The results just obtained, suggest that, if it is possible to find a particular reference input-price vector $\mathbf{w}^*$ such as $[C^1(\mathbf{w}^*,\mathbf{y}^1)/C^0(\mathbf{w}^*,\mathbf{y}^0)] = \mathbf{w}^* \cdot \lambda \cdot \mathbf{x}^0 / \mathbf{w}^* \cdot \mathbf{x}^0 = \lambda,$ then an economic input-quantity index that is *locally* linearly homogeneous can be found. However, also this particular economic index number would not be satisfactory, since the dual (implicitly derivable) input-price index, which is to be consistent with the (weak) factor-reversal test, would not satisfy, in general, the Fisher-Samuelson-Swamy extended homogeneity property. To see this, let us consider the following implicit input price index number obtained as the ratio between the nominal-value index of total costs and a linearly homogeneous input-quantity index number, independently constructed with reference to the price vector $\mathbf{w}^*$:

(4.16) $$\widetilde{W}_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}^*) \equiv \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \cdot \frac{1}{X(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}^*)}$$

In contrast with the economic input-price index numbers, this index fails, in general, the linear-homogeneity test. If all the elementary input prices change by a scalar λ between period 0 and period 1, then the Paasche- and Laspeyers-weighted economic index numbers change by the same factor λ

(4.17) $$\frac{C^1(\lambda \cdot \mathbf{w}^0,\mathbf{y}^1)}{C^1(\mathbf{w}^0,\mathbf{y}^1)} = \frac{C^0(\lambda \cdot \mathbf{w}^0,\mathbf{y}^0)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} = \lambda$$

whereas, in general, from (4.16) we have

(4.18) $$\widetilde{W}_C(\mathbf{w}^0,(\lambda \cdot \mathbf{w}^0),\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}^*) \equiv \frac{C^1(\lambda \cdot \mathbf{w}^0,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \cdot \frac{1}{X(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}^*)} \neq \lambda$$

since

(4.19) $$C^1(\mathbf{w}^0,\mathbf{y}^1) = \mathbf{w}^0 \cdot \mathbf{x}^1(\mathbf{w}^0,\mathbf{y}^1)$$
$$\neq C^0(\mathbf{w}^0,\mathbf{y}^0) \cdot X(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}^*)$$

5. "Superlative" index numbers: can they solve the "non-aggregability"—"non-invariance" problem?

> "*In contrast to the case of a linearly homogeneous aggregator function where the cost function takes the simple form* C*(*u,**p***)* = c*(***p***)*u*,* [...] *the* [*Törnqvist*] *index number* $P_0(\mathbf{p}^0,\mathbf{p}^1;\mathbf{x}^0,\mathbf{x}^1)$ *is exact for functional forms for* C(u,**p**) *other than the translog*".
>
> "[…] *Thus the same* [*Törnqvist*] *price index P_0 is exact for more than one functional form (and reference utility level) for the true cost of living*".
>
> Diewert (1976, pp. 122-123)

In a another paper (Milana, 2005), we have found severe problems in practical application of Diewert's (1976) superlative index numbers when these are defined as approximations to true unknown index numbers. This is also related to the fact that superlative index numbers may vary widely, even by far beyond the Laspeyres-Paasche index number spread (see Robert Hill, 2006a). In fact, they turn out to be applicable only when they are "exact" for the true supporting function. Moreover, Diewert (1976, pp. 122-123) himself, while finding that in the general non-homothetic case the Törnqvist price index is "exact" for an underlying translog function, he recognized that this is not "an if and only if result". He had admitted the possibility that the Törnqvist price index number is exact for functional forms *other than the translog* and concluded more explicitly that the same Törnqvist price index is exact for more than one functional form. In the case of non-homothetic changes in tastes or technology, the Törnqvist index number and all the other index numbers defined to be superlative by Diewert could also be interpreted as being "exact" for first-order (and not second-order!) approximating functions in the space of the examined economic variables. In other words, in the non-homothetic case, "superlative" index numbers might be not at all superlative in Diewert's sense. In order to avoid the severe problems that are related to the approximation, in what follows we assume that the index numbers are "exact" not for an approximating function but for a true economic function[13].

[13] Within the so-called "functional or economic approach" in the index number theory, certain functional forms of the aggregator function can be associated with certain index number formulas. In other words, these can be defined "exact" for particular aggregator functions if they are equal to the ratio of the values of these functions at an initial and final point. The concept of the "exact" index numbers has been used in the economic literature at least since the contributions of Bennet (1920), Bowley (1928, 1938), Frisch (1936, pp. 27-29), and Wald (1939, p. 329). All these authors were unknown to Konüs and Byushgens (1926), who wrote independently in Russian language and were brought to the attention of the international scientific community by Schultz (1939, p. 8). They used extensively duality theory and index numbers that they found to be "exact" for a variety of aggregating functional forms. These include the quadratic function for which the Fisher ideal index number is "exact" (see, Diewert, 1993, pp. 46-50

The requirements that an index number formula is to be invariant with respect to the reference variables of the underlying true function is equivalent to the requirement of separability and aggregability of the elementary arguments of this function independently from the reference variables taken as given. Therefore, if this requirements are not satisfied, then the economic meaning and the existence itself of such an index number are unfounded. The theoretical developments in this field during the last thirty years have tried to find a way to weaken the conditions under which certain index numbers could be constructed under the conditions of non-homothetic changes. Diewert (1980) showed that, at least one of the class of Diewert's (1976) superlative index numbers, the Törnqvist, can be regarded to solve, at least locally and with an approximation, the non-separability—non-invariance problem. Therefore, the use of this index formula has been justified for being the most general among the known index formulas, since it permits us to take into account any possible price- or technology-induced input-output substitution.

Diewert (1976, pp. 129-136) has shown that a set of superlative index numbers can be derived as special cases from the *Quadratic-mean-of-order-r index*, which is itself a superlative index number and is "exact" for a *Quadratic-mean-of-order-r* aggregator function[14]. Fisher's ideal and the Törnqvist index numbers belong to this class: the former is "exact"" to the Konüs-Byushgens (1926) homogeneous quadratic aggregator function (corresponding to the *Quadratic-of-order-2* aggregator function) and the latter is "exact" for the *translog* aggregator function (corresponding to a *Quadratic-of-order-r* function, with *r* approaching zero). Diewert (1976, p. 118) has also established the *Quadratic approximation lemma,* which has been used to show that a Törnqvist index number is "exact" for a *translog* unit cost function. As it may be noted, a Törnqvist index number is "exact" for the translog unit cost function *if and only if* this function is linearly homogeneous. (In a more recent contribution, Diewert, 2000b, pp. 8-10 has generalized the *Quadratic Identity* to the *quadratic-mean-of-order-r* functional forms.)

Diewert (1976, pp. 123-124) also has shown that the Törnqvist input-quantity index is "exact" for the geometric mean of two Malmquist input-quantity indexes when the two underlying functions are both translog with different parameters in their zero- and first-order

for an historical description of these early remarkable developments, and Diewert, 1976, 1978 and Lau, 1979 for a modern treatment of the concept of "exact" index numbers).

[14] This functional form is due to McCarthy (1967), Kadiyala (1971-72), Denny (1972, 1974), and Hasenkamp (1973).

terms. This result has been later found with the Törnqvist cost-of-living or input-price index by Caves, Christensen, and Diewert (1982, pp. 1409-1411), who have extended Diewert's (1976, p. 118) Quadratic approximation lemma by establishing the *Translog Identity* (see Caves *al.*, 1982, pp. 1412-1413). They have been able to show that the Törnqvist index number is "exact" for the geometric mean of two translog functions differing in the parameters of their zero- and first-order terms. More recently, it has been shown that the Törnqvist index number is "exact" for a weighted geometric mean of two translog functions differing in all their parameters (see Milana, 2005).

The Törnqvist index is, therefore, regarded as being equally valid for measuring aggregate relative changes in input quantities or prices under assumptions of homothetic and non-homothetic changes. Caves *et al.* (1982, p. 1411) claim: "This result implies that the Törnqvist index is superlative in a considerably more general sense than shown by Diewert. We are not aware of other indexes that can be shown to be superlative in this more general sense". However, since all superlative indexes are supposed to be numerically approximate each other, they conclude: "any superlative index (in the sense of Diewert, 1976) will be approximately equal to the geometric mean of two Malmquist indexes based on the translog form".

In a successive work, where these results were extended to the output index numbers, Diewert (1983) recognized that a quantity index number obtained implicitly by deflating the index of total nominal revenues by means of an economic price index may not result to be linearly homogeneous in the elementary quantities. This may occur even if the deflator is the Törnqvist index. The cost-based Törnqvist index of input prices is given by:

(5.1) $$W_T \equiv \prod_{n=1}^{N} (w_n^1 / w_n^0)^{\frac{1}{2}(s_n^0 + s_n^1)}$$

where $s_n^r \equiv \dfrac{w_n^r \cdot x_n^r}{\sum_i^N w_i^r \cdot x_i^r}$, for $r = 0,1$.

Following Milana (2005), it can be shown that

(5.2) $$W_T = (W_{C_T}^0)^{(1-\lambda)} \cdot W_{C_T}^1)^{\lambda} \equiv \left[\frac{C_T^0(\mathbf{w}^1, \mathbf{y}^0)}{C_T^0(\mathbf{w}^0, \mathbf{y}^0)} \right]^{(1-\lambda)} \left[\frac{C_T^1(\mathbf{w}^1, \mathbf{y}^1)}{C_T^1(\mathbf{w}^0, \mathbf{y}^1)} \right]^{\lambda}$$

where $\lambda \equiv \dfrac{\frac{1}{2}(C_T^1 - C_T^0) + \frac{1}{4}(\mathbf{w}^1 - \mathbf{w}^0)'(\mathbf{A}^1 - \mathbf{A}^0)(\mathbf{w}^1 - \mathbf{w}^0)}{(C_T^1 - C_T^0)}$, with $\mathbf{A}^0$ and $\mathbf{A}^1$ being the matrices of second-order parameters of the translog functions C_T^0 and C_T^1. If $\mathbf{A}^1 = \mathbf{A}^0$, then λ must be equal to 1/2 in order for the Törnqvist index number W_T to be exact for the geometric mean of the two translog functions (this is the particular case considered by Caves, Christensen, and Diewert, 1982 in their well-known formulation of the translog identity).

Similarly, we can define a conditional revenue-based Törnqvist index number of output prices.

Samuelson and Swamy (1974, p. 576) observed that, in the general non-homothetic case, the conditional economic price index number does not satisfy the requirements of zero-degree homogeneity in the reference conditional variables. In other words, the conditional economic price index number fails to be, in such conditions, "invariant" with respect to the reference variables. This also applies to the geometric mean of the economic price indexes calculated at two different levels of those reference variables[15]. In the terminology of Samuelson and Swamy (1974, p. 570), "[t]he invariance of the price index is seen to imply and to be implied by the invariance of the quantity index from its reference price base". The homotheticity of the underlying economic function is a necessity as well as a sufficiency of the invariance of the economic index numbers. Moreover, Samuelson and Swamy (1974, p. 576) observed that, in the general non-homothetic case, the corresponding quantity index obtained implicitly by deflating the nominal cost by means of the economic price index fails to satisfy the requirements of the linear homogeneity test.

It is, therefore, straitforward to show that, in the general non-homothetic case, the Törnqvist price index number is not invariant with respect to the reference variables in the underlying function. Moreover, the corresponding implicit Törnqvist quantity index is not linearly homogeneous. We have in fact

[15] In defining the economic index numbers it should be kept in mind the distinction of the *reference variables* of the underlying economic function (for example, the outputs $\mathbf{y}^r$ in the case of the cost function) from the *weights* used to construct the index number formula (for example, the input quantities $\mathbf{x}^r$ or shares $\mathbf{s}^r$ in the case of cost functions). In the homothetic separability case, the relative contribution of the arguments to the value of the economic function is not affected by the reference variables and this is reflected in the zero-homogeneity in the reference variables of the corresponding economic index numbers.

(5.3) $$\widetilde{X}_T \equiv \frac{C_T^1(\mathbf{w}^1,\mathbf{y}^1)}{C_T^0(\mathbf{w}^0,\mathbf{y}^0)} / W_T = (X_{C_T}^0)^{\lambda} \cdot (X_{C_T}^1)^{1-\lambda}$$

where

$$X_{C_T}^0 \equiv \frac{C_T^1(\mathbf{w}^0,\mathbf{y}^1)}{C_T^0(\mathbf{w}^0,\mathbf{y}^0)} = \frac{C_T^1(\mathbf{w}^1,\mathbf{y}^1)}{C_T^0(\mathbf{w}^0,\mathbf{y}^0)} / \frac{C_T^1(\mathbf{w}^1,\mathbf{y}^1)}{C_T^1(\mathbf{w}^0,\mathbf{y}^1)} \text{ and } X_{C_T}^1 \equiv \frac{C_T^1(\mathbf{w}^1,\mathbf{y}^1)}{C_T^0(\mathbf{w}^1,\mathbf{y}^0)} = \frac{C_T^1(\mathbf{w}^1,\mathbf{y}^1)}{C_T^0(\mathbf{w}^0,\mathbf{y}^0)} / \frac{C_T^0(\mathbf{w}^1,\mathbf{y}^0)}{C_T^0(\mathbf{w}^0,\mathbf{y}^0)}$$

are the Laspeyres- and Paasche-weighted translog economic index numbers of input quantities respectively. Following Pollak (1971), Samuelson and Swamy (1974, 576-77), and Fisher's (1988) reasoning reported above, we note that both $X_{C_T}^0$ and $X_{C_T}^1$ fail to satisfy the linear homogeneity test in the non-homothetic case and so does also their (weighted) geometric mean. This conclusion definitely rejects the possibility of aggregation in the non-homothetic case and, consequently, also the general validity of Diewert's (1976) superlative index numbers, including the Törnqvist index, in decomposing the observed changes in economic value (or production) functions, into aggregated changes in prices (or quantities) and a "residual" component.

Similarly, following Archibald (1977), it turns out that, in the non-homothetic case, also any implicit economic output quantity index constructed by deflating the conditional revenue function fails to satisfy the homogeneity requirements for aggregation[16].

In searching a way out from this impasse, Diewert (1983) constructed a revenue-based direct Törnqvist price index and a direct Törnqvist quantity index. This last index is justified for being identically equal to a geometric mean of two Malmquist indexes, which, in turn, are based on technically (not economically) defined distance functions. This procedure has been accepted by Russell's comments. These Törnqvist price and quantity index numbers turn out to satisfy the linear homogeneity requirement, but at the cost of failing to satisfy the requirements of the factor-reversal test (stating that the price index multiplied by the quantity index should equal the index of total nominal revenues or costs). Samuelson and Swamy (1974, p. 576) clearly observed: "If, like Pollak, one employs a quantity definition that satisfies Fisher's (*i**) [linear homogeneity test], then [given the imposed linear homogeneity of the price index] one of the other tests, such as (*v**) [weak factor reversal test], will fail in the nonhomothetic case". They spelled out this outcome even more clearly in another example (p. 577, fn. 10): "Afriat favors the

16 This conclusion is immediate if one considers that the economic index numbers that are derived from a non-homothetic function could never satisfy, by construction, the homogeneity requirements.

linear Engel-curve approximation: $e(P;Q) = \theta(P)\phi(Q) + \mu(P)$", where the last additive term is a residual not captured by the price index multiplied by the quantity index.

6. An alternative approach based on the unrestricted net profit function

> *"The profit function takes the high ground; it is the most sophisticated representation of the technology"*
>
> R. Färe and Primont (1995, p. 149)

We have noted earlier that the transformation functions $T^t(\mathbf{y},\mathbf{x}) = 0$ and $\pi^t(\mathbf{p},\mathbf{w})$ defined respectively in the quantity and price spaces are linear homogeneous in their arguments, that is $T^t(\mathbf{y},\mathbf{x}) = \lambda^{-1}T^t(\lambda\mathbf{y},\lambda\mathbf{x})$ and $\pi^t(\mathbf{p},\mathbf{w}) = \lambda^{-1}\pi^t(\lambda\mathbf{p},\lambda\mathbf{w})$. This property suggest us to explore an avenue that has been seldom travelled (Archibald, 1977 and Balk, 1998 being the exceptions) probably because the null value of these two functions do not allow the construction of index numbers in terms of ratios, in contrast for example with the conditional revenue and cost functions. However, when outputs are not separable from inputs, thus reflecting non-constant returns to scale with non-homothetic effects on the underlying functional structure, the general form of these functions may permit us to obtain invariant indicators of quantity or price components of their absolute change along with their respective residual complements having the meaning of technical change.

Let us define the nominal net profit function normalized with respect to the value of the main output, say y_h^t, so that $\pi^{*t}(\mathbf{p}^*,\mathbf{w}^*) \equiv \pi^t(\mathbf{p},\mathbf{w})/p_h \cdot y_h$ for t = 0,1, with $\mathbf{p}^* \equiv \mathbf{p}/p_h$, $\mathbf{w}^* \equiv \mathbf{w}/p_h$, $\mathbf{y}^* \equiv \mathbf{y}/y_h = \nabla_{\mathbf{p}}\pi^t(\mathbf{p},\mathbf{w})/y_h$ and $\mathbf{x}^* \equiv \mathbf{x}/y_h = \nabla_{\mathbf{w}}\pi^t(\mathbf{p},\mathbf{w})/y_h$ by Hotelling's lemma (see, for example, Luenberger, 1995, pp. 77-78 for a description of this notion of normalized profit function[17]). We note that $[\mathbf{y}^*\,\mathbf{x}^*]'$ with $y_i^* = \dfrac{\partial\pi^{*t}}{\partial p_i^*} = \dfrac{\partial\pi^t}{\partial p_i}\cdot\dfrac{\partial\pi^{*t}}{\partial\pi^t}\cdot\dfrac{\partial p_i}{\partial p_i^*}$ and $x_i^* = \dfrac{\partial\pi^{*t}}{\partial w_i^*} = \dfrac{\partial\pi^t}{\partial w_i}\cdot\dfrac{\partial\pi^{*t}}{\partial\pi^t}\cdot\dfrac{\partial w_i}{\partial w_i^*}$, is the vector of the technical input-output coefficients for y_h.

[17] The terminology "normalized profit" is originally due to Jorgenson and Lau (1974a)(1974b) (see also Lau, 1978).

We decompose the observed absolute difference $(\pi *^1 - \pi *^0)$ into price and technical change components as follows:

$$\pi *^1 (\mathbf{p}^{*1}, \mathbf{w}^{*1}) - \pi *^0 (\mathbf{p}^{*0}, \mathbf{w}^{*0}) = P_{\pi}^{0,1} + T_{\pi}^{0,1} \tag{6.1}$$

where $P_{\pi}^{0,1} \equiv$ price change component and $T_{\pi}^{0,1} \equiv$ technical change component defined, respectively, as

(6.2) $P_{P-\pi}^{0,1} \equiv \pi *^1 (\mathbf{p}^{*1}, \mathbf{w}^{*1}) - \pi *^1 (\mathbf{p}^{*0}, \mathbf{w}^{*0})$ (Paasche-weighted price component)

(6.3) $T_{L-\pi}^{0,1} \equiv \pi *^1 (\mathbf{p}^{*0}, \mathbf{w}^{*0}) - \pi *^0 (\mathbf{p}^{*0}, \mathbf{w}^{*0})$ (Laspeyres-weighted technical change component)

or,

(6.4) $P_{L-\pi}^{0,1} \equiv \pi^{*0} (\mathbf{p}^{*1}, \mathbf{w}^{*1}) - \pi^{*0} (\mathbf{p}^{*0}, \mathbf{w}^{*0})$ (Laspeyres-weighted price component)

(6.5) $T_{P-\pi}^{0,1} \equiv \pi^{*1} (\mathbf{p}^{*1}, \mathbf{w}^{*1}) - \pi^{*0} (\mathbf{p}^{*1}, \mathbf{w}^{*1})$ (Paasche-weighted technical change component)

The indicators $(-T_{L-\pi}^{0,1})$ and $(-T_{P-\pi}^{0,1})$ are alternative measures of the relative rate of technical change between the observation points $t = 0$ and $t = 1$..

If the technical input-output coefficients are fixed and there are no price-induced substitution effects, we can apply the following formulas:

$$\pi *^0 (\mathbf{p}^{*1}, \mathbf{w}^1) = \mathbf{p}^{*1} \cdot \mathbf{y}^{*0} - \mathbf{w}^{*1} \cdot \mathbf{x}^{*0} \tag{6.8}$$

$$\pi *^1 (\mathbf{p}^{*0}, \mathbf{w}^0) = \mathbf{p}^{*0} \cdot \mathbf{y}^{*1} - \mathbf{w}^{*0} \cdot \mathbf{x}^{*1} \tag{6.9}$$

In particular cases where the profit functions $\pi *^0$ and $\pi *^1$ have more flexible functional forms we can take account of price-induced substitution effects in the calculation of the price component and, then, calculate the technical change component as a residual. Let us consider the Quadratic-mean-of-order-*r* functional form established by Diewert (1976):

$$\pi_{Q_r}^{*t} (\mathbf{z}) \equiv \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij}^t z_i^{\frac{1}{r}} z_j^{\frac{1}{r}} \right]^{\frac{1}{r}} \quad \text{where} \quad \mathbf{z} \equiv [\mathbf{p}^* \;\; \mathbf{w}^*]' \tag{6.10}$$

where the parameters α_{ij}^t are allowed to change under the zero-homogeneity constraint. If technical change is jointly Hicks-cost-profit neutral, the function (6.10) becomes

$$\text{(6.11)} \qquad \pi_{Q_r}^{*t}(\mathbf{z}) \equiv \sigma^t \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} z_i^{\frac{1}{r}} z_j^{\frac{1}{r}} \right]^{\frac{1}{r}}$$

where σ^t is a technological parameter (see, for example, Chandler, 1988, pp. 224-228 for different types of technical change with profit functions).

If the two functions $\pi_{Q_r}^{*t}(\mathbf{z})$ with t = 0,1 have the functional form (6.10) or (6.11) with r = 1, corresponding to the *quadratic-mean-of-order-1* (Generalized Leontief) profit functions, then (see Milana, 2005)

$$\text{(6.11)} \qquad P_{Q_1}^{0,1} \equiv \sum_{i=1}^{N} \left\{ \frac{(z_i^0)^{\frac{1}{2}} q_i^0}{(z_i^0)^{\frac{1}{2}} + (z_i^1)^{\frac{1}{2}}} + \frac{(z_i^1)^{\frac{1}{2}} q_i^1}{(z_i^0)^{\frac{1}{2}} + (z_i^1)^{\frac{1}{2}}} \right\} (z_i^1 - z_i^0) \quad \text{where} \quad \mathbf{q}^t \equiv [\mathbf{y}^{*t} \; (-\mathbf{x}^{*t})]'$$

If the two profit functions $\pi_{Q_r}^{*t}(\mathbf{z})$ with t = 0,1 have the functional form (6.10) or (6.11) with r = 2, corresponding to the *quadratic-mean-of-order-2* (Konüs-Byushgens) functional form, then (see Milana, 2005)

$$\text{(6.14)} \qquad P_{Q_2}^{0,1} \equiv \sum_{i=1}^{N} \left\{ \frac{z_i^0 \cdot \pi_{Q_2}^{*0}}{\pi_{Q_2}^{*0} + \pi_{Q_2}^{*1}} + \frac{z_i^1 \cdot \pi_{Q_2}^{*1}}{\pi_{Q_2}^{*0} + \pi_{Q_2}^{*1}} \right\} (z_i^1 - z_i^0)$$

and $\pi^{*t} \equiv \mathbf{z}^t \cdot \mathbf{q}^t$ with $t = 0,1$.

It is possible to show that, with any pair of quadratic-mean-of-order-r functional forms differing in parameters as in (6.12) or (6.13)

$$\text{(6.15)} \qquad P_{Q_r}^{0,1} = (1-\lambda) P_{L-Q_r}^{0,1} + \lambda P_{P-Q_r}^{0,1}$$

where $P_{L-Q_r}^{0,1} \equiv \pi_{Q_r}^{*0}(\mathbf{p}^{*1}, \mathbf{w}^1) - \pi_{Q_r}^{*0}(\mathbf{p}^{*0}, \mathbf{w}^0)$

$$P_{P-Q_r}^{0,1} \equiv \pi_{Q_r}^{*1}(\mathbf{p}^{*1}, \mathbf{w}^1) - \pi_{Q_r}^{*1}(\mathbf{p}^{*0}, \mathbf{w}^0)$$

where, similarly to the case of the translog function,

(6.16) $$\lambda \equiv \frac{\frac{1}{2}(\pi_{Q_r}^{*1} - \pi_{Q_r}^{*0}) + \frac{1}{4}(\mathbf{w}^1 - \mathbf{w}^0)'(\mathbf{A}^1 - \mathbf{A}^0)(\mathbf{w}^1 - \mathbf{w}^0)}{(\pi_{Q_r}^{*1} - \pi_{Q_r}^{*0})},$$

where $\mathbf{A}^r \equiv [\alpha_{ij}^t]$ is the symmetric matrix of second-order parameters of the function. If these parameters are constant over the examined period, that is $\mathbf{A}^1 = \mathbf{A}^0$, then $\lambda = 1/2$.

Using (6.11) or (6.12), the residual component representing the relative rate of technical change is given by

(6.17) $$T_\pi^{0,1} \equiv (\pi_{Q_r}^{*1} - \pi_{Q_r}^{*0}) - P_\pi^{0,1} \qquad \text{for } \pi_{Q_r}^{*1} = \pi_{Q_1}^{*1} \text{ or } \pi_{Q_2}^{*1}, \text{ respectively.}$$

This results should be contrasted with those originally obtained by Caves, Christensen, and Diewert (1982), who contended that only the Törnqvist index among the known index formulas is exact for a non-homothetic function (the translog), whereas all other Diewert's superlative index numbers are not consistent with non-homothetic changes. The reason for having overlooked these properties stem from the particular methods of derivation that imposed unnecessary separability conditions. The problem with the general non-homothetic case is that all indexes (including Diewert's class of superlative indexes) are exact for combinations of *non-invariant* economic index numbers, thus reflecting that the aggregates that they are intended to measure do not really exist. For this reason, in applying the index number formulas presented here that allow us to overcome the input-output non-homothetic separability, we shall assume an Hicks-neutral technical change reflecting homothetic changes in parameters. Thus, if technical change is not Hicks-neutral (a number of empirical studies seem to confirm this hypothesis[18]), then our measure of technical change does not provide us with an "aggregate" of technical

[18] See, for example, Takayama (1974), who found empirical evidence of a biased technical change in the U.S. in a paper that appeared on the same issue of the *AER* where Samuelson and Swamy's (1974) article was published. These authors had, instead, favoured the homotheticity hypothesis in production theory rather than in consumer theory. They claimed in fact: "Fortunately, in the case of production theory […] homotheticity is not always so unrealistic" (p. 577, fn. 10). Other examples of empirical evidence of a biased technical change are those of Jorgenson and Fraumeni (1981) and Jorgenson, Gollop, and Fraumeni (1987, 211-260) for the US.

effects. In this case, our indicator of price changes is not homogeneous of degree zero with respect to parameter changes, reflecting its dependence on the particular technology path between the two situations under comparison while the technology change indicator is not homogeneous of degree one thus reflecting a distortion.

7. Empirical results

The theoretical discussion on measurement problems with economic index numbers can be confronted with the empirical analysis using the available structural data on production activities. The analytical module of the database set up for the EUKLEMS project recently funded by the EU Commission is particularly suitable for our purpose. This database provides us with time series of price and quantity indexes of outputs and inputs within supply and use input-output tables at the level of disaggregation of 72 industries as well as stocks and services of durable capital goods used in production. It has been constructed in close collaboration with national statistical agencies and is fully consistent with the official national accounts following the directives of Eurostat. In order to save space, we present only the results obtained for Italy at aggregate level.

The assumptions of input-output separability and constant returns to scale are taken into account by using cost-based index numbers of total factor productivity. The results on TFP obtained under the alternative Leontief, Generalized Leontief, and Konüs-Byushgens cost functions are shown in Table 1. The implicit Laspeyres and implicit Paasche index numbers are exact for the Leontief (fixed coefficients) technology. In aggregating the elementary input-price changes, they use as weights the technical coefficients observed at the base and current years, respectively. The ideal Fisher index number is, instead, exact for the Konüs-Byushgens cost function, and is usually interpreted as a close approximation of the Generalized Leontief (up to the second order). The indicators constructed are relative differences rather than ratios, so that they represent directly rates of change and not index numbers.

The results in Table 1 show that there was a wide variation in the Laspeyres-Paasche spread. A large spread may reveal that the true indicator (if it exists as an aggregate indicator) may be far from being close to measures constructed here. The implicit Paasche and Laspeyres indicators turn out to be very close during the years 1993-1997 and 2001-2003, but relatively far from each other during the decades of the seventies and the eighties. This is not surprising, considering the relatively intense restructuring activities that had taken place in Italy and other European countries after the first and second oil shocks. Intense energy saving technological change and price-induced input substitutions were reflected in the immediate reply of cost-reducing policies within the firms and governments in those periods (see, however, Hill, 2006b for a discussion on the conditions for chaining to reduce the Paasche-Laspeyres spread).

Moreover, a reverse position in the ranking of numerical values of the Laspeyres- and Paasche-type indicators with respect to the indications of the theory of bounds of economic index numbers, may suggest that non-homotheticity has taken place in all years, except three. The implicit Laspeyres-type (implicit Paasche-type) quantity index, corresponding to the total nominal costs deflated by the direct Laspeyres (direct Paasche) price index, is, in fact, a direct Paasche-type (direct Laspeyres-type) quantity index . The theory of bounds that we have recalled above suggests that, in the homothetic case, the direct Laspeyres index (which is always the upper bound of the Laspeyres-weighted "true" index) is comparable to and higher than the direct Paasche index (which is always the lower bound of the Paasche-weighted "true" index). If, instead, the direct Laspeyres index turns out to be lower than the direct Paasche index, then a non-homothetic change situation may have occurred. In Table 1, the direct Laspeyres turns out to be substantially higher than the direct Paasche TFP index growth in only 2 years in the whole period 1971-2003, thus indicating that non-homotheticity effects have been the norm rather than the exception.

The Konüs-Byushgens (KB) indicator corresponds to the arithmetic average of the Laspeyres and Paasche-type indicators and, therefore, is always found between their bounds by construction. Moreover, the Konüs-Byushgens and the Generalized Leontief (GL) indicators are found to be very close to each other thus confirming that they always perform in close approximation (see Hill, 2006a). Moreover, the fact that these two indicators are both found to be, respectively, a perfect and a close approximation to the arithmetic average of the two Laspeyers and Paasche indicators is rather problematic in the case of severe non-homothetic changes, where the true index is brought beyond the Laspeyres-Paasche interval in a very asymmetrical way.

The results obtained by considering a separable cost function based on the input-output separability assumption can be contrasted with those obtained with the indicators derived from a profit function in the input-output non-separability case. Figure 1, showing the technical-change measures obtained with indicators based on the GL and KB cost functions, can be contrasted with Figure 2, showing the technical-change measures obtained with indicators based on the GL and KB profit functions. We must recognize, however, that these results are not fully comparable, since the functional forms of the cost and profit functions are not "self-dual", meaning that the profit function corresponding to a GL (or KB) cost function does not have a GL (or KB) functional form, and vice versa. The consequence of this is that we are comparing the results obtained under different hypotheses on input-output separability *combined with* different hypotheses on functional forms. However, in cases where the spread between different formulas (as those given in Table 1) is not too wide, the difference in results may be mainly due to the different separability hypotheses.

Figure 3 compares the results obtained with cost- and profit-based indicators shown in Figures 1 and 2. We note that the cost-based TFP measure and the profit-based technical change measures are surprisingly different in many years of the examined period. In 11 out of 33 years the difference has been found to be at least greater than 50 per cent. The recent productivity slowdown observed in Italy after the year 2000 seems reduced to more than half within the picture obtained with the more general framework that allows us to take account of non-constant returns to scale in a period of reduced pace of economic growth. This should be contrasted with the years 1999 and 2000, where the higher dynamics of production has led the cost-based measure of TFP growth to be lower than the profit-based measure of technical change. These results suggest that the Italian economy is characterized by non-constant returns to scale and is affected by various constraints that hinder the full exploitation of its factor employment.

Figures 4 and 5 show, respectively, the effects that TFP and technical changes have brought about on real factor rewards during the more recent period 2000-2003. It can be seen that, during the year 2000, the high increase in energy prices (notably crude oil prices) during a worldwide economic expansion has absorbed the whole TFP gain achieved in that year and required also losses in the real labour compensation and services, while the positive short-run performance in production has allowed some small gains in the real capital rewards (both ICT and non-ICT). These movements in real factor prices appear all amplified in the results obtained with the more general framework based on profit-based indicators.

The same Figures 4 and 5 permit us also to contribute to the current debate on productivity slowdown in Italy. During the period 2001-2003, we observe that this productivity slowdown does not appear to be related to efficiency losses as much as they seem if we look at more traditional indicators. These appear to be theoretically unfounded since the hypotheses on input-output separability and constant returns to scale on which they are based are not, in fact, confirmed by the results obtained using more general models. Efficiency losses, for example, turn out to be negligible and the estimated productivity downfall may be due to measurement errors as much as to a real phenomenon.

8. Conclusion

Economic index numbers of outputs, inputs, and productivity are theoretically derived from production-related functions. In practice, they are constructed by means of traditional index numbers that turn out to be "exact" for those functions when (as it is usually the case) these are not known and cannot be used directly. However, economic index numbers have little or no meaning when the reference variables have an influence on the changes in the elementary items subject to aggregation. Much of the progress made during the last thirty years in the theory of economic index numbers has been devoted to this serious problem. The discovering of the "superlative" index numbers, which are "exact" for flexible functional forms of the underlying economic functions, has seemed to open a way towards an "invariant" aggregation methodology. In fact, the Törnqvist index number has been found to be "exact" for the geometric average of two translog functional forms that are different in a non-homothetic way. As we have shown in a previous paper, all superlative index numbers are "exact" for flexible functions subject to non-homothetic changes. However, in the very non-homothetic case, as it was already well known with the Törnqvist index number, also any other superlative index number may be "exact" for more than one specific functional form, including a linear functional form! Consequently, with non-homothetic changes aggregation of outputs or inputs is an arbitrary procedure.

A partial solution to the non-separability problem in technical change measurement may be found by aggregating outputs and inputs together using the so-called transformation functions. The profit function can be considered as a transformation function in the space of prices and may be used under the hypothesis that the observed data are optimal from the point of view of long-run equilibrium. A decomposition procedure has been devised to decompose changes in the value of the profit function into a technical change component and a price component without imposing any assumption on input-output separability. This method has been applied empirically to the case of the Italian industries using the newly built database of the EUKLEMS project. Homotheticity seems to have been the exception rather than the rule in Italy during the period 1970-2003 and the results obtained have been contrasted with those of traditional approaches that assume input-output separability. Although these alternative measures are not fully comparable, we conclude that the TFP decline recently reported in Italy is not confirmed in size and direction by our findings on technical change.

An arbitrary solution is, however, applied also in this approach when technical change effects are non-separable from outputs and inputs. No decomposition procedure based on index numbers can be univocal when technical change is non-homothetic. Therefore, no way is still open to a full definite solution to the problem of non-invariant index numbers. We conclude that statistical agencies should be aware that no index number formula is superior to others when the internal structure of the underlying functions is not known. In constructing economic aggregates, it would be better to indicate, when possible, the range of plausible measures just as it is traditionally done in other contexts and for other reasons (for example, in the field of econometrics where confidence intervals are usually constructed around point estimates of the unknown parameters). This is, however, conditional to our tastes and habits, which are rarely expected to change promptly.

Table 1. Alternative measures of TFP changes based on different cost functions (in percentage)

All industries in the Italian economy

Year	Implicit Laspeyres (direct Paasche) (1)	Implicit Konüs-Byushgens (ideal Fisher) (2)	Implicit Generalized Leontief (3)	Implicit Paasche (direct Laspeyres) (4)	Direct Paasche/Direct Laspeyres ratio (5) = (1)/(4)	Difference between direct Paasche and direct Laspeyres (6) = (1) - (4)
1971	0.65	0.47	0.48	0.30	2.20	0.35
1972	-1.33	-1.49	-1.8	-1.64	0.82	0.30
1973	2.93	2.86	2.86	2.78	1.05	0.15
1974	1.95	1.79	1.78	1.64	1.19	0.32
1975	-3.30	-3.45	-3.44	-3.61	0.91	0.31
1976	1.51	1.46	1.46	1.41	1.07	0.11
1977	-0.61	-0.65	-0.65	-0.68	0.89	0.07
1978	-0.06	-0.12	-0.12	-0.17	0.34	0.11
1979	-0.82	-0.93	-0.93	-1.05	0.78	0.23
1980	0.58	0.35	0.35	0.12	4.86	0.46
1981	-1.46	-1.50	-1.50	-1.54	0.94	0.09
1982	-0.70	-0.71	-0.71	-0.72	0.97	0.02
1983	0.17	0.14	0.14	0.12	1.35	0.04
1984	0.22	0.21	0.21	0.19	1.15	0.03
1985	1.68	1.66	1.66	1.63	1.03	0.05
1986	0.60	0.64	0.64	0.68	0.88	-0.08
1987	0.56	0.49	0.49	0.43	1.32	0.14
1988	1.00	0.98	0.98	0.95	1.05	0.05
1989	0.29	0.26	0.26	0.24	1.23	0.05
1990	-0.32	-0.35	-0.35	-0.38	0.83	0.06
1991	-0.34	-0.31	-0.31	-0.28	1.23	-0.06
1992	0.93	0.89	0.88	0.84	1.11	0.09
1993	0.94	0.94	0.94	0.94	1.00	0.00
1994	1.65	1.64	1.64	1.63	1.01	0.02
1995	1.20	1.20	1.20	1.21	0.99	-0.02
1996	-0.26	-0.26	-0.26	-0.26	1.00	0.00
1997	0.54	0.52	0.52	0.50	1.07	0.03
1998	-0.29	-0.30	-0.30	-0.30	0.97	0.01
1999	-0.08	-0.09	-0.09	-0.10	0.79	0.02
2000	0.73	0.63	0.62	0.53	1.36	0.19
2001	-0.31	-0.31	-0.31	-0.31	0.98	0.01
2002	-0.34	-0.34	-0.34	-0.35	0.96	0.01
2003	-0.42	-0.42	-0.42	-0.42	0.99	0.00

Figure 1. Technical-change measures based on GL and KB cost functions

All industries in the Italian economy

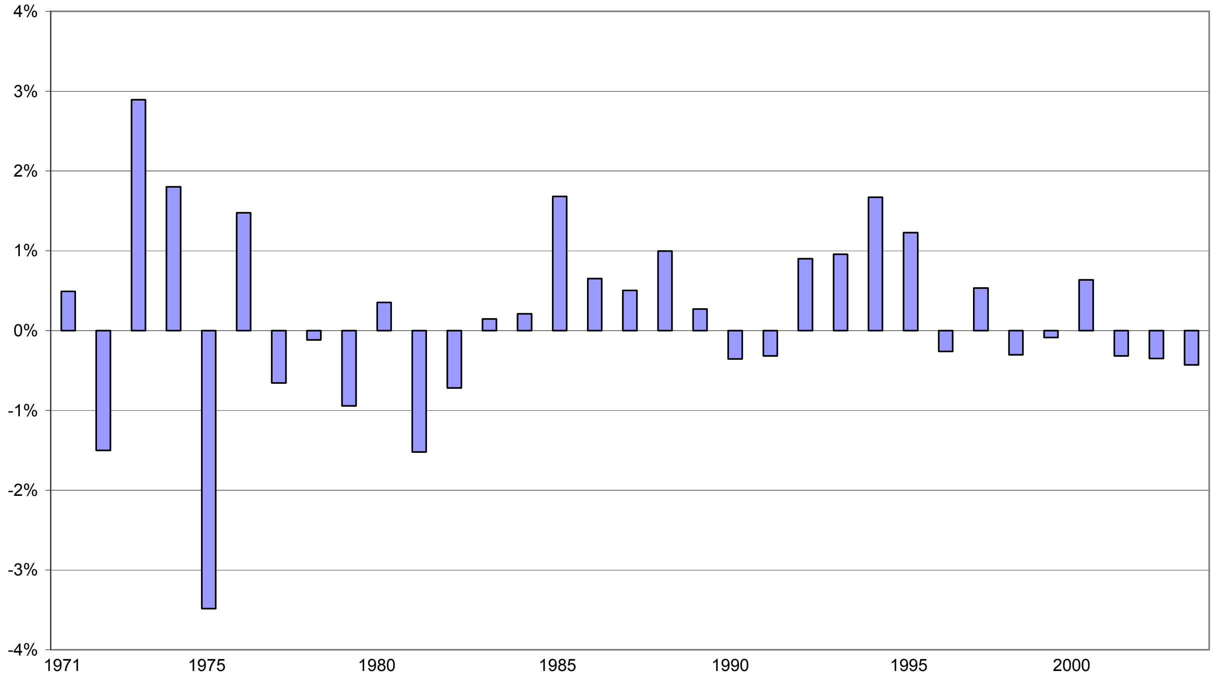


Figure 2. Technical-change measures based on GL and KB profit functions

All industries in the Italian economy

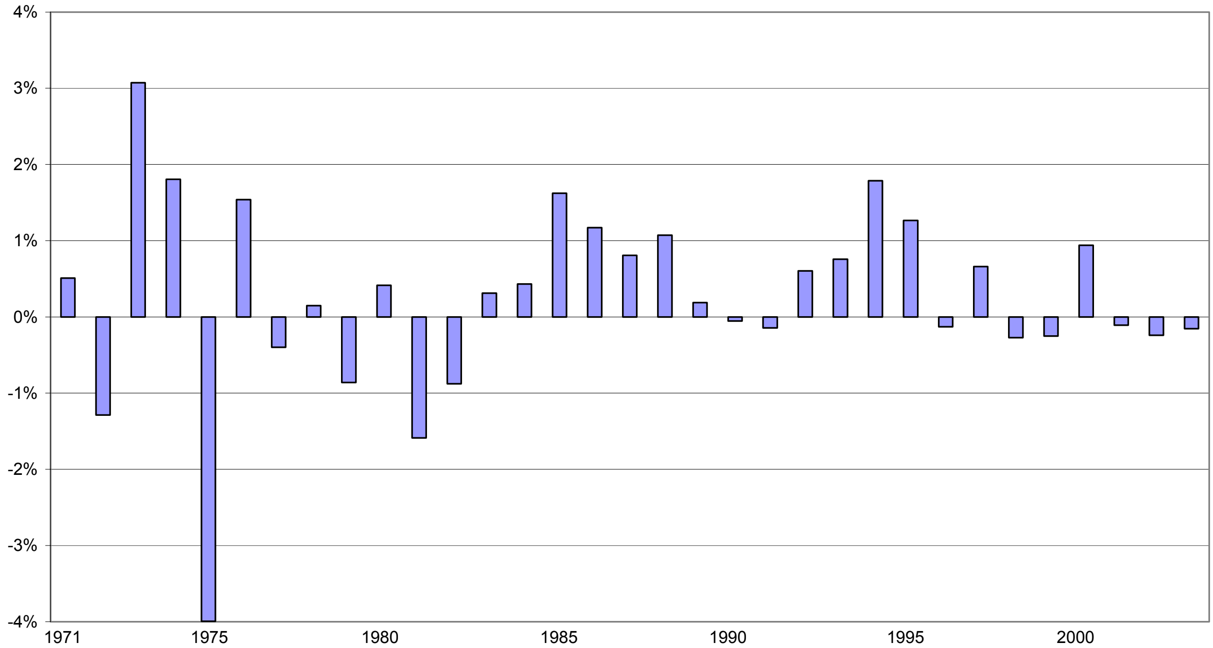


Figure 3. Relative differences bewteen GL (or KB) cost- and profit-based measures of TFP

All industries in the Italian economy

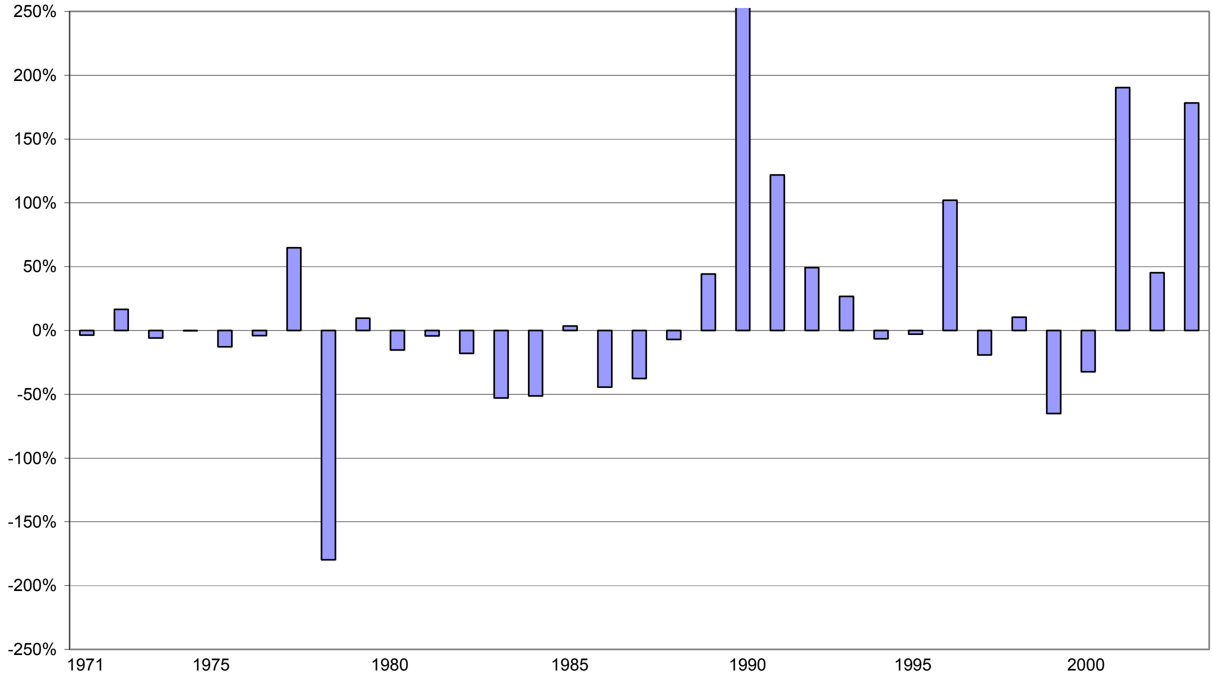


Figure 4. Measures of effects of TFP growth on real factor prices, based on the GL and KB cost functions

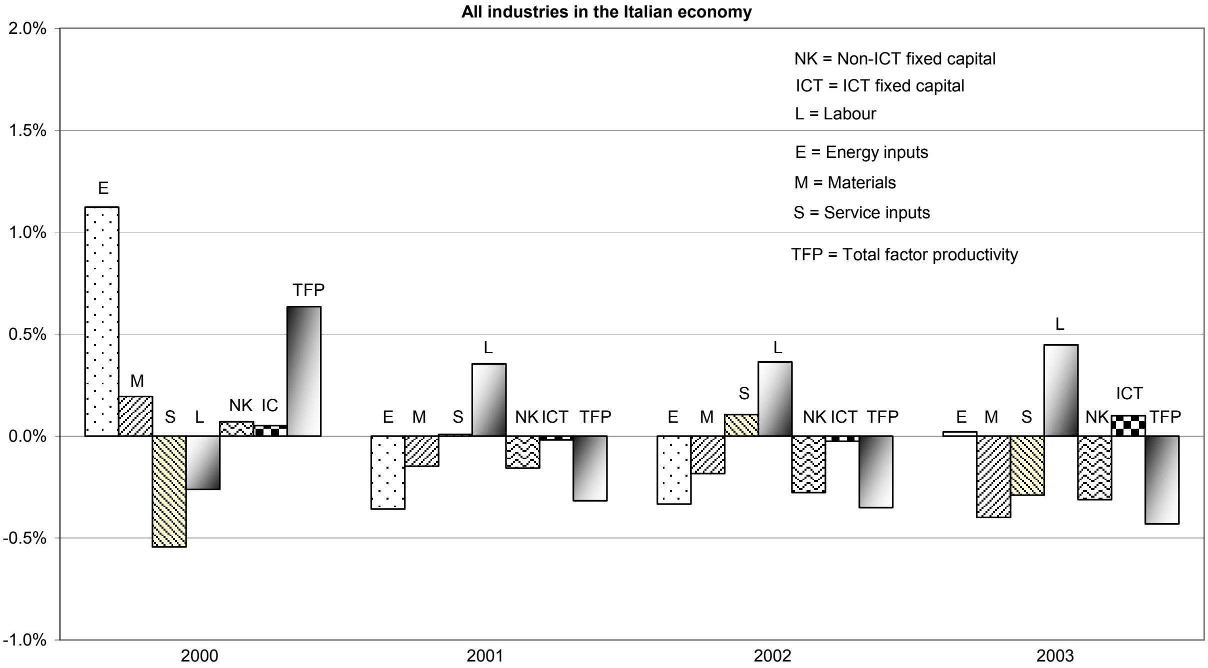
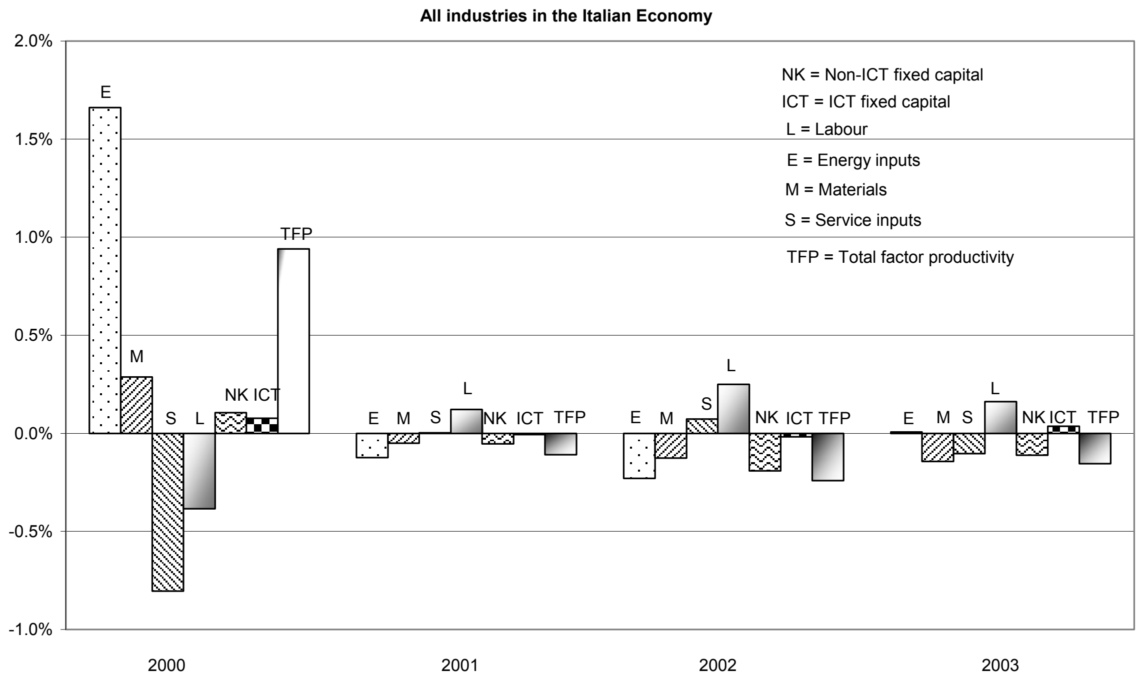


Figure 5. Measures of effects of TFP growth on real factor prices, based on the GL and KB profit functions



APPENDIX A

A list of alternative economic index numbers based on revenue and cost functions is presented in Table A1. These economic index numbers can be used to define some index number formulas that are "exact" for those functions and can be implemented using observed data. The revenue- and cost-based index numbers of quantities, prices, and productivity should be contrasted with the respective profit-based indicators developed in section 6.

Table A1. Alternative economic index numbers based on revenue and cost functions

Revenue-function based index numbers	Cost-function based index numbers
Nominal revenue (A1) $V_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1) \equiv \frac{R^1(\mathbf{p}^1,\mathbf{x}^1)}{R^0(\mathbf{p}^0,\mathbf{x}^0)}$	**Nominal cost** (A16) $V_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1) \equiv \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)}$
Output prices *Direct index number*: (A2) $P_R(\mathbf{p}^0,\mathbf{p}^1;\overline{\mathbf{x}}^r,\overline{T}^r) \equiv \frac{\overline{R}^r(\mathbf{p}^1,\overline{\mathbf{x}}^r)}{\overline{R}^r(\mathbf{p}^0,\overline{\mathbf{x}}^r)}$ *Implicit index number*: (A3) $\widetilde{P}_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\overline{\mathbf{p}}^r)$ $\equiv \frac{V_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1)}{Y_R(\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\overline{\mathbf{p}}^r)}$ $= \frac{R^1(\mathbf{p}^1,\mathbf{x}^1)}{R^0(\mathbf{p}^0,\mathbf{x}^0)} \Big/ \frac{R^1(\overline{\mathbf{p}}^r,\mathbf{x}^1)}{R^0(\overline{\mathbf{p}}^r,\mathbf{x}^0)}$, where Y_R is defined by (A4).	**Output prices** *Direct index number*: (A17) $P_C(\mathbf{w}^0,\mathbf{w}^1,T^0,T^1;\overline{\mathbf{y}}^r) \equiv \frac{C^1(\mathbf{w}^1,\overline{\mathbf{y}}^r)}{C^0(\mathbf{w}^0,\overline{\mathbf{y}}^r)}$ *Implicit index number*: (A18) $\widetilde{P}_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}^r)$ $\equiv \frac{V_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1)}{Y_C(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{w}}^r)}$ $= \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \Big/ \frac{\overline{C}^r(\overline{\mathbf{w}}^r,\mathbf{y}^1)}{\overline{C}^r(\overline{\mathbf{w}}^r,\mathbf{y}^0)} \cdot \frac{1}{RS_C}$ where Y_C is defined by (A19) and RS_C is defined by (A29).
Output quantities *Direct index number*: (A4) $Y_R(\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\overline{\mathbf{p}}^r) \equiv \frac{R^1(\overline{\mathbf{p}}^r,\mathbf{x}^1)}{R^0(\overline{\mathbf{p}}^r,\mathbf{x}^0)}$ *Implicit index number*: (A5) $\widetilde{Y}_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\overline{\mathbf{x}}^r,\overline{T}^r)$ $\equiv \frac{V_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1;T^0,T^1)}{P_R(\mathbf{p}^0,\mathbf{p}^1;\overline{\mathbf{x}}^r,\overline{T}^r)}$ $= \frac{R^1(\mathbf{p}^1,\mathbf{x}^1)}{R^0(\mathbf{p}^0,\mathbf{x}^0)} \Big/ \frac{\overline{R}^r(\mathbf{p}^1,\overline{\mathbf{x}}^r)}{\overline{R}^r(\mathbf{p}^0,\overline{\mathbf{x}}^r)}$	**Output quantities** *Direct index number*: (A19) $Y_C(\mathbf{y}^1,\mathbf{y}^0;\overline{\mathbf{w}}^r,\overline{T}^r) \equiv \frac{C^r(\overline{\mathbf{w}}^r,\mathbf{y}^1)}{C^r(\overline{\mathbf{w}}^r,\mathbf{y}^0)} \cdot RS_C$ where RS_C is the returns to scale effect measured by (A29). *Implicit index number*: (A20) $\widetilde{Y}_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\overline{\mathbf{y}}^r,\overline{T}^r)$ $\equiv \frac{V_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1)}{P_C(\mathbf{w}^0,\mathbf{w}^1;\overline{\mathbf{y}}^r,\overline{T}^r)}$ $= \frac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \Big/ \frac{C^1(\mathbf{w}^1,\overline{\mathbf{y}}^r)}{C^0(\mathbf{w}^0,\overline{\mathbf{y}}^r)}$

Table A1. (*continued*)

Revenue-function based index numbers	Cost-function based index numbers
Input prices *Direct index number*: (A6) $W_R(\mathbf{p}^0,\mathbf{p}^1;\bar{\mathbf{x}}^r,T^0,T^1) \equiv \dfrac{R^1(\mathbf{p}^1,\bar{\mathbf{x}}^r)}{R^0(\mathbf{p}^0,\bar{\mathbf{x}}^r)}$ *Implicit index number*: (A7) $\widetilde{W}_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\bar{\mathbf{p}}^r,\bar{T}^r)$ $\equiv \dfrac{V_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1;T^0,T^1)}{X_R(\mathbf{x}^0,\mathbf{x}^1;\bar{\mathbf{p}}^r,\bar{T}^r)}$ $= \dfrac{R^1(\mathbf{p}^1,\mathbf{x}^1)}{R^0(\mathbf{p}^0,\mathbf{x}^0)} \Big/ \dfrac{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^1)}{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^0)} \cdot RS_R$ where RS_R is defined by (A14).	**Input prices** *Direct index number*: (A21) $W_C(\mathbf{w}^1,\mathbf{w}^0;\bar{\mathbf{y}}^r,\bar{T}^r) \equiv \dfrac{\bar{C}^r(\mathbf{w}^1;\bar{\mathbf{y}}^r)}{\bar{C}^r(\mathbf{w}^0;\bar{\mathbf{y}}^r)}$ *Implicit index number*: (A22) $\widetilde{W}_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{w}}^r,\bar{T}^r)$ $\equiv \dfrac{V_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1)}{X_C(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{w}}^r)}$ $= \dfrac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \Big/ \dfrac{C^1(\bar{\mathbf{w}}^r,\mathbf{y}^1)}{C^0(\bar{\mathbf{w}}^r,\mathbf{y}^0)}$
Input quantities *Direct index number*: (A8) $X_R(\mathbf{x}^0,\mathbf{x}^1;\bar{\mathbf{p}}^r,\bar{T}^r) \equiv \dfrac{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^1)}{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^0)} \cdot \dfrac{1}{RS_R}$ where RS_R is the returns to scale effect given by (A14). *Implicit index number*: (A9) $\widetilde{X}_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\bar{\mathbf{x}}^r)$ $\equiv \dfrac{V_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1)}{W_R(\mathbf{p}^0,\mathbf{p}^1,T^0,T^1;\bar{\mathbf{x}}^r)}$ $= \dfrac{R^1(\mathbf{p}^1,\mathbf{x}^1)}{R^0(\mathbf{p}^0,\mathbf{x}^0)} \Big/ \dfrac{R^1(\mathbf{p}^1,\bar{\mathbf{x}}^r)}{R^0(\mathbf{p}^0,\bar{\mathbf{x}}^r)}$	**Input quantities** *Direct index number*: (A23) $X_C(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{w}}^r) \equiv \dfrac{C^1(\bar{\mathbf{w}}^r;\mathbf{y}^1)}{C^0(\bar{\mathbf{w}}^r;\mathbf{y}^0)}$ *Implicit index number*: (A24) $\widetilde{X}_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{y}}^r)$ $\equiv \dfrac{V_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1;T^0,T^1)}{W_C(\mathbf{w}^0,\mathbf{w}^1,T^0,T^1;\bar{\mathbf{y}}^r)}$ $= \dfrac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \Big/ \dfrac{\bar{C}^r(\mathbf{w}^1,\bar{\mathbf{y}}^r)}{\bar{C}^r(\mathbf{w}^0,\bar{\mathbf{y}}^r)}$
Technical change *Direct index number*: (A10) $TC_R\left(T^0,T^1;\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r\right) \equiv \dfrac{R^1(\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r)}{R^0(\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r)}$ *Implicit index number*: (A11) $T\widetilde{C}_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r,\bar{T}^r)$ $\equiv \dfrac{V_R(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1,T^0,T^1)}{P_R(\mathbf{p}^0,\mathbf{p}^1;\bar{\mathbf{x}}^r,\bar{T}^r)\cdot X_R(\mathbf{x}^0,\mathbf{x}^1;\bar{\mathbf{p}}^r,\bar{T}^r)}$ $= \dfrac{R^1(\mathbf{p}^1,\mathbf{x}^1)}{R^0(\mathbf{p}^0,\mathbf{x}^0)} \Big/ \left(\dfrac{\bar{R}^r(\mathbf{p}^1,\bar{\mathbf{x}}^r)}{\bar{R}^r(\mathbf{p}^0,\bar{\mathbf{x}}^r)} \cdot \dfrac{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^1)}{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^0)} \right)$	**Technical change** *Direct index number*: (A25) $TC_C(T^0,T^1;\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r) \equiv 1 \Big/ \dfrac{C^1(\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r)}{C^0(\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r)}$ *Implicit index number*: (A26) $T\widetilde{C}_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r,\bar{T}^r)$ $\equiv \dfrac{V_C(\mathbf{w}^0,\mathbf{w}^1,\mathbf{y}^0,\mathbf{y}^1;T^0,T^1)}{P_C(\mathbf{w}^0,\mathbf{w}^1;\bar{\mathbf{y}}^r,T^r)\cdot X_C(\mathbf{y}^0,\mathbf{y}^1;\bar{\mathbf{w}}^r,\bar{T}^r)}$ $= 1/ \dfrac{C^1(\mathbf{w}^1,\mathbf{y}^1)}{C^0(\mathbf{w}^0,\mathbf{y}^0)} \Big/ \left(\dfrac{\bar{C}^r(\mathbf{w}^1,\bar{\mathbf{y}}^r)}{\bar{C}^r(\mathbf{w}^0,\bar{\mathbf{y}}^r)} \cdot \dfrac{\bar{C}^r(\bar{\mathbf{w}}^r,\mathbf{y}^1)}{\bar{C}^r(\bar{\mathbf{w}}^r,\mathbf{y}^0)} \right)$

Table A1. (*continued*)

Revenue-function based index numbers	Cost-function based index numbers
Total factor productivity *Ratio of price index numbers*: (A12) $TFP_R^{W/P}(\mathbf{p}^0,\mathbf{p}^1,T^0,T^1;\bar{\mathbf{x}}^r,\bar{T}^r)$ $\equiv \frac{W_R(\mathbf{p}^0,\mathbf{p}^1,T^0,T^1;\bar{\mathbf{x}}^r)}{P_R(\mathbf{p}^0,\mathbf{p}^1;\bar{\mathbf{x}}^r,\bar{T}^r)}$ $= \frac{R^1(\mathbf{p}^1,\bar{\mathbf{x}}^r)}{R^0(\mathbf{p}^0,\bar{\mathbf{x}}^r)} \Big/ \frac{\bar{R}^r(\mathbf{p}^1,\bar{\mathbf{x}}^r)}{\bar{R}^r(\mathbf{p}^0,\bar{\mathbf{x}}^r)}$ *Ratio of quantity index numbers*: (A13) $TFP_R^{Y/X}(\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\bar{\mathbf{p}}^r,\bar{T}^r)$ $\equiv \frac{Y_R(\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\bar{\mathbf{p}}^r)}{X_R(\mathbf{x}^0,\mathbf{x}^1;\bar{\mathbf{p}}^r,\bar{T}^r)}$ $= \frac{R^1(\bar{\mathbf{p}}^r,\mathbf{x}^1)}{R^0(\bar{\mathbf{p}}^r,\mathbf{x}^0)} / \left(\frac{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^1)}{\bar{R}^r(\bar{\mathbf{p}}^r,\mathbf{x}^0)}\cdot\frac{1}{RS_R}\right)$ where RS_R is defined by (A14).	**Total factor productivity** *Ratio of price index numbers*: (A27) $TFP_C^{W/P}(\mathbf{w}^0,\mathbf{w}^1,T^0,T^1;\bar{\mathbf{y}}^r,\bar{T}^r)$ $\equiv \frac{W_C(\mathbf{w}^0,\mathbf{w}^1;\bar{\mathbf{y}}^r,\bar{T}^r)}{P_C(\mathbf{w}^0,\mathbf{w}^1,T^0,T^1;\bar{\mathbf{y}}^r)}$ $= \frac{\bar{C}^r(\mathbf{w}^1,\bar{\mathbf{y}}^r)}{\bar{C}^r(\mathbf{w}^0,\bar{\mathbf{y}}^r)} \Big/ \frac{C^1(\mathbf{w}^1,\bar{\mathbf{y}}^r)}{C^0(\mathbf{w}^0,\bar{\mathbf{y}}^r)}$ *Ratio of quantity index numbers*: (A28) $TFP_C^{Y/X}(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{w}}^r,\bar{T}^r)$ $\equiv \frac{Y_C(\mathbf{y}^0,\mathbf{y}^1;\bar{\mathbf{w}}^r,T^r)}{X_C(\mathbf{y}^0,\mathbf{y}^1;\bar{\mathbf{w}}^r,T^0,T^1)}$ $= \left(\frac{C^r(\bar{\mathbf{w}}^r,\mathbf{y}^1)}{C^r(\bar{\mathbf{w}}^r,\mathbf{y}^0)}\cdot RS_C\right) / \frac{C^1(\bar{\mathbf{w}}^r,\mathbf{y}^1)}{C^0(\bar{\mathbf{w}}^r,\mathbf{y}^0)}$ where RS_C is defined by (A29).
Effects of returns to scale on TFP *Ratio of price-based TFP to TC*: (A14) $RS_R^{W/P}(\mathbf{p}^0,\mathbf{p}^1,T^0,T^1;\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r,\bar{T}^r)$ $\equiv \frac{TFP_R(\mathbf{p}^0,\mathbf{p}^1,T^0,T^1;\bar{\mathbf{x}}^r,\bar{T}^r)}{TC_R\left(T^0,T^1;\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r\right)}$ $= \left(\frac{R^1(\mathbf{p}^1,\bar{\mathbf{x}}^r)}{R^0(\mathbf{p}^0,\bar{\mathbf{x}}^r)} \Big/ \frac{\bar{R}^r(\mathbf{p}^1,\bar{\mathbf{x}}^r)}{\bar{R}^r(\mathbf{p}^0,\bar{\mathbf{x}}^r)}\right) \Big/ \frac{R^1(\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r)}{R^0(\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r)}$ *Ratio of quantity-based TFP to TC*: (A15) $RS_R^{Y/X}(\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r,\bar{T}^r)$ $= \frac{TFP_R^{Y/X}(\mathbf{x}^0,\mathbf{x}^1,T^0,T^1;\bar{\mathbf{p}}^r,\bar{T}^r)}{TC_R\left(T^0,T^1;\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r\right)}$ $= TFP_R^{Y/X} \Big/ \frac{R^1(\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r)}{R^0(\bar{\mathbf{p}}^r,\bar{\mathbf{x}}^r)}$	**Effects of returns to scale on TFP** (*) *Ratio of price-based TFP to TC*: (A29) $RS_C^{W/P}(\mathbf{w}^0,\mathbf{w}^1,T^0,T^1;\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r,\bar{T}^r)$ $\equiv \frac{TFP_C(\mathbf{w}^0,\mathbf{w}^1,T^0,T^1;\bar{\mathbf{y}}^r,\bar{T}^r)}{TC_C\left(T^0,T^1;\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r\right)}$ $= \left(\frac{\bar{C}^r(\mathbf{w}^1,\bar{\mathbf{y}}^r)}{\bar{C}^r(\mathbf{w}^0,\bar{\mathbf{y}}^r)} \Big/ \frac{C^1(\mathbf{w}^1,\bar{\mathbf{y}}^r)}{C^0(\mathbf{w}^0,\bar{\mathbf{y}}^r)}\right) \Big/ \frac{C^1(\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r)}{C^0(\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r)}$ *Ratio of quantity-based TFP to TC*: (A30) $RS_C^{Y/X}(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{w}}^r,\bar{\mathbf{x}}^r,\bar{T}^r)$ $= \frac{TFP_C^{Y/X}(\mathbf{y}^0,\mathbf{y}^1,T^0,T^1;\bar{\mathbf{w}}^r,\bar{T}^r)}{TC_C\left(T^0,T^1;\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r\right)}$ $= TFP_C^{Y/X}\cdot\frac{C^1(\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r)}{C^0(\bar{\mathbf{w}}^r,\bar{\mathbf{y}}^r)}$ (*) The *degree* of returns to scale, when defined in terms of the cost function, can be measured as inverse *cost flexibility* as follows, in the one-output case (see, for example, Nelson, 1998, p.7, and Milana, 2005, Appendix A): $\frac{C^t(\mathbf{w},y)}{(\partial C^t/\partial y)y} = \frac{\sum_{i=1}^{N}(\partial f/\partial x_i)x_i}{f}$ which is equal to 1 with constant returns to scale.

Legenda: $\mathbf{x}^0$: input quantities at period 0; $\mathbf{x}^1$: input quantities at period 1; $\bar{\mathbf{x}}^r$: reference input quantities; $\mathbf{w}^0$: input prices at period 0; $\mathbf{w}^1$: input prices at period 1; $\bar{\mathbf{w}}^r$: reference input prices;

$\mathbf{y}^0$: output quantities at period 0; $\mathbf{y}^1$: output quantities at period 1; $\bar{\mathbf{y}}^r$: reference output quantities ; $\mathbf{p}^0$: output prices at period 0; $\mathbf{p}^1$: output prices at period 1; $\bar{\mathbf{p}}^r$: reference output prices;

T^0 = technology at period 0; T^1 = technology at period 1; $\bar{T}^r$ = reference technology.

References

Afriat, S. N. (1977), *The Price Index*, Cambridge, U.K., Cambridge University Press.

Allen, R.G.D. (1949), "The Economic Theory of Index Numbers", *Economica*, *N.S.* 16: 197-203.

Allen, R.G.D. (1963), "Price Index Numbers", *Review of the International Statistical Institute* 31: 281-297.

Allen, R.G.D. (1975), *Index Numbers in Theory and Practice*, London, Macmillan.

Archibald, R. (1975), "On the Theory of Industrial Price Measurement: Input Price Indexes", *Annals of Economic and Social Measurement* 6: 57-72.

Arrow, Kenneth J. (1958), "The Measurement of Price Changes", in U.S. Congress, Joint Economic Committee, *The Relationship of Prices to Economic Stability and Growth*, Washington, D.C., Government Printing Office, pp. 77-88.

Arrow, Kenneth J. (1974), "The Measurement of Real Value Added", in P.A. David and M. W. Reder (eds.), *Nations and Households in Economic Growth*, New York, Academic Press, pp. 3-19.

Balk, Bert M. (1998), *Industrial Price, Quantity, and Productivity Indices*. *The Micro-Economic Theory and An Application*, Boston, Dordrecht, London, Kluwer Academic Publishers.

Balk, Bert M.; Rolf Färe; and Shawna Grosskopf (2000), "The Theory of Economic Price and Quantity Indicators", mimeo available at *http:// people.fbk.eur.nl/bbalk/personal/ publications/2001e.pdf.*

Basu, Susanto and John G. Fernald (1997), "Returns to Scale in U.S. Production: Estimates and Implications", *Journal of Political Economy* 105: 249-283.

Berndt, Ernst R. and Laurits R. Christensen (1973), "The Internal Structure of Functional Relationship: Separability, Substitution, and Aggregation", *Review of Economic Studies* 40(3): 403-410.

Berndt, Ernst R. and Mohammed S. Khaled (1979), "Parametric Productivity Measurement and Choice of Flexible Functional Forms", *Journal of Political Economy* 87: 1220-1245.

Blackorby, Charles and R. Robert Russell (1977), "Indices and Subindices of the Cost of Living and the Standard of Living", International Economic Review 18: 229-240.

Blackorby, Charles; Daniel Primont; and R. Robert Russell (1978), *Duality, Separability, and Functional Structure: Theory and Economic Applications*, Amsterdam, North-Holland Publishing Co.

Blackorby, Charles and William E. Schworm (1988), "The Existence of Input and Output Aggregates in Aggregate Production Functions", *Econometrica* 56: 613-643.

Blackorby, Charles; William E. Schworm; and Timothy C.G. Fisher (1986), "Testing for the Existence of Input Aggregates in an Economy Production Function", University of British Columbia, Department of Economics, Discussion paper 86-26, Vancouver, B.C., Canada.

Bliss, C. J. (1975), *Capital Theory and the Distribution of Income*, New York, North-Holland Publ. Co./American Elsevier.

Caves, Douglas W.; Laurits R. Christensen; and Joseph A. Swanson (1981), "Productivity Growth, Scale Economies and Capacity Utilization in U.S. Railroads, 1955-74", *American Economic Review* 71: 994-1002.

Caves, Douglas W.; Laurits R. Christensen; and W. Erwin Diewert (1982), "The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity, *Econometrica* 50: 1393-1414.

Chambers, Robert G. (1988), *Applied Production Analysis*, Cambridge, U.K., Cambridge University Press.

Debreu, G. (1959), *Theory of Value*, New York, Wiley.

Denny, Michael (1980), "Comment on Aggregation Problems in the Measurement of Capital", in D. Usher (ed. by), *The Measurement of Capital*, Studies in Income and Wealth, Vol. 45, National Bureau of Economic Research, Chicago, University of Chicago Press, pp. 528-538.

Diewert, W. Erwin (1971), "An Application of the Shepard's Duality Theorem: A Generalized Leontief Production Function", *Journal of Political Economy* 79: 481-507.

Diewert, W. Erwin (1973), "Functional Forms for Profit and Transformation Functions", *Journal of Economic Theory* 6: 284-316.

Diewert, W. Erwin (1974), "Applications of Duality Theory", in M. Intriligator and D.A. Kendrick (eds.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam, North-Holland Publ. Co., pp. 106-206.

Diewert, W. Erwin (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4: 115-145.

Diewert, W. Erwin (1980), "Aggregation Problems in the Measurement of Capital", in D. Usher (ed. by), *The Measurement of Capital*, Studies in Income and Wealth, Vol. 45, National Bureau of Economic Research, Chicago, University of Chicago Press, pp. 433-528.

Diewert, W. Erwin (1981), "The Economic Theory of Index Numbers: A Survey", in A. Deaton (ed. by), *Essays in the Theory and Measurement of Consumer Behavior in Honor of Sir Richard Stone*, Cambridge, U.K., Cambridge University Press, pp. 163-208.

Diewert, W. Erwin (1983a), "The Theory of the Cost-of-Living Index and Measurement of Welfare Change", in W.E. Diewert and C. Montmarquette (eds.), *Price Level*

Measurement: Proceedings from a Conference Sponsored by Statistics Canada, Ottawa, Minister of Supply and Services Canada, pp. 163-233.

Diewert, W. Erwin (1983a), "The Theory of the Output Price Index and the Measurement of the Real Output Change", in W.E. Diewert and C. Montmarquette (eds.), *Price Level Measurement: Proceedings from a Conference Sponsored by Statistics Canada*, Ottawa, Minister of Supply and Services Canada, pp. 1049-1113..

Diewert, W. Erwin (1987), "Index Numbers", in J. Eatwell, M. Milgate, and P. Newman (eds.), *The New Palgrave: A Dictionary of Economics*, Vol. 2, The Macmillan Press, pp. 767-780.

Diewert, W. Erwin (1993), "The Early History of Price Index Research", in W.E. Diewert and A.O. Nakamura (eds.), *Essays in Index Number Theory*, Vol. I, Amsterdam, North-Holland Publ. Co./Elsevier Science Publishers B.V., pp. 33-65.

Diewert, W. Erwin (1998), "Index Number Theory Using Differences Rather Than Ratios", University of British Columbia, Department of Economics, Discussion Paper No. 98-10, Vancouver, B.C., Canada (published in *American Journal of Economics and Sociology* 64 (2005): 311-360).

Diewert, W. Erwin (2000a), "Productivity Measurement Using Differences Rather Than Ratios: A Note", University of New South Wales, School of Economics, Discussion Paper 2000/1, Sydney, NSW, Australia.

Diewert, W. Erwin (2000b), "The Quadratic Approximation Lemma and Decompositions of Superlative Indexes", University of British Columbia, Department of Economics, Discussion Paper No. 00-15 (published in *Journal of Economic and Social Measurement* 28 (2002): 63-88).

Diewert, W. Erwin and Kevin J. Fox (2005), "Malmquist and Törnqvist Productivity Indexes: Returns to Scale and Technical Progress with Imperfect Competition", Centre for Applied Economic Research, Working Paper 2005/03, The University of New South Wales, School of Economics, Sydney, Australia.

Diewert, W. Erwin and Alice O. Nakamura (2002), "The Measurement of Aggregate Total Factor Productivity Growth", in J.J. Heckman and E. Leamer (eds.), *Handbook of Econometric Methods,* Amsterdam, North-Holland Publ. Co.

Diewert, W. Erwin and Alice O. Nakamura (2003), "Index Number Concepts, Measures and Decomposition of Productivity Growth" *Journal of Productivity Analysis* 19: 127-159.

Diewert, W. Erwin, M. B. Reinsdorf, and C. Ehemann (2000), "Additive Decompositions for Fisher, Törnqvist and Geometric Mean Indexes", University of British Columbia, Department of Economics, Discussion Paper No. 01-01 (published in *Journal of Economic and Social Measurement* 28 (2002): 51-61).

Färe, Rolf and Daniel Primont (1995), *Multi-Output Production and Duality: Theory and Applications*, Norwell, MA, Kluwer Academic Publishers.

Farrell, M.J. (1957), "The Measurement of Productivity Efficiency", *Journal of the Royal Statistical Society Series A*, 120: 253-90.

Fenchel, W. (1949), "On Conjugate Convex Functions", *Canadian Journal of Mathematics* 1: 73-77.

Fenchel, W. (1953), *Convex Cones, Sets, and Functions*. Department of Mathematics, Princeton University, Princeton, NJ.

Fisher, Irving (1911), *The Purchasing Power of Money*. London, Macmillan.

Fisher, Iring (1922), *The Making of Index Numbers*, Boston, MA, Houghton-Mifflin.

Fisher, Franklin M. (1988), "Production-Theoretic Input Price Indices and the Measurement of Real Aggregate Input Use", in W. Eichhorn (ed. by), *Measurement in Economics*, Heidelberg, Physica-Verlag, pp. 87-98.

Fisher, Franklin M. (1995), "The Production-Theoretic Measurement of Input Price and Quantity

Indices”, *Journal of Econometrics 65*: 155-174.

Fisher, Franklin M. and K. Shell (1972), *The Economic Theory of Price Indices*, New York, Academic Press.

Fisher, Franklin M. and Karl Shell (1998), *Economic Analysis of Production Price Indexes*, Cambridge, U.K., Cambridge University Press.

Fox, Kevin J. (2005), “Returns to Scale, Technical Progress and Total Factor Productivity Growth in New Zealand Industries”, New Zealand Treasury, Working Paper 05/04, Wellington, New Zealand (available at http://www.treasury.govtnz).

Frisch, R. (1936), "Annual Survey of General Economic Theory: The Problem of Index Numbers", *Econometrica* 4: 1-39.

Funke, H. (1988), “Mean Value Properties of the Weights of Linear Price Indices”, in W. Eichhorn (ed. by), *Measurement in Economics: Theory and Applications of Economic Indices*, Heidelberg, Physica-Verlag, pp. 99-115.

Gorman, W. M. (1959), “Separable Utility and Aggregation”, *Econometrica* 27: 469-481.

Gorman, W. M. (1968), "The Structure of Utility Functions", *Review of Economic Studies* 35: 369-390.

Green, H. A. J. (1964), *Aggregation in Economic Analysis: An Introductory Survey*, Princeton, Princeton University Press.

Griliches, Zvi (1963), “The Sources of Measured Productivity Growth: United States Agriculture, 1940-1960”, *Journal of Political Economy* 71: 331-346.

Hasenkamp, G. (1973), *Specification and Estimation of Multiple Output Production Functions*, Ph. D. Dissertation, University of Wisconsin at Madison.

Hicks, J. R. (1958), “The Measurement of Real Income”, *Oxford Economic Papers* 10: 125-162.

Hill, Robert (2006a), "Superlative Index Numbers: Not All of Them Are Super", *Journal of Econometrics*, 103: 25-43.

Hill, Robert (2006b), "When Does Chaining Reduce the Paasche-Laspeyres Spread? An Application to Scanner Data", *Review of Income and Wealth* 52: 309-325.

Hotelling, H. (1935), "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions", *Journal of Political Economy* 40: 577-616.

Jacobsen, S.E. (1970), "Production Correspondences", *Econometrica* 38: 754-770.

Jacobsen, S. E. (1972), "On the Shephard's Duality Theorem", *Journal of Economic Theory* 4: 458-464.

Jorgenson, Dale W. and Barbara M. Fraumeni (1981), "Relative Prices and Technical Change", in Ernst R. Berndt and Barry C. Field (eds.), *Modelling and Measuring Natural Resource Substitution*, Cambridge, MA, The MIT Press, pp. 17-47.

Jorgenson, Dale W., Frank M. Gollop, and Barbara M. Fraumeni (1987), *Productivity and U.S. Economic Growth*, Cambridge, MA, Harvard University Press.

Jorgenson, Dale W. and Lawrence J. Lau (1974a), "The Duality of Technology and Economic Behavior", *Review of Economic Studies* 41: 181-200.

Jorgenson, Dale W. and Lawrence J. Lau (1974b), "Duality and Differentiability in Production", *Journal of Economic Theory* 9: 23-42.

Kadiyala, K.R. (1971-72), "Production Functions and the Elasticity of Substitution", *Southern Economic Journal* 38: 281-284.

Lau, Lawrence J. (1969), "Duality and the Structure of Utility Functions", *Journal of Economic Theory* 1(4): 374-396.

Lau, Lawrence J. (1972), "A Note on Separable Cost Functions", mimeo.

Lau, Lawrence J. (1974) "Comments on Diewert's 'Applications of Duality Theory'", in M. Intriligator and D.A. Kendrick (eds.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam, North-Holland Publ. Co., pp. 176-99.

Lau, Lawrence J. (1978) "Applications of Profit Functions", in M. Fuss and D. McFadden (eds.), *Production Economics: A Dual Approach to Theory and Applications.* Vol. I, Amsterdam, North-Holland Publ. Co., pp. 133-216..

Lau, Lawrence J. (1979), "On Exact Index Numbers", *Review of Economics and Statistics* 61: 73-82.

Leontief, Wassily W. (1936), "Composite Commodities and the Problem of Index Numbers", *Econometrica* 4: 39-59.

Leontief, Wassily W. (1947a), "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives", *Bulletin of the American Mathematical Society* 53: 343-350.

Leontief, Wassily W. (1947b), "Introduction to a Theory of the Internal Structure of Functional Relationships", *Econometrica* 15: 361-373.

Luenberger, David G. (1995), *Microeconomic Theory*, New York, McGraw-Hill, Inc.

McFadden, Daniel (1966), "Cost, Revenue, and Profit Functions: A Cursory Review", Working Paper No. 86, IBER, Department of Economics, University of California, Berkely.

McFadden, Daniel (1978a), "Cost, Revenue, and Profit Functions", in M. Fuss and D. McFadden (eds.), *Production Economics: A Dual Approach to Theory and Applications.* Vol. I, Amsterdam, North-Holland Publ. Co., pp. 3-109.

McFadden, Daniel (1978b), "The General Linear Profit Function", in M. Fuss and D. McFadden (eds.), *Production Economics: A Dual Approach to Theory and Applications.* Vol. I, Amsterdam, North-Holland Publ. Co., pp. 269-286.

Milana, Carlo (1993), "Numeri indici", in *Enciclopedia Italiana*, Istituto dell'Enciclopedia Italiana founded by Giovanni Treccani, Rome, 5th Appendix (1979-1992), pp. 704-709.

Milana, Carlo (2000), "Review of *Economic Analysis of Production Price Indexes* by Franklin M. Fisher and Karl Shell", *Economic Systems Research* 12: 433-436.

Milana, Carlo (2001), "The Input-Output Structural Decomposition Analysis of 'Flexible' Production Systems", in Michael L. Lahr and Erik Dientzenbacher (eds.), *Input-Output Analysis: Frontiers and Extensions*, New York, Palgrave, pp. 349-80.

Milana, C. (2005), "The Theory of Exact and Superlative Index Numbers Revisited", EUKLEMS Project Working Paper No. 3, http://www.euklems.net.

Nakajima, Takanobu, Alice Nakamura, and Masao Nakamura (2002), "Technical Progress and Returns to Scale in Japanese Manufacturing Industries Before and After the Burst of the 1990 Financial Bubble", mimeo.

Nelson, Carl H. (1998), "Returns to Scale, Homogeneity, and Homotheticity", Department of Agricultural and Consumer Economics, University of Illinois, Urbana, IL, mimeo.

Park, Seung-Rok and Jene K. Kwon (1995), "Rapid Economic Growth with Increasing Returns to Scale and Little or No Productivity Growth", *Review of Economics and Statistics* 77: 332-351.

Rockafellar, R.T. (1970), "Conjugacy Convex Functions in Optimal Control and the Calculus of Variations", *Journal of Mathematical Analysis and Applications* 32: 411-427.

Russell, R. Robert, (1983), "Comments on Diewert's 'The Theory of the Cost-of-Living Index and Measurement of Welfare Change'", in W.E. Diewert and C. Montmarquette (eds.), *Price Level Measurement: Proceedings from a Conference Sponsored by Statistics Canada*, Ottawa, Minister of Supply and Services Canada, pp. 234-239.

Samuelson, Paul A. (1947), *Foundations of Economic Analysis,* Cambridge, MA, The President and Fellows of Harvard College.

Samuelson, Paul A. (1950), "Evaluation of Real National Income", *Oxford Economic Papers* 2: 1-29.

Samuelson, Paul A. (1953-54), "Prices of Factors and Goods in General Equilibrium", *Review of Economic Studies* 21: 1-20.

Samuelson, Paul A. and S. Swamy (1974), "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis", *American Economic Review* 64: 566-593.

Sato, R. (1976), "The Meaning and Measurement of the Real Value Added Index", *Review of Economics and Statistics* 58: 434-442.

Shephard, Ronald W. (1953), *Cost and Production Functions*, Princeton, Princeton University Press.

Shephard, Ronald W. (1970), *The Theory of Cost and Production Functions*, Princeton, Princeton University Press.

Shephard, Ronald W. (1974), "Comments on E. W. Diewert, 'Applications of Duality Theory'", in M.D. Intriligator and D.A. Kendrick (1974), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam, North-Holland Publ. Co., pp. 200-206.

Sono, M. (1945), "The Effect of Price Changes on the Demand and Supply of Separable Goods" (In Japanese), *Kokumin Keisai Zasshi* 74: 1-51.

Sono, M. (1961), "The Effect of Price Changes on the Demand and Supply of Separable Goods", *International Economic Review* 2: 239-271.

Stigum, B.P. (1967), "On certain Problems of Aggregation", *International Economic Review* 8: 349-367.

Strotz, R. (1959), "The Utility Tree: A Correction and Further Appraisal", *Econometrica* 27: 482-488.

Swamy, S. (1985), "Theoretical Aspects of Index Numbers", Harvard Institute for Economic Research Discussion Paper no. 1192, Boston, MA, Harvard University.

Takayama, Akira (1974), "On Biased Technological Progress", *American Economic Review* 64: 631-639.

Uzawa, H. (1964), "Duality Principles in the Theory of Cost and Production", *International Economic Review* 5: 216-220.

Vogt, A. (1980), "Der Zeit und der Factorumkehrtest als 'Finders of Test'" *Statistiche Hefte* 21: 66-71.

Westfield, Fred M. (1966), "Technical Progress and Returns to Scale", *Review of Economics and Statistics* 48: 432-441.

Wold, H. (1943, 1944), "A Synthesis of Pure Demand Analysis", *Skandinavisk Aktuarietidskrift* 26: 85-144 and 220-275; 27: 69-120.

Young, Allyn A. (1928), "Increasing Returns and Economic Progress", *Economic Journal* 38: 527-542.