Session Number 4B
Session Title: Econometric issues in Multidimentional Measurement and Comparison of Economic Well-being
Session Organizers:Garner, Duclos Osberg.
Discussant: Maria Ana Lugo

Paper Prepared for the 28[th] General Conference of
The International Association for Research in Income and Wealth
Cork, Ireland, August 22 - 28, 2004

Indices and Tests for Multidimensional Inequality: Multivariate Generalizations of the Gini Coefficient and Kolmogorov-Smirnov Two Sample Test.

Gordon Anderson

For additional information please contact:
Gordon Anderson
anderson@chass.utoronto.ca

Introduction

There are many circumstances in which the appropriate comparison instruments for evaluating welfare and inequality are many in number. The general (and diverse) functionings and capabilities approach of Sen (1995) and the lifetime wealth approach of Atkinson (1983) are examples. The Atkinson (1983) example depends not only on an agents income stream, but also her life span, the rate at which she discounts happiness through time and the rate at which income may be exchanged through time. Unfortunately, since the appropriate function of these many instruments is not directly known or observable, recourse is usually made to unidimensional comparison tests or indices (usually transformations of annual income or consumption measures are used as proxies for lifetime wealth) in order to make statements regarding the progress of welfare and inequality. Much may be lost in ignoring the multidimensional nature of the issue by abstracting from the discounting and life expectancy aspects of lifetime economic wealth for example, and the progress of welfare and inequality may be misconstrued as a consequence.

Multivariate welfare indices have already been developed (Tsui(1995) and Maasoumi(1993)) though the most popular inequality index, the Gini coefficient, has not been extended to the multivariate environment. Extension of univariate empirical welfare comparison stochastic dominance techniques to many dimensions has been achieved in theory (Atkinson and Bourguignon (1987)) but has yet to be implemented statistically in practice. In the univariate domain two types of omnibus statistical comparison instrument exist, tests comparing differences between two functions at a sequence of points (Anderson(1996), Davidson and Duclos(1997),(2000), Formby et.al. (2000)) and tests examining the maximum distance between two functions (Barrett and Donald(2003), Linton et.al.(2002)). The extension of these classes of tests to a multi-dimensional environment is problematic in practice. In the former case whilst the distribution of the statistic is well defined and understood, its calculation runs into the problem of the curse of dimensionality familiar in non-parametric statistics. In the latter case the statistic ceases to have a well defined non-parametric distribution (in the sense that its distribution now depends on the nature of the probability distributions underlaying the data) though it does not appear to run into the curse of dimensionality problem.

This paper introduces and employs multidimensional extensions to existing comparison techniques to examine whether or not ignoring multidimensional aspects result in misleading inferences. The techniques are employed in the context of a sample of nations, in essence each country in the sample is represented by an agent characterized by the per capita GNP of that country, the GNP growth rate of that country and the average life expectancy in that country. Multidimensional techniques lead to substantially different conclusions from those drawn from the use of unidimensional measures. Section 1 introduces a variety of Multidimensional Gini Coefficients, including extensions to the family of polarization indices introduced in Esteban and Ray (1994), and compares them with existing multidimensional inequality indices. Tests for the comparison of multidimensional distributions in small samples are introduced in section 2 and Section 3 outlines the theoretical basis for multidimensional comparisons in the application. Section 4 reports the results and some conclusions are drawn in section 5.

1. The Multidimensional Gini Inequality Index.

Popularized by Dalton (1920), the GINI coefficient of inequality (Gini (1912)) has been, and is still, the most extensively employed statistic in studies of income diversity[1]. In terms of a sample of n agents indexed i = 1 to n with incomes $x_i > 0$, the GINI coefficient is usually written as:

$$GINI = \frac{1}{2\bar{x}n^2}\sum_{i=1}^{n}\sum_{i=1}^{n}|x_i - x_j|$$

where:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}x_i$$

---

[1] This in spite of little theoretical support for its use (Atkinson (1970), Newberry (1970) but see Sheshinski (1972))

The popularity of the Gini coefficient is due in large part to its intuitive appeal as the average distance between all pairs of mean standardized incomes and to its relationship with the Lorenz curve which maps the share of aggregate income held by a given proportion of the population to that population proportion. As the ratio of the area between the Lorenz curve and the complete equality (45°) line to the area below the 45° line, the Gini coefficient is a number bounded between 0 and 1 whose closer proximity to 1 indicates greater inequality. Extending it measuring inequality amongst agents characterized in more than one dimension is facilitated by writing it in equivalent form as:

$$GINI = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{i=1}^{n}\sqrt{\left(\frac{x_i - x_j}{\bar{x}}\right)^2}$$

and interpreting it as the average distance between all pairs of mean standardized one dimensional vectors of incomes. Let the k'th characteristic (k=1,..,K) of the i'th agent be x_{ik} (assume $x_{ik} \geq 0$ for all i and k with the strict inequality holding for at least one i for every k). GINIM, the Gini coefficient for a sample of n agents characterized by many (K) characteristics, is given by:

$$GINIM = \frac{1}{2K^{0.5}n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{\sum_{k=1}^{K}\left(\frac{x_{ik} - x_{jk}}{\bar{x}_k}\right)^2}$$

where:

$$\bar{x}_k = \frac{1}{n}\sum_{i=1}^{n} x_{ik} \quad k=1,..,K$$

Obviously GINI can be seen as a special case where K=1.

Existing Multidimensional Inequality Measures.

Dissatisfaction with the Gini (arising from the lack of a social welfare function rationalizing its use (see Atkinson (1970), Newberry (1970)) and the fact that information theory suggests other forms of measures (Maasoumi (1993))) prompted development of "Ethically Based" and Information Theory based inequality measures. Multidimensional generalizations of these measures have already been developed. By postulating a set of "desirable" axioms that a social evaluation function should satisfy, and properties that an inequality index should possess, Tsui (1995) provides Relative and Absolute multidimensional measures which are generalizations of the Atkinson(1970) - Kolm (1966) - Sen (1997) "Ethically Based" family[2]. The relative measures are of the following form:

$$TR1 = 1 - \left(\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{K}\left(\frac{x_{ik}}{\mu_k}\right)^{r_k}\right)^{\frac{1}{\Sigma r_k}}$$

$$TR2 = 1 - \prod_{i=1}^{n}\left(\prod_{k=1}^{K}\left(\frac{x_{ik}}{\mu_k}\right)^{\frac{r_k}{\Sigma r_k}}\right)^{\frac{1}{n}}$$

where r_k's are chosen to reflect increasing and strictly concave preferences. Measures responding to information theoretic concerns are provided by the following Generalized Entropy Family (D) developed by Maasoumi (1986), (1999):

$$D_\beta(S,X,\theta) = \sum_{j=1}^{m}\theta_j\left(\frac{\sum_{i=1}^{n} S_i\left(\left(\frac{S_i}{X_{ij}}\right)^{\beta}-1\right)}{\beta(\beta+1)}\right)$$

where S_i is an "optimal" aggregation function summarizing the welfare of individual i which could be of the following forms[3]:

---

[2] Following Tsui (1995) GINIM is axiomatically consistent with a) Continuity, b) Normalization to Perfect Equality, c) Symmetry d) Lorenz Dominance e) Scale and Location invariance and is a member of the family of Atkinson-Kolm-Sen relative indices (Blackorby, Bossert and Donaldson (1999)).

[3]In effect the index S_i corresponds to the Utility function of individual i.

$$S_i = \left(\sum_{j=1}^{m} \theta_j X_{ij}^{-\beta}\right)^{-\frac{1}{\beta}} ; \quad \beta \neq 0,-1$$

$$= \prod_{j=1}^{m} X_{ij}^{\theta_j} ; \quad \beta = 0$$

$$= \sum_{j=1}^{m} \theta_j X_{ij} ; \quad \beta = -1.$$

D then reflects the dispersion of these felicity indices. When the social evaluation function is strongly separable the indices derived by Tsui are similar to those derived by Maasoumi.

Extensions to the Multidimensional Gini, (characteristic and population weighting).

Suppose the different characteristics have differing degrees of importance in analysis (as in the indices proposed by Maasoumi (1987) and Tsui (1995) referred to above), then an appropriately weighted multivariate Gini may be employed, consider:

$$GINIMCW = \frac{1}{2K^{0.5}n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{\sum_{k=1}^{K} \alpha_k \left(\frac{x_{ik} - x_{jk}}{\bar{x}_k}\right)^2}$$

where the α_k are "importance" weights chosen by the investigator such that:

$$\alpha_k > 0, \ k=1,..,K \ \textit{together with} \ \sum_{k=1}^{K} \alpha_k = K.$$

so that GINIM is GINIMCW with $\alpha_k = 1$ for all k. Similarly suppose that sampling is stratified and that a population that contains N agent types (note that this is distinct from n, the number of agents in the sample) with weights γ_i, i= 1,..,N where the population weights are such that:

$$\gamma_i > 0, \ i=1,..,N \ \textit{together with} \ \sum_{i=1}^{N} \gamma_i = N.$$

Then a population weighted Gini (GINIMPW) may be written as[4]:

$$GINIMPW = \frac{1}{2K^{0.5}(\sum_{i=1}^{n}\gamma_i)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i^{1+\theta}\gamma_j\sqrt{\sum_{k=1}^{K}(\frac{x_{ik}-x_{jk}}{\bar{x}_k})^2}$$

where the θ, $(0 \leq \theta < 1.6)$ is a polarization parameter (Esteban and Ray (1994)) indicating the degree of polarization reflected in the index ($\theta = 0$ corresponds to a standard sample weighted multivariate Gini), hence GINIM is GINIMPW with $\gamma_i = 1$ for all i. This extends the class of Polarization indices proposed in Estaban and Ray (1994) to many dimensions and brings the multidimensional Gini into that class. Clearly GINIMPW and GINIMCW may be augmented into a population and characteristic weighted multi-variate Gini which may be written as:

$$GINIMPCW = \frac{1}{2K^{0.5}(\sum_{i=1}^{n}\gamma_i)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_i^{1+\theta}\gamma_j\sqrt{\sum_{k=1}^{K}\alpha_k(\frac{x_{ik}-x_{jk}}{\bar{x}_k})^2}$$

2. Inequality Tests for Multi Variate Distributions with limited sample sizes.

Multidimensional Inequality Indices refer to particular features of the multivariate distribution of characteristics of interest, tests for differences in these distributions would be useful in establishing whether trends in the indices reflect significant changes in the underlying distributions.

---

[4]Like GINIM the Tsui indices referred to above can be modified to accommodate population weighting schemes as in GINIMPW. In particular the first measure takes the form:

$$TR1PW = 1 - (\frac{1}{\sum_{i=1}^{n}\gamma_i})\sum_{i=1}^{n}\gamma_i\prod_{k=1}^{K}(\frac{x_{ik}}{\mu_k})^{r_k})^{\frac{1}{\Sigma r_k}}$$

where the γ_i are as before.

In the univariate domain, omnibus tests for distributional differences come in two flavours, tests which compare differences between two functions at a sequence of points (like Pearson Goodness of Fit tests) and tests which examine the maximum distance between two functions (like Kolmogorov - Smirnov tests), Rao (1973) for example discusses the two tests. Anderson (2001a) provides a Monte-Carlo based comparison of the two techniques suggesting, amongst other things, superior properties for the latter in detecting location differences and superior properties for the former in detecting scale differences. Generally the choice between the two tests is informed by arguments a) that the former test is potentially inconsistent whereas the latter is not[5] (Barrett and Donald(2003)) and b) that the former test is more informative than the latter in highlighting whereabouts in the distribution discrepancies occur (Rao(1973)). More specific unidimensional inequality tests relating to Lorenz, Generalized Lorenz, Stochastic and Polarization Dominance of various orders have been provided in Beach and Davidson (1983), Anderson (1996)(2004)(2004a), Andrews (1997), Davidson and Duclos (1997) (2001), Formby et. al. (2000), Linton et. al. (2002), McFadden(1989), Barrett and Donald(2003).

Crawford (1999) and Ibbott (1997) extend analogues of the Pearson Goodness of Fit test to the multivariate domain in introducing techniques for making statistical multidimensional stochastic dominance comparisons following the work of Atkinson and Bourguignon (1982). They partition the support of the joint density into many cells and, having estimated the probability of falling in each cell, estimate the relevant cumulative joint and marginal distributions via appropriate aggregation matrices. Unfortunately a curse of dimensionality problem arises since, for precision and power reasons in estimating cell probabilities, it is recommended that partitions should be chosen to ensure a sufficient number of observations (c) are expected to fall into each cell (see Boero et. al. (2004) for a discussion). Since each dimension requires several cells[6] (say r), the demands on the sample size are of the order cr^K where K is the number of dimensions.

---

[5]however see Anderson (2001)

[6]This issue also present problems for multidimensional extensions of the techniques proposed in Beach and Davidson (1983), Davidson and Duclos (1997) (2000).

The Kolmogorov -Smirnov Two Sample Statistic does not suffer this problem since it does not depend upon cell probability calculations. It is readily extended to the multi-variate population weighted framework by using an empirical distribution function $F_n(\gamma_x,X,z)$ defined as:

$$F_n(\gamma,X,z) = \frac{1}{\sum_{i=1}^{n}\gamma_i}\sum_{i=1}^{n}\gamma_i I(x_i \leq z)$$

where γ_x is a vector of population weights γ_i , X is an N x K matrix of stacked vectors x_i i=1,..,n corresponding to the sample of vectors of K characteristics, z is a vector of the same dimension as x_i, I( ) is a standard indicator function and "≤ " corresponds to a vector inequality relationship. Defining $G_m(\gamma_y,Y,z)$ in a similar fashion and letting $n^* = nm/(n+m)$, inference can be carried out using $D = \max_z (F_n(X,z)-G_m(Y,z))$ or $D_a = \max_z |F_n(X,z)-G_m(Y,z)|$. The latter is used to examine unspecified differences in distributions, the former can be used to explore a first order stochastic dominance relationship between F and G by considering D and $DR=\max_z (-F_n(X,z)+G_m(Y,z))$. Distribution G first order dominates F when $F(x_1, x_2, .., x_K) \geq G(x_1, x_2, .., x_K)$ for all possible $(x_1, x_2, .., x_K)$ with strict inequality holding for some $(x_1, x_2, .., x_K)$. Thus D being in the rejection region and DR not being in the rejection region constitutes evidence of first order dominance of G over F.

For situations where K = 1 and the x_i's are independent drawings with weights γ_i from f( ) (and similarly for y_j's) the asymptotic distribution of D and D_a in large samples[7] under a null of identical distributions is given by:

[7]The exact distribution of D is known for K=1 and sample sizes smaller than those in the present application.

$$\lim_{n,m\to\infty} P(\sqrt{n^*}D < \lambda) = 1 - e^{-2\lambda^2}, \ \lambda > 0.$$

$$= 0 \ otherwise$$

$$\lim_{n,m\to\infty} P(\sqrt{n^*}D_a < \lambda) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2(k\lambda)^2}, \ \lambda > 0.$$

$$= 0 \ otherwise$$

rendering testing non parametric since these distributions are independent of F. In the case of D the distribution is particularly simple coming from the family of Rayleigh distributions with scale parameter 0.5 (essentially the underlying random variable would be the positive square root of a $\chi^2(2)$ random variable re-scaled by the standard deviation of the underlying normal variate which in this case is 0.5).

Unfortunately in the present circumstances K > 1 and, since a panel is being used, it is extremely unlikely that the sample of x's will be independent of the sample of y's, in both cases the distribution of D and D_a is not known (recently Linton et al (2002) have attacked the within and between sample dependence issue in K-S tests via a sub-sampling approach). Most of the theory on D and D_a has been worked out for the one sample Kolmogorov - Smirnov Test (where G( ) in the above becomes a theoretical rather than empirical distribution function) and extended to D by invoking Glivenko-Cantelli theorems on the convergence of empirical G to theoretical G asymptotically. Dvoretsky, Kiefer and Wolfowitz (1956) established a probability bound for D in the one sample case when K = 1 and the realizations are i.i.d, the bound is of the form:

$$P(\sqrt{n}D > d) \leq Ce^{-2d^2} \qquad n = 1,2,..$$

Keifer and Wolfowitz (1958) established the existence of a distribution function for the D's when K > 1 but found that generally it depended upon F. Kiefer(1961) revisited the bounds issue for situations where K > 1 and established a bound of the form:

$$P(\sqrt{n}D > d) \leq C(\varepsilon,K)e^{-(2-\varepsilon)d^2} \qquad \varepsilon > 0, \quad n = 1,2,..$$

interestingly he observes that $C(\varepsilon,K) < C$ suggesting that estimates of $P(\sqrt{n}D > d)$ for the univariate case would provide conservative (i.e. larger) estimates of the true values when $K > 1$. Preliminary simulation studies support this view and suggest that positive correlation between x and y samples has a similar shrinking effect on the distribution of D. Indeed the simulation evidence for positive correlation between the samples is even stronger suggesting that the distribution of D remains in the Rayliegh class and simply reduces the scale parameter[8].

In studying relative inequality, the quantities $w_{jk} = x_{jk} / \mu_k$ $j=1,..,n$ $k=1,..,K$ are instrumental and appear in some form in all of the indices above. It is their joint distribution that governs the behavior of relative inequality indices and in what follows $f(w_1,w_2,..,w_3)$ and $g(w_1,w_2,..,w_3)$ will refer to the appropriately transformed distributions. By definition such distributions will not suffer period to period mean location shifts since the expected values of each of their arguments is always the unit vector and a strict first order dominance relation will never exist[9]. Univariate techniques for studying stochastic dominance in these circumstances have been developed (Formby et. al. (2000)) but again suffer the curse of dimensionality problem alluded to earlier when extended to the multi-dimensional environment. What is relevant for inequality comparisons of the form "f( ) is

---

[8]These results are available from the author on request. The Intuition behind this is that positively correlated paired elements in the samples would be closer together than if they were independent, reducing the chance of observing large differences in the respective empirical cumulative distribution functions at given points.

[9] This is easily demonstrated for uni-variate distributions f(w) and g(w) confined to the positive orthant since:

$$E(w|f) = E(w|g) \rightarrow \int_0^\infty [(1-F(w))-(1-G(w))]dw = 0 \Rightarrow \int_0^\infty (G(w)-F(w))dw = 0$$

for the last equality to hold $G(w) \geq F(w)$ for all $0 \leq w$ with strict equality for some w cannot hold and similarly neither can $F(w) \geq G(w)$ for all $0 \leq w$ with strict equality for some w.

more equal than g( )" is that the terms $\prod_k w_{jk}$ are in some sense closer to 1 under f( ) than g( ). If there is a significant change in the degree of inequality reflected in the difference between f( ) and g( ) D_a can be expected to be significant but it will not indicate the direction of change in inequality. Similarly both D and DR can be expected to be in the rejection region ruling out 1st order dominance in any direction.

The issue of greater equality under f( ) than g( ) can be examined by considering the inequalities:

$$\int_0^{w^*_1}\int_0^{w^*_2}\cdots\int_0^{w^*_K} (f(w_1,w_2,..,w_K)-g(w_1,w_2,..,w_K))dw_1,dw_2,..,dw_K \le 0 \textit{ for all vectors } w^* \textit{ s. t. } \prod_1^K w^*_i \le 1$$

together with

$$\int_{w^*_1}^{\infty}\int_{w^*_2}^{\infty}\cdots\int_{w^*_K}^{\infty} (f(w_1,w_2,..,w_K)-g(w_1,w_2,..,w_K))dw_1,dw_2,..,dw_K \ge 0 \textit{ for all vectors } w^* \textit{ s. t. } \prod_1^K w^*_i > 1$$

where strict inequality on the integral conditions holds at least somewhere. Satisfying this condition essentially confirms that the multivariate distribution of w is more dense around the unit vector under f( ) than under g( ). By considering the transformation z = -w, where $f^-(z)$ and $g^-(z)$ are the appropriate transformations of f(w) and g(w), the second condition may be written as:

$$\int_{-\infty}^{z^*_1}\int_{-\infty}^{z^*_2}\cdots\int_{-\infty}^{z^*_K} (f^-(z_1,z_2,..,z_K)-g^-(z_1,z_2,..,z_K))dz_1,dz_2,..,dz_K \le 0 \textit{ for all vectors } z^* \textit{ s. t. } \prod_{i=1}^K w^*_i > 1$$

which, when taken with the first condition, can be seen to be equivalent to two 1st order stochastic dominance relationships over limited ranges of the relevant distributions which can be examined using the unidirectional versions (D and DR) of the Kolmogorov-Smirnov test outlined above in each case.

3. An Application, Lifetime Wealth Comparisons.

In unidimensional empirical welfare and inequality comparisons the choice as to which instrument to use depends upon the balance of the arguments for income (that it is more accurately measured) and for consumption (that it more adequately reflects an agents welfare via the consumption smoothing property associated with the permanent income hypothesis see Modigliani and Brumberg (1954), Friedman (1957)). The consumption smoothing property emerges from an agents solution to maximizing lifetime utility constrained by their lifetime income (see Browning and Lusardi(1996)). The formulation most favored by researchers assumes a constant relative risk aversion form of an instantaneous utility function (which itself remains unchanged over time) together with the conditions necessary for income smoothing and yields a consumption process of the form:

$$C_t = e^{\frac{(g^*-r^*)t}{\zeta}} C_0 = e^{gt}C_0 \quad t = 0,1,\ldots,T$$

Where the long run consumption growth rate g* (which is equal to the long run interest rate under the golden rule Phelps (1961)), the rate of time preference r* and the coefficient of relative risk aversion ζ, imply an incremental consumption augmenting rate of $g = (g^* - r^*)/\zeta$.

A more appropriate instrument for welfare and inequality comparisons is an agents lifetime wealth or happiness (Atkinson (1983)). Given a life length T, no bequests and an initial consumption level C, lifetime wealth W at t = 0 may be written as[10]:

---

[10]Similarly, writing the instantaneous utility function as U(C(t)) lifetime utility at t = 0 may be written as:

$$U = \int_0^T U(C(t))e^{-r^*t}dt$$

which, for U(C) homogenous of degree one, can be seen to be proportionate to wealth.

$$W = \int_0^T Ce^{gt}dt = \frac{C}{g}(e^{gT} - 1) \qquad [1]$$

Since r*, ζ and hence g are fundamentally unobservable, W or U cannot be calculated, however assuming r* and ζ are constant and the same for all agents, we can examine how they are distributed in terms of the constituent variable components C, T and g*. In particular we can assess inequality of the distribution of W by assessing the joint inequality amongst its constituent components via the multidimensional inequality indices and tests.

The weights for a characteristic weighted Gini can be motivated by a simple Taylor series expansion of [1] showing that the mean standardized deviation of wealth from its value at the characteristic sample means is a weighted sum of the mean standardized mean deviations of the variates with their respective elasticities as weights. Note the vector of wealth partials with respect to C, T and g are respectively:

$$\frac{1}{g}(e^{gT} - 1)$$

$$Ce^{gT}$$

$$\frac{C}{g^2}((gT - 1)e^{gT} + 1)$$

In terms of orders of magnitude, for a wealth augmenting rate of 1% and an expected lifetime of 60 (roughly the orders of magnitude found in the samples), the partials of W with respect to C, T and g yield elasticities of 1, 1.33 and 0.33. The matrix of cross partials may be written as:

$$\begin{matrix} 0 & . & . \\ e^{gT} & CTe^{gT} & . \\ \frac{1}{g^2}((gT-1)e^{gT}+1) & CTe^{gT} & \frac{C}{g^3}[(gT)^2e^{gT}-2((gT-1)e^{gT}+1)] \end{matrix}$$

and:

$$\frac{\partial^3 W}{\partial C \partial T \partial g} = Te^{gT}$$

all of which are positive for pertinent values of the variables.

Following Atkinson and Bourguignon (1982), successive integration by parts and by letting $\Delta f(x_1, x_2, x_3) = f(x_1, x_2, x_3) - g(x_1, x_2, x_3)$ and similarly $\Delta F(x_1, x_2, x_3) = F(x_1, x_2, x_3) - G(x_1, x_2, x_3)$, yields the expected change in the objective function extended to the three variable case as:

$$\begin{aligned}
\int_0^{a_1}\int_0^{a_2}\int_0^{a_3} W(x_1,x_2,x_3)\Delta f(x_1,x_2,x_3)dx_1dx_2dx_3 &= W(a_1,a_2,a_3)\int_0^{a_1}\int_0^{a_2}\int_0^{a_3} \Delta f(x_1,x_2,x_3)dx_1dx_2dx_3 \\
&- \int_0^{a_1} W_1(x_1,a_2,a_3)\Delta F(x_1,a_2,a_3)dx_1 - \int_0^{a_2} W_2(a_1,x_2,a_3)\Delta F(a_1,x_2,a_3)dx_2 \\
&- \int_0^{a_3} W_3(a_1,a_2,x_3)\Delta F(a_1,a_2,x_3)dx_3 + \int_0^{a_1}\int_0^{a_2} W_{12}(x_1,x_2,a_3)\Delta F(x_1,x_2,a_3)dx_1dx_2 \\
&+ \int_0^{a_1}\int_0^{a_3} W_{13}(x_1,a_2,x_3)\Delta F(x_1,a_2,x_3)dx_1dx_3 + \int_0^{a_2}\int_0^{a_3} W_{23}(a_1,x_2,x_3)\Delta F(a_1,x_2,x_3)dx_2dx_3 \\
&- \int_0^{a_1}\int_0^{a_2}\int_0^{a_3} W_{123}(x_1,x_2,x_3)\Delta F(x_1,x_2,x_3)dx_1dx_2dx_3
\end{aligned}$$

The first term in this sum is 0 by definition, thus for objective functions in the class $W_i \geq 0$, $i=1,2,3$, $W_{ij} \leq 0$ $i,j = 1,2,3$ and $W_{123} \geq 0$ a sufficient condition for the expected value of the objective function to be no lower is:

$$\Delta F(x_1, x_2, x_3) \leq 0 \text{ for all } x_1, x_2, x_3.$$

For objective functions in the class $W_i \geq 0$, $i=1,2,3$, $W_{ij} \geq 0$ $i,j = 1,2,3$ and $W_{123} \geq 0$ a sufficient condition for the expected value of the objective function to be no lower is:

$$\Delta[F(x_1,x_2,x_3)-F(x_1,x_2,a_3)-F(x_1,a_2,x_3)-F(a_1,x_2,x_3)+F(x_1,a_2,a_3)+F(a_1,x_2,a_3)+F(a_1,a_2,x_3)]$$
$$= K(x_1,x_2,x_3)-J(x_1,x_2,x_3) \leq 0 \text{ for all } x_1,x_2,x_3$$

where K( ) is the linear combination of "F" functions and J( ) is the corresponding linear combination of "G" functions. If both K( ) and J( ) behave as regular smooth continuous cumulative density functions with respect to x the Kolmogorov-Smirnov and other distribution comparison tests would be applicable Appendix 1 shows that this will be the case if F( ) and G( ) are smooth and continuous cumulative density functions.

Results.

Data from the World Bank World Development Indicator series on per capita purchasing power parity GDP in constant 1995 $US together with Population Size and Life Expectancy were collected for 135 countries for the years 1987, 1990, 1992, 1995, 1997 and 1999 (Appendix 2 contains a list of countries in the sample). GDP per capita growth rates were calculated as the annual average over years since the preceding observation. The inequality indices require all comparator variables be non-negative, hence growth rates were standardized in each observation year by deducting the lowest average growth rate in that year from all others so that the variable may be interpreted as the growth rate differential over the minimum growth rate for that observation year. Sample weighting was based upon relative population size each year. Table 1 presents summary statistics for the raw data (i.e. growth rates have not yet been standardized).

Broadly speaking mean lnGDP and median life expectancy grew throughout the period in both weighted and un-weighted terms (there was a slight diminution of un-weighted lnGDP in 1992 and of un-weighted life expectancy in 1999). The trend in median lnGDP was somewhat less categorical, it also fell in 1999. Average and median growth rates increased and then diminished over the period. The effect of sample weighting is evident in diminished lnGDP and life expectancy and increased growth rate means reflecting the increased weight of less developed countries in the

Table 1. Sample Statistics

Year	LnGDP	lnGDP (sample w.)	GDPgrowth	GDPgrowth (sample w.)	Life Expect	Life Expect (sample w.)
Means						
1990	8.3389	8.0357	0.0014	0.0166	63.9751	65.4408
1992	8.3316	8.0851	-0.0037	0.0291	64.3698	65.9142
1995	8.3499	8.1996	0.0061	0.0416	64.6455	66.3759
1997	8.3713	8.2411	0.0107	0.0249	64.9619	66.7446
1999	8.3777	8.2681	0.0032	0.0175	64.7490	66.7853
Medians						
1990	8.3856	7.3709	0.0028	0.0169	68.0839	61.7073
1992	8.3623	7.5668	0.0060	0.0101	68.1024	62.7073
1995	8.3938	7.8941	0.0116	0.0215	68.4610	64.1459
1997	8.4193	8.0193	0.0091	0.0149	69.0366	65.1049
1999	8.4118	7.8934	0.0041	0.0116	69.4776	65.7244
Sample Standard Deviations						
1990	1.0770	1.0491	0.0372	0.0280	10.8973	7.8862
1992	1.0823	1.0194	0.0586	0.0518	11.1240	7.7998
1995	1.1072	0.9762	0.0447	0.0505	11.3641	7.8112
1997	1.1153	0.9690	0.0473	0.0283	11.6624	8.0017
1999	1.1299	0.9577	0.0460	0.0373	12.4382	8.5735

sample. The sample standard deviations, being a crude inequality measure, provide an interesting perspective of increasing inequality in un-weighted lnGNP and diminishing inequality in Weighted lnGNP. Growth rate inequality is all over the map! With respect to life expectancy inequality increases in the un-weighted sample whilst the weighted sample exhibits a "U" shaped profile.

Table 2 reports the absolute welfare implications of the data in table 1 via 1st order stochastic dominance tests following Atkinson and Bourguignon (1982). Both un-weighted and weighted (*)

samples indicate welfare improvements between 1990 and 1992 and 1997 and 1999. Welfare deterioration between 1992 and 1995 and 1995 and 1997 are indicated by the un-weighted sample but not in the weighted sample. This suggests the multidimensional comparison is greatly influence by the growth rate variable which has the largest coefficient of variation by a considerable margin since it follows the progress of that instrument. It is probably also reinforced by the weak performance of lnGDP in 1992 and both lnGDP and life expectancy in 1999.

Table 3 reports results for the various Multidimensional Inequality Indices and their unidimensional counterparts (for the purposes of these results the growth rates have been standardized). The unidimensional inequality indices for lnGDP and Life Expectancy generally follow the patterns indicated by their corresponding sample standard deviation counterparts whereas the inequality indices for growth rates do not, but recall that these have been standardized in each period for the inequality calculations whereas the standard deviations are based upon raw data. Sample weighting again has a profound effect upon the conclusions drawn, effectively reversing the trends observed in lnGDP and Life Expectancy. Since in welfare and inequality calculations sample weighting by population size is appropriate, it is these results that we now focus upon. It is interesting to note that reliance upon any single aspect of the lifetime wealth calculation would lead to quite different inferences to that derived from any other. Population Weighted Gini and Polarization coefficients for lnGDP would suggest a continued and sustained decline of both inequality and polarization whereas inequality and polarization time profiles are "U" shaped for life expectancy and inverted "U" shaped for growth. What conclusions are to be drawn when the various trends are taken together? With regard to the multivariate indices it should at first be observed that the Mgini and Tsui / Maasoumi indices are in broad agreement, 1995 population weighted and population and characteristic weighted indices being the exception. Generally the Tsui / Maasoumi indices exhibit more variability than the corresponding Mgini index. Given the diverse trends of the components it is no surprise that no solid underlying trend is

Table 2. Multivariate Kolmogorov-Smirnov 1st Order Dominance Tests*.

Null Hypothesis	90 v 92	90 v 92$^{\#}$	92 v 95	92 v 95$^{\#}$	95 v 97	95 v 97$^{\#}$	97 v 99	97 v 99$^{\#}$
2nd yr $\prec$ 1st yr	0.0000	0.0000	0.6588	0.0002	0.8187	0.0002	0.0000	0.0000
1st yr $\prec$ 2nd yr	0.9780	0.9519	0.0048	0.0000	0.0000	0.0000	0.8860	0.9252

*The upper tail probabilities reported are conservative estimates based upon the assumption that the one sided K-S test is Rayliegh distributed with a scale factor of 0.5.
$^{\#}$Based upon population weighted samples.

Table 3. Multidimensional Inequality Coefficients*

Year	GINIM (TR1)	GINIMPW (TR1PW)	GINIMCW (TR1CW)	GINIMPCW (TR1CPW)	MPOLE	MPOLECW
1990	0.1270	0.0938	0.1109	0.0838	1.0679	0.9762
	(0.0157)	(0.0142)	(0.0101)	(0.0142)		
1992	0.1107	0.0921	0.1029	0.0812	1.2154	1.0134
	(0.0138)	(0.0121)	(0.0080)	(0.0123)		
1995	0.1062	0.0999	0.1014	0.0845	1.3251	1.0500
	(0.0122)	(0.0106)	(0.0065)	(0.0106)		
1997	0.1315	0.0968	0.1159	0.0828	1.2578	1.0085
	(0.0167)	(0.0150)	(0.0109)	(0.0151)		
1999	0.1178	0.0918	0.1106	0.0817	1.0886	0.9376
	(0.0127)	(0.0109)	(0.0069)	(0.0110)		

* Reported respectively in the first four columns are the Multidimensional Gini Coefficients un-weighted, population weighted, characteristic weighted and population and characteristic weighted. Below in brackets are the correspondingly weighted Tsui / Maasoumi indices. The last two columns report non-normalized multivariate versions of the Esteban and Ray index un-weighted and weighted by characteristic employing a polarization parameter value of 1

Unidimensional Gini (un-weighted and weighted) and Esteban-Ray Indices**

	Ln Gross Domestic Product			GDP growth			Life Expectancy		
Year	GINI	GINW	POL	GINI	GINW	POL	GINI	GINW	POL
1990	0.0741	0.0658	0.7301	0.1539	0.1064	1.1673	0.0949	0.0672	0.7929
1992	0.0744	0.0648	0.7190	0.1163	0.1070	1.5838	0.0955	0.0651	0.7481
1995	0.0759	0.0627	0.6903	0.1030	0.1261	1.8832	0.0974	0.0641	0.7170
1997	0.0763	0.0622	0.6828	0.1546	0.1199	1.7484	0.0991	0.0646	0.7068
1999	0.0771	0.0615	0.6699	0.1205	0.1068	1.3823	0.1053	0.0685	0.7467

discernable in the multi-variate indices except for the polarization indices which both exhibit an inverted "U" time profile.

Table 4. Multivariate Stochastic Dominance Tests (Weighted Mean Standardized Distributions)*.

Comparison		1990 v 1992	1992 v 1995	1995 v 1997	1997 v 1999
Overall Distrib.	2nd ≺ 1st	0.0099	0.0000	0.0001	0.0001
	1st ≺ 2nd	0.0973	0.9463	0.0877	0.0846
Lower Tail	2nd ≺ 1st	0.8921	0.3028	0.9704	0.3997
	1st ≺ 2nd	0.2912	0.9818	0.3550	0.6891
Upper Tail	2nd ≺ 1st	0.2685	0.6401	0.9731	0.1006
	1st ≺ 2nd	0.0964	0.0020	0.0841	0.1115

*The upper tail probabilities reported are conservative estimates based upon the assumption that the one sided K-S test is Rayliegh distributed with a scale factor of 0.5.

There appears to be stronger evidence of inequality trends in the dominance tests reported in Table 4. Contrary to expectations, the general 1st order dominance tests reject dominance in only one direction whereas the upper and lower tail tests (with the exception of the 92-95 comparison) have to be deemed inconclusive at the 5% level[11]. In every case the overall comparison implies the latter observation period dominates the preceding period and since 1st order dominance implies 2nd order dominance this implies increasing equality over the sample period.

Conclusions.

Indices and tests facilitating the examination of inequality in many dimensions have been introduced which, in an example of an international representative agent lifetime welfare model, indicate that substantially different conclusions may be inferred from those drawn from univariate

[11]The lack of power in the "tail" directed tests is probably due to the reduction in the sample size (it is roughly halved) implicit in focusing on the tails.

comparisons. The index techniques introduced are simple to use generalizations of the familiar univariate Gini coefficient which also extend the Esteban and Ray (1994) class of polarization indices. These were found to follow closely the behavior of ethically based and generalized entropy multidimensional inequality indices. The testing techniques are multivariate extensions of the Kolmogorov-Smirnov two sample test (Rao (1972)) together with a simple adaptation to examine the "multidimensional tail behavior" of mean standardized data employed in inequality studies.

In an application studying income, growth and life expectancy, three variables that influence lifetime wealth, the variables exhibit quite distinct and different inequality and polarization trends over the observation period. It is then perhaps no surprise that the multivariate indices which in essence report an agglomeration of the three indices indicate no such discernable trend except for the "U" shaped time profile of the polarization indices over the period of study. It is equally unsurprising that the unequivocal welfare improvements inferred from a univariate analysis (see for example Anderson (2004)) become much less clear in the multivariate paradigm.

Acknowledgements.
Many thanks are due to Charles Beach, Essie Maasoumi, Lars Osberg and Russell Davidson for advice in the early stages of this work and to Ying Ge for his helpful research assistance The work has been carried out with the support of the SSHRC under grant numbers 4100000732 and 410040254. All errors are the responsibility of the author.

References.

Anderson G.J. (1996) "Non-Parametric Tests of Stochastic Dominance in Income Distributions" Econometrica 64 1183-1193.

Anderson G.J. (2001) "A Note on the Inconsistency of Tests Employing Point-wise Comparisons for the Equality of Two Functions" mimeo Economics Department University of Toronto

Anderson G.J. (2001a) "The Power and Size of Nonparametric Tests for Common Distributional Characteristics" Econometric Reviews 20 1-30.

Anderson G.J. (2004) "Towards an Empirical Analysis of Polarization" forthcoming Journal of Econometrics.

Anderson G.J. (2004a) "Making Inferences About The Polarization, Welfare and Poverty of Nations: A Study of 101 Countries 1970-1995" forthcoming Journal of Applied Econometrics.

Andrews D.W.K. (1997) "A Conditional Kolmogorov Test" Econometrica 65 1097-1128.

Atkinson A.B.(1983) The Economics of Inequality 2nd Edition Clarendon Press Oxford.

Atkinson A.B. (1970) "On the Measurement of Inequality" Journal of Economic Theory 2

Atkinson A.B. and F. Bourguignon (1982) "The Comparison of Multi-Dimensioned Distributions of Economic Status" Review of Economic Studies 49 183-201.

Barrett G and Donald S. (2003) "Consistent Tests for Stochastic Dominance" Econometrica 71, 71-103.

Beach, C.M. and R. Davidson (1983) "Unrestricted Statistical Inference with Lorenz Curves and Income Shares" Review of Economic Studies, 50, 723-735.

Blackorby, C., Bossert W. and D Donaldson (1999) "Income Inequality Measurement: the Normative Approach" chapter 3 Handbook of Income Inequality Measurement J Silber editor Kluwer Academic Publishers Boston.

Boero, G., J. Smith and K.F. Wallis (2004) "Sensitivity of the Chi-Squared Goodness of Fit Test to the Partitioning of Data" mimeo Economics Department University of Toronto.

Browning M and A Lusardi (1996) "Household Saving: Micro Theories and Micro Facts" Journal of Economic Literature 34 1797-1855.

Crawford I. (1999) Nonparametric Tests of Stochastic Dominance in Bivariate Distributions, With an Application to UK Data. Institute of Fiscal Studies.

Dalton H (1920) "The measurement of Inequality of Incomes" Economic Journal 30

Davidson R. and J-Y Duclos (1997) "Statistical Inference for the Measurement of the Incidence of Taxes and Transfers" Econometrica 65, 1453-1466.

Davidson, R and J-Y Duclos (2000) "Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality. Econometrica 68 1435-1464.

Dvoretzky, A. J. Kiefer and J. Wolfowitz (1956) "Asymptotic Minimax Character of the Sample Distribution Function and of the Classical Multinomial Estimator" Annals of Mathematical Statistics 27 352-374.

Esteban, J-M and D Ray (1994) "On the Measurement of Polarization" Econometrica 62, 819-851.

Formby J.P., W. J. Smith, B. Zheng and V. Chow (2000) "Inequality Orderings, Normalized Stochastic Dominance and Statistical Inference" Journal of Business and Economic Statistics v18

Friedman M (1957) A Theory of the Consumption Function. Princeton University Press

Gini C (1912) Variabilita e mutablita. Bologna

Kiefer, J. (1961) "On Large Deviations of the Empiric D.F. Of Vector Chance Variables and a Law of the Iterated Logarithm" Pacific Journal of Mathematics 11 649 - 660.

Kiefer, J. and J. Wolfowitz (1958) "On the Deviations of the Empiric Distribution Function of Vector Chance Variables" Transactions of the American Mathematical Society 87 173-186.

Kolm S-C. (1966) "The Optimal Production of Social Justice" in "Colloques Internationaux du Centre National de la Recherche Scientifique 2-9 Septembre 1966".

Lambert P.J. (1993) The Distribution and Redistribution of Income: A Mathematical Analysis 2nd Edition Manchester University Press.

Linton O., E. Maasoumi and Y-J Whang (2002) "Consistent Tests for Stochastic Dominance: A Subsampling Approach" mimeo LSE.

McFadden, D. (1989) "Testing for Stochastic Dominance" in Studies in the Economics of Uncertainty: In Honor of Josef Hadar ed. Bt T. Fomby and T.K. Seo, Springer New York

Maasoumi E. (1986) "The Measurement and Decomposition of Multidimensional Inequality" Econometrica 54 771-779

Maasoumi E. (1993) "A Compendium of Information Theory in Economics and Econometrics" Econometric Reviews 12 1-49.

Maasoumi E. (1999) "Multidimensioned Approaches to Welfare Analysis" chapter 15 Handbook of Income Inequality Measurement J Silber editor Kluwer Academic Publishers Boston.

Modigliani F. and R Brumberg (1954) "Utility Analysis and the Consumption Function: An Interpretation of Cross Section Data" in K.K. Kurihara ed. Post Keynesian Economics Rutgers University Press New Brunswick New Jersey

Newberry D.M.G. (1970) "A Theorem on the Measurement of Inequality" Journal of Economic Theory 2

Phelps, E.S. (1961) "The Golden Rule of Accumulation: A Fable for Growthmen" American Economic Review 51 pp.638-643.

Rao R.C. (1973) Linear Statistical Inference and Its Applications. Wiley New York

Sen A. (1995) Inequality Reexamined Harvard University Press

Sen (1997) On Economic Inequality (Expanded Edition) with J.E.Foster Oxford University Press.

Sheshinski (1972) "Relation between the Social Welfare Function and the Gini Index of Inequality" Journal of Economic Theory 4.

Tsui K-Y (1995) "Multidimensional Generalizations of the Relative and Absolute Inequality Indices: The Atkinson-Kolm-Sen Approach. Journal of Economic Theory 67 pp251-265.

Appendix 1.

Note that if the underlying F( ) and G( ) distributions are smooth and continuous then so will K( ) and J( ) be. The remaining issue is whether K( ) and J( ) are bounded between 0 and 1 and non-decreasing functions. That they are is demonstrated for K( ) (and may be for J( ) in a similar fashion) by letting $x = (x_1\ x_2\ x_3)$, $a^i = x$ with a_i substituted for x_i, $i = 1,..,3$, $a^{ij} = a^i$ with a_j substituted for x_j, $i,j = 1,..,3\ i \neq j$ and $a = (a_1\ a_2\ a_3)$ and noting that:

$$0 \leq F(x) \leq F(a^i) \leq F(a^{ij}) \leq F(a) = 1 \textit{ for all } x,\ a^i,\ a^{ij},\ i,\ j = 1,..,3\ i \neq j$$

it readily follows that $0 \leq K(x) \leq 1$. To show that K(x) is non-decreasing for all x, consider $\partial K(x)/\partial x_1 = \partial F(x)/\partial x_1 - \partial F(a^3)/\partial x_1 - \partial F(a^2)/\partial x_1 + \partial F(a^{23})/\partial x_1$ which may be written as:

$$\begin{aligned}
\frac{\partial K(x)}{\partial x_1} &= \int_0^{x_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 - \int_0^{a_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 - \int_0^{x_2}\!\!\int_0^{a_3} f(x_1,z_2,z_3)dz_3dz_2 + \int_0^{a_2}\!\!\int_0^{a_3} f(x_1,z_2,z_3)dz_3dz_2 \\
&= \int_0^{x_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 - \int_0^{x_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 - \int_{x_2}^{a_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 - \int_0^{x_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 \\
&\quad - \int_0^{x_2}\!\!\int_{x_3}^{a_3} f(x_1,z_2,z_3)dz_3dz_2 + \int_0^{x_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 + \int_{x_2}^{a_2}\!\!\int_0^{x_3} f(x_1,z_2,z_3)dz_3dz_2 + \int_0^{x_2}\!\!\int_{x_3}^{a_3} f(x_1,z_2,z_3)dz_3dz_2 \\
&\quad + \int_{x_2}^{a_2}\!\!\int_{x_3}^{a_3} f(x_1,z_2,z_3)dz_3dz_2 = \int_{x_2}^{a_2}\!\!\int_{x_3}^{a_3} f(x_1,z_2,z_3)dz_3dz_2 \geq 0 \textit{ for all } x,\ 0 \leq x \leq a.
\end{aligned}$$

$\partial K(x)/\partial x_2 \geq 0$ and $\partial K(x)/\partial x_3 \geq 0$ may be demonstrated the same way. Hence Kolmogorov-Smirnov 2 sample statistics may be applied to K( )-J( ).

Appendix 2.

Countries represented in the sample.

Algeria Angola Antigua and Barbuda Argentina Australia Austria Azerbaijan Bangladesh Barbados Belarus Belgium Belize Benin Bolivia Botswana Brazil Bulgaria Burkina Faso Burundi Cambodia Cameroon Canada Cape Verde Central African Republic Chad Chile China Colombia Comoros Congo Costa Rica Cote d'Ivoire Cyprus Dominica Dominican Republic Ecuador Egypt El Salvador Equatorial Guinea Estonia Ethiopia Fiji Finland France French Polynesia Gabon Gambia Ghana Greece Guatemala Guinea Guinea-Bissau Guyana Haiti Honduras Hong Kong Hungary Iceland India Indonesia Iran Ireland Israel Italy Jamaica Japan Jordan Kazakhstan Kenya South Korea Kyrgyz Lao Latvia Lesotho Luxembourg Macao Madagascar Malawi Malaysia Mali Mauritania Mauritius Mexico Mongolia Morocco Mozambique Namibia Nepal Netherlands New Caledonia New Zealand Nicaragua Niger Nigeria Norway Pakistan Panama Papua New Guinea Paraguay Peru Philippines Portugal Romania Rwanda Samoa Saudi Arabia Senegal Sierra Leone Singapore Slovak Republic Solomon Islands South Africa Spain Sri Lanka St. Kitts and Nevis St. Lucia St. Vincent and the Grenadines Swaziland Sweden Switzerland Syrian Arab Republic Thailand Togo Trinidad and Tobago Tunisia Turkey Uganda Ukraine United Kingdom United States Uruguay Vanuatu Venezuela Zambia Zimbabwe