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Additivity, Matrix Consistency and a New Method for International Comparisons of Real Income and Purchasing Power Parities

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Abstract

The paper re-examines the concepts of additivity and matrix consistency, invoked in the context of methods for international comparisons, provides a more formal treatment of these concepts and distinguishes them using the numerical values of the eigen values implicit in these concepts. Given the importance attached to the concept of additivity within the International Comparison Project/Program (ICP), the paper provides a framework for easily identifying methods satisfying additivity.

The main objective of the paper is to provide an answer in the negative to a recent conjecture of Rao (1997) which postulates that the Geary-Khamis method may be the only multilateral aggregation method that is additively consistent and transitive. In this paper a new method of aggregation, a variant of the Kurabayashi-Sakuma (1981) class of methods, is proposed which is shown to satisfy the properties of commensurability, transitivity and additivity. A numerical illustration showing the results from the new method using 1990 benchmark results for the OECD countries is also provided in the paper.

JEL Classification:

1. Introduction

International comparisons of real income and purchasing power parities of currencies are a major activity and concern of several international organisations including the United Nations, O.E.C.D., the World Bank, E.U., FAO, and more recently the I.M.F. Much of the work on international comparisons from the expenditure side of the national accounts has been under the auspices of the International Comparison Project/Program (ICP). The ICP had originated at the University of Pennsylvania under the guidance of Professor Kravis, and then shifted to the United Nations and other bodies. In addition considerable effort of individual researchers and groups has been channelled into international comparisons of real output and productivity, principally from the Food and Agriculture Organisation focusing on agriculture sector comparisons and from the ICOP group at the University of Groningen initiated by Professor Angus Maddison.

Two major components can be identified in any international comparisons exercise. The first component involves the painstaking process of gathering detailed price data from all the countries in the comparisons and the second component requires proper aggregation of the price and quantity or value data compiled leading to meaningful estimates of purchasing powers of currencies and real incomes in different countries. The issue of aggregation formed the subject matter for research by several researchers over the last three decades. These concentrated research efforts over time have led to a large range of aggregation or index number methods available for purposes of international comparisons.

The problem of choice of a suitable method for international comparisons from a variety of index number methods obviously points towards the need for criteria for selection of a suitable method. A number of properties, that can be used in assessing the suitability of a given aggregation method, have been discussed in the literature (see Kravis et al, 1982). These include among others: transitivity, base invariance, commensurability, additivity, matrix consistency and characteristicity. Diewert (1986) and Balk (1996) provide a more formal list of such properties in the form of tests, similar to the Fisher’s tests for index numbers, for comparisons over time, and assess several competing aggregation methods.

One of the most important of these properties is the property of additivity. Within the general description of additivity only of the many aggregation procedures, the Geary-Khamis (GK) method seems to the be the only aggregation procedure satisfying additivity. For a number of years several international organisations have debated the relative merits of the Geary-Khamis method and the Elteto-Koves-Szulc (EKS) method, the EKS method does not satisfy additivity but supposed to preserve characteristicity in the context of multilateral international comparisons. Recently, in 1996, the OECD has decided to publish international comparison results based on both the GK and EKS methods.

A property very close to additivity is the property of matrix consistency, so close that the distinction between these two properties is often blurred. Even the exposition in Kravis et al (1982) does not offer a detailed discussion of these two properties. A number of aggregation methods based on the matrix consistency property have been proposed in the literature, the most significant are the Kurabayashi-Sakuma and the van IJeren class of methods.

An important objective of the paper is to offer formal definitions and descriptions of additivity and matrix consistency and discuss the link between these two concepts.

In the literature, Geary-Khamis method has been the only method possessing the additivity property so far. The present paper provides a new method for aggregation, which may be described as a variant of the Kurabayashi-Sakuma class of methods of aggregation, that satisfies the additivity and matrix consistency properties. Various properties of the method are investigated in the paper.

A brief outline of the paper is as follows. Section 2 establishes the notation used in the paper. Section 3 deals with the additivity and matrix consistency properties used in international comparisons and discusses the link between these two properties. In Section 4 a new aggregation method for international comparisons of prices and real income is outlined and a numerical illustration based on the 1990 OECD benchmark comparisons is provided. Results from the new method are presented and contrasted with the results derived using the Geary-Khamis method.

2. Notation

The following notation is used throughout the paper. As the paper deals with multilateral comparisons of price and quantities, it is assumed that price and quantity data are available for N countries and M commodities. Let

P be a M x N matrix of prices;

Q be a M x N matrix of quantities;

p_{ij} and q_{ij} represent, respectively, price and quantity of i-th commodity in j-th country;

$\overline{p}$ be a column vector of order M consisting of elements $\overline{p}_i$ representing the international average price of i-th commodity;

PPP_j representing the purchasing power parity between the currency of country j and the currency of the base or numeraire country (number of units of currency of country j equivalent in purchasing power to one unit of numeraire currency);

q be a column vector of order N consisting of elements of q_j representing the aggregate quantity measure associated with country j;

V be the value matrix of order M x N with typical element $v_{ij} = p_{ij}\ q_{ij}$; and

i be a column vector consisting of elements that are all equal to unity.

A few comments are in order. The prices for any given commodity are expressed in respective national currency units and, therefore, cannot be added across countries. Thus it is necessary to either normalise prices in each country or convert each country prices into a common currency unit using a currency conversion factor such as a purchasing power parity (PPP) associated with a given currency. The quantity measure, q_j, associated with country j, reflects the total volume of country j relative to a numeraire country or the world or simply expressed in a form that is directly comparable to the quantity measure of another country. Ratios of quantity measures can be interpreted as quantity index numbers.

3. Additivity and Matrix Consistency

This section deals with the two concepts central to the paper, these concepts have played a major role in the derivation and construction of a number of aggregation methods for international comparisons. As stated in the introduction, these two concepts are intricately connected and the distinction between these two methods has not always been clear. For purpose of exposition, the concept of matrix consistency is dealt with first and then the additivity concept is described.

Matrix Consistency

The idea of matrix consistency was explained in the Phase III ICP report by Kravis et al (1982) as “Quantities expressed in value terms in matrix form, defined for *n* countries and *m* goods categories, should be stated in such a way that (1) the values for any category will be directly comparable between countries, and (2) the values of for any country wlll be directly comparable between categories.” (p. 72).

The first requirement states that quantity measures can be derived at different levels of aggregation such that the quantity measures can be compared across countries, including the level that includes the full set of commodities. The second requirement allows the possibility of defining quantity measures at any desired level of aggregation that can be used in the first requirement.

The requirements of matrix consistency require the use of a set of international average prices, defined for each of the commodities in the complete list of commodities using observed national prices, such that the ratios of quantity measures q_k/q_j for two countries represent a quantity index number. The quantity measures q_j for each country j are determined upto a factor of proportionality as:

$$q_j = \lambda \sum_{i=1}^{M} \bar{p}_i q_{ij} \text{ , } j = 1, 2, \ldots, N \tag{1}$$

where $\bar{p}_i$ represents the international average price of i-th commodity such that the prices are all expressed in units that are comparable across countries and commodity groups. Matrix consistency requires that equation (1) holds for all sub-groups of commodities.

While matrix consistency requires that international average prices are defined, there is no explicit mention of the need to obtain measures for price comparison such as the purchasing power parities (PPPs) of currencies. However, the PPPs can be defined indirectly as

$$PPP_j = \frac{\sum_{i=1}^{M} p_{ij} q_{ij}}{q_j} = \frac{1}{\lambda} \frac{\sum_{i=1}^{M} p_{ij} q_{ij}}{\sum_{i=1}^{M} \bar{p}_i q_{ij}} \tag{2}$$

Equation (2) implies that purchasing power parities of currencies can also be defined as ratios of the value aggregates at national and international average prices. A further implication of matrix consistency that follows from equation (2) is that the national value aggregate for each country converted into a common currency unit using PPP_j is equal to the constant price value of the country-j quantity vector augmented by a constant. That is

$$\sum_{i=1}^{M} \bar{p}_i q_{ij} = \frac{1}{\lambda} \frac{\sum_{i=1}^{M} p_{ij} q_{ij}}{PPP_j} \tag{3}$$

The LHS of equation (3) is the quantity measure required for purposes of matrix consistency.

An other point to be made about matrix consistency is that to make the concept operational, that is to be able to actually derive internationally comparable quantity measures, it is still necessary to define international average prices. This objective is achieved by different methods in different ways. The Geary-Khamis method defines international average prices directly using PPP_js in equation (2). The Kurabayashi-Sakuma (1981) approach defines average prices using normalised national price data. Similar approach is also used in the van IJeren class of methods.

The concept of additivity and its relationship between matrix consistency and additivity are provided below.

Additivity or Additive Consistency

Following several papers of Khamis (1984 and 1996), additive consistency is defined as the requirement that for each country the national value aggregate expressed in national currency units converted into a common currency unit using purchasing power parities should be equal to the value aggregate derived by valuing the quantity vector of a country at international average prices, $\bar{p}_i$ for i = 1, 2, …, M. In algebraic form,

$$\frac{\sum_{i=1}^{M} p_{ij} q_{ij}}{PPP_j} = \sum_{i=1}^{M} \bar{p}_i q_{ij} \tag{4}$$

Equation (4) ensures that national accounts presented at constant prices are consistent with value aggregates adjusted for price differentials using purchasing power parities. Note that the requirement in equation (4) is stipulated at the aggregate level only. That is what makes it different from the concept of *full additivity* discussed in Kravis et al (1982).

Equation (4) is very similar to equation (3), derived from the matrix consistency concept, the only difference being that equation (3) requires the equality to hold with a factor of proportionality constant $1/\lambda$. The relationship between the additivity and matrix consistency concepts can be highlighted using the following result.

Result: Additivity implies matrix consistency but the converse is not true.

Additivity implies matrix consistency trivially with the proportionality constant $\lambda = 1$. The fact that the converse is not true can be established using the following example drawn from Rao (1997).

For purposes of illustration consider a simple van IJzeren system based on a simple additive specification for $\bar{p}_i$'s. The system is defined for i=1,2,...,N and j = 1,2,...,M, as

$$\bar{p}_i = \frac{1}{N}\sum_{j=1}^{N} p_{ij} / PPP_j \qquad \text{and} \qquad \frac{1}{PPP_j} = \frac{\sum_i \bar{p}_i q_{ij}}{\sum_i p_{ij} q_{ij}} \tag{5}$$

In proposing this method, it is argued that it has a unique solution when solved iteratively, at each iteration normalising e_j . It can be shown that such an iterative procedure converges to a unique positive solution, and therefore it can be considered as a viable additively consistent method.

However, a simple substitution of $\bar{p}_i$ into $1/PPP_j$ leads to a system for which no non-trivial solution exists. This observation is in contrast to the statement just above which claims to provide a positive solution to the system.

An explanation for this paradox is as follows. Additivity constraint in equation (4) together with a linear function for $\bar{p}_i$'s will lead to a system of equations for $1/PPP_j$'s in the form

$$Ax = x \qquad \text{or} \quad (I-A)x = 0 \tag{6}$$

Non-trivial solutions exist if (I- A) is singular. This in turn implies that the matrix A has “unity” as an eigen value.

However if additivity restriction is replaced by matrix consistency requirement in equation (3)

$$\sum_{i=1}^{M} \bar{p}_i q_{ij} = \frac{1}{\lambda}\frac{\sum_{i=1}^{M} p_{ij} q_{ij}}{PPP_j} \qquad \text{or} \qquad \frac{1}{PPP_j} = \lambda \frac{\sum \bar{p}_i q_{ij}}{\sum_i p_{ij} q_{ij}} \tag{7}$$

Then it leads to a system

$$Ax = \lambda x$$

($\lambda \neq 1$) which can be shown to have a positive solution. But for ($\lambda = 1$) only trivial solution is feasible.

This example clearly demonstrates the crucial nature of the difference between the concept of matrix consistency and additivity in the context of discussing aggregation methods for international comparisons. This result has prompted Rao to postulate the following conjecture.

In general most systems, like those due to Ikle, Gerardi and van IJzeren, additivity condition is imposed with a factor of proportionality built into it as shown in (7). From the discussion, it appears that if additivity in the form specified in the original work of Geary and Khamis, and as described in (4), then the Geary-Khamis system may be conjectured to be the only additively consistent system. This discussion shows that it is nearly impossible to construct additively consistent systems similar to the GK system.

In the ensuing Section of the paper, this answer to the conjecture is shown to be negative by providing an alternative to the Geary-Khamis system which has additive consistency property.

4. A New Aggregation Method

In this section, a new aggregation method which is derived as a variant of the Kurabayashi-Sakuma (KS) class of methods is outlined. The method uses the same basic framework as that used in the KS class of methods, but satisfies additional properties such as commensurability and additivity property discussed in Section 3.

Following the KS framework, a world price vector, $\bar{p}$, and a quantity measure vector, q, with elements q_j representing the relative size of the country j quantity. In order to simplify notation, the total value in each country is denoted by GDP_j, such that

$$GDP_j = \sum_{i=1}^{M} p_{ij} q_{ij}$$

A feature of the KS method is that the purchasing power parities are not defined directly from the method, rather indirectly as

$$PPP_j = \frac{GDP_j}{q_j} \tag{8}$$

Once the quantity measures are derived, then PPP_j can be derived using equation (8). The world price and quantity vectors are defined using the following system of interdependent equations.

$$\begin{aligned} Q'\bar{p} &= \lambda q \\ Pq &= \mu\bar{p} \end{aligned} \tag{9}$$

These two equations lead to the following equations, each of which imply an eigen value problem. The equations are:

$$PQ'\bar{p} = \lambda\mu\,\bar{p} = \alpha\,\bar{p} \tag{10}$$

$$Q'P\,q = \lambda\mu q = \alpha\,q \tag{11}$$

In order to make the KS method operational, it is necessary to normalise the price matrix so that Pq in equation (8) is well defined. In actual empirical applications, matrix is usually replaced by a normalised price matrix S, leading to

$$SQ'\bar{p} = \lambda\mu\,\bar{p} = \alpha\,\bar{p}$$

$$Q'S\,q = \lambda\mu q = \alpha\,\bar{p}$$

Existence of unique (up to a factor of proportionality) positive solutions for q and $\bar{p}$ is shown using the Frobenius Perron Theorem and the eigen value α is the Frobenis Perron root. Within the framework of the KS specification of equations (8), there is no guarantee that the resulting eigen value is equal to unity. In fact in most cases, the eigen value is not equal to 1.

Several specifications of the S matrix were considered in the past, including the numeraire and simplex methods of normalising the national prices. However, both of these normalisations fail to satisfy the property of commensurability that is considered to be a standard requirement for index numbers. In this paper, a new normalisation is proposed leading to a new method for international comparisons.

4.1 A New Method for International Comparisons

Consider the following normalisation of the price data for different countries. The normalisation proposed here normalises price of each commodity in a given country by the total value of the global quantities so that the normalsed global value is set to unity. The S matrix of normalised prices is given by

$$S = P \cdot \widehat{P'\overline{q}}^{-1} \tag{12}$$

where $\overline{q}$ is the vector of aggregate quantities such that $\overline{q}_i = \sum_{j=1}^{N} qij$.

Given the new normalisation, the world price vector, $\overline{p}$, and quantity measure vector, q, are obtained as solutions from the following set of equations.

$$Q' P \cdot \widehat{P'\overline{q}}^{-1} q = \alpha q \tag{13}$$

and

$$P \cdot \widehat{P'\overline{q}}^{-1} Q' \overline{p} = \alpha \overline{p} \tag{14}$$

The following sequence of theorems establish the most important properties of the new method.

<u>Theorem 1</u>: The systems of equations in (13) and (14) both have a unit eigen value.

Proof: Consider first, equation system (13). We have

$$Q' P \cdot \widehat{P'\overline{q}}^{-1} q = \alpha q$$

which can be written as

$$Q' \hat{\overline{q}}^{-1} \hat{\overline{q}} P \widehat{P' \hat{\overline{q}} i}^{-1} q = \alpha q .$$

We note, firstly, that column sums of the matrix $Q' \hat{\overline{q}}^{-1}$ are all equal to 1. Secondly, it can also be seen that column sums of $\hat{\overline{q}} P \widehat{P' \hat{\overline{q}}}^{-1}$ are all equal to one. These two conclusions follow from the structure of the matrices.

Next, we note that if all the column sums (or row sums) of a square matrix A are equal to 1, then A has a unit eigen value. Proof of this follows from the fact that eigen values of A, denoted by $\lambda(A)$, satisfy the following inequalities.

$$\min r_i \le \lambda(A) \le \max r_i$$
$$\min s_j \le \lambda(A) \le \max s_j$$

where $r_i = \sum_{j=1}^{n} a_{ij}$ and $s_j = \sum_{i=1}^{n} a_{ij}$ denote respectively the row and column sums. See Nikaido (1968) for a proof of the statement (page 108).

Further we note that, if B is a matrix of size NxM and C is a matrix whose size is MxN, and if all column sums of B and all column sums of C are equal to 1, then all column sums of the (product) matrix D = BC are also equal to 1. The proof follows from the fact that the column sums can be written as:

$$\sum_{i=1}^{N} d_{ij} = \sum_{i=1}^{N}\sum_{k=1}^{M} b_{ik} \, . c_{kj} = \sum_{k=1}^{M} c_{kj} \left[\sum_{i=1}^{N} b_{ik} \right] = 1,$$

Given all the observations above, it follows that the equation (13) can be written as

$$Aq = \alpha q$$

where matrix A has all column sums equal to 1, and with $\lambda(A)$ equal to 1. This concludes proof of Theorem 1. Similar proof can be constructed to show that eigen value in equation (14) is equal to unity.

Corollary: The new aggregation method derived using the matrix consistency property satisfies additivity since it satisfies the Rao-Khamis unit root constraint for additivity.

The Corollary to Theorem 1 shows that answer to the conjecture in Rao (1997) is in the negative since these results show that the new aggregation method satisfies additivity property as stipulated in Rao (1997) with a unit eigen value.

Theorem 2: International comparisons from the solutions of (13) and (14) satisfy the commensurability property.

Proof: International comparisons are said to satisfy commensurability if the volume and price comparisons derived using the method, in the form of vectors q and $\overline{p}$ are invariant to changes in the unit of measurement. Suppose units of measurement are changed so that the new price and quantity matrices are denoted by P^* and Q^*. Then

$$P^* = \hat{\omega} P \quad \text{and} \quad Q^* = \hat{\omega}^{-1} Q$$

Then by direct substitution of P^* and Q^* into equations (13) and (14) it can be shown that the resulting quantity comparisons q^* are identical to q prior to the change in the units of measurement.

<u>Theorem 3</u>: In the case of binary comparisons, the new method with N=2 provides the following explicit solutions for the PPPs and the aggregate quantity measures,

$PPP_1 = 1$ and $PPP_2 = \dfrac{(F_{12})^2}{[EM_{12}]}$ where F and EM respectively denote the Fisher and Edgeworth-Marshall index number formulae.

Proof: Consider equation (13). We have

$$Q'P \cdot \widehat{P'\bar{q}}^{-1} q = \alpha q .$$

It is easy to see that

$$Q'P = \begin{bmatrix} \sum_{i=1}^{M} p_{i1}q_{i1} & \sum_{i=1}^{M} p_{i2}q_{i1} & \cdots & \sum_{i=1}^{M} p_{in}q_{i1} \\ \sum_{i=1}^{M} p_{i1}q_{i2} & \sum_{i=1}^{M} p_{i1}q_{i2} & \cdots & \sum_{i=1}^{M} p_{i1}q_{i2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{M} p_{i1}q_{iM} & \sum_{i=1}^{M} p_{i1}q_{iM} & & \sum_{i=1}^{M} p_{i1}q_{iM} \end{bmatrix}$$

and

$$\widehat{P'\bar{q}}^{-1} = \begin{bmatrix} 1/\sum_{i=1}^{M} p_{i1}\bar{q}_i & 0 & \cdots & 0 \\ & 1/\sum_{i=1}^{M} p_{i1}\bar{q}_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/\sum_{i=1}^{M} p_{i1}\bar{q}_i \end{bmatrix}$$

Using these two expressions, the system of equations in (13) can be written as:

$$L^{**} q = q \qquad (15)$$

where $L^{**} = \left(\ell_{ij}^{**}\right) = \left(\dfrac{\sum_{k=1}^{M} p_{kj}q_{ki}}{\sum_{k=1}^{M} p_{kj}\bar{q}_k} \right)$.

Now consider the special case of the general system above with N=2 where the number of countries involved is equal to 2. Then the required vector of quantity ratios, q, can be solved from the following system of two liner homogeneous equations in as many unknowns.

$$\begin{bmatrix} 1-\dfrac{\sum_{i=1}^{M} p_{i1}q_{i1}}{\sum_{i=1}^{M} p_{i1}(q_{i1}+q_{i2})} & -\dfrac{\sum_{i=1}^{M} p_{i2}q_{i1}}{\sum_{i=1}^{M} p_{i1}(q_{i1}+q_{i2})} \\ -\dfrac{\sum_{i=1}^{M} p_{i1}q_{i2}}{\sum_{i=1}^{M} p_{i2}(q_{i1}+q_{i2})} & 1-\dfrac{\sum_{i=1}^{M} p_{i2}q_{i2}}{\sum_{i=1}^{M} p_{i2}(q_{i1}+q_{i2})} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix on the LHS is singular since each column sum is equal to zero, which shows that there is a non-trivial solution for the unknown (q_1, q_2). Suppose we let q_2 = 1 and then solve for the unknown q_1, we get

$$q_1 = \frac{\sum_{i=1}^{M} p_{i2}q_{i1} / \sum_{i=1}^{M} p_{i2}(q_{i1}+q_{i2})}{\sum_{i=1}^{M} p_{i1}q_{i1} / \sum_{i=1}^{M} p_{i1}(q_{i1}+q_{i2})}$$

$$= \frac{1/(1+P_{12})}{1/(1+P_{21})}$$

where P_{12} and P_{21} are Paasche quantity index numbers for countries 2 and 1 respectively with the remaining country as the base. It can be shown similarly that if q_1 = 1, then solution for q_2 is given by

$$q_2 = \frac{1/(1+P_{21})}{1/(1+P_{12})} \tag{16}$$

Now we turn to the formulae for the purchasing power parities (PPPs) for the currencies of the two countries. The parities can be derived using the relationship between volume indicators q_j (j=1 and 2) and PPP_j (1 and 2) given by:

$$q_j = \frac{GDP_j}{PPP_j} \text{ for } j = 1 \text{ and } 2 \tag{17}$$

where GDPj denotes the gross domestic product of country j, expressed in the currency units of country j. Given equation (17), we can derive expressions for PPP_j using equation (16). Using currency of country 1 as the numeraire, we have

$$PPP_2 = \frac{GDP_2 / q_2}{GDP_1 / q_1} = \frac{\sum_{i=1}^{M} p_{i2} q_{i2} / q_2}{\sum_{i=1}^{M} p_{i1} q_{i1} / q_1}$$

$$= \frac{\sum_{i=1}^{M} p_{i2} q_{i2}}{\sum_{i=1}^{M} p_{i1} q_{i1}} \cdot \frac{\sum_{i=1}^{M} p_{i2} q_{i1} / \sum_{i=1}^{M} p_{i2} (q_{i1} + q_{i2})}{\sum_{i=1}^{M} p_{i1} q_{i2} / \sum_{i=1}^{M} p_{i1} (q_{i1} + q_{i2})}$$

$$= \frac{P_{12}^{price} . L_{12}^{price}}{\sum_{i=1}^{M} p_{i2} (q_{i1} + q_{i2} / \sum_{i=1}^{M} p_{i1} (q_{i1} + q_{i2})}$$

$$= \frac{\left(F_{12}^{price}\right)^2}{EM_{12}^{price}}$$

where F denotes the Fisher index number and EM denotes the Edgeworth-Marshall price index numbers for country 2 with country 1 as the base.

The analytical expressions in Theorem 3 for the volume indices and the underlying PPPs serve as a guide to the results derived under the new method outlined in this paper. The PPPs are linked to the Fisher index of prices as well as the well-known Edgeworth-Marshall price index which can also be expressed as a function of the Laspeyres and Paasche index numbers.

5. Empirical Illustration

In this section we provide an empirical illustration of the new method, termed the commensurable Kurabayashi-Sakuma (CKS) method, described in Section 4. The illustration uses data drawn from the international comparisons exercised undertaken at the OECD for the benchmark year 1993. Since the results shown are for purposes of illustration only, the aggregation undertaken here considers data at a very aggregate level. The following is the list of countries considered in the illustration.

Country name	**Abbreviation used**
Germany	DEU
France	FRA
Italy	ITA
Netherlands	NLD
Belgium	BEL
Luxembourg	LUX
United Kingdom	GBR
Ireland	IRL
Denmark	DNK
Greece	GRE
Spain	ESP
Portugal	PRT
Austria	AUT
Switzerland	CHE
Finland	FIN
Iceland	ISL
Norway	NOR
Sweden	SWE
Turkey	TUR
Australia	AUS
New Zealand	NZL
Japan	JPN
Canada	CAN
United States	USA

The following is a listing of the commodity groups for purposes of aggregation. Only summary categories are selected for the purpose of this illustration. The categories are:

1. Food, Beverages and Tobacco
2. Clothing and Footwear
3. Gross Rent, Fuel and Power
4. House Furnishing and Operations
5. Medical Care
6. Transport and Communications
7. Recreation and Education
8. Other Expenditure
9. Construction
10 Producer durables
11. Net Exports
12. Government

Since these categories represent commodity groups, prices for these items are in the form of purchasing power parities derived at the category level within the international comparison exercise at the OECD. The quantities for each of these groups is then obtained by converting the nominal expenditures (in national currencies) into a common currency unit using the purchasing power parities of currencies for each of the groups. In this exercise, the item “net exports” is dropped since the net exports can be either negative or positive unlike the rest of the items. Thus treatment of such an item necessarily depends upon the conventions used. So the results presented below are based on all the items with the exception of net exports.

For purposes of comparison, we have also computed the Geary-Khamis purchasing power parities and, more importantly, the underlying real volumes which can be compared across countries. Results from the CKS method and the Geary-Khamis method are presented in the Table below.

Table 1: Volume Comparisons between OECD Countries

CKS and GK Methods

Country	CKS Method			Geary-Khamis Method		
	Volume Ratio	Population	Per capita Volume	Volume Ratio	Population	Per capita Volume
Germany	0.204	63.232	0.813	0.205	63.232	0.814
France	0.183	56.42	0.815	0.183	56.42	0.816
Italy	0.172	57.663	0.749	0.172	57.663	0.749
Netherlands	0.042	14.947	0.705	0.042	14.947	0.707
Belgium	0.030	9.967	0.750	0.030	9.967	0.752
Luxembourg	0.001	0.381	0.896	0.001	0.381	0.897
United Kingdom	0.164	57.411	0.719	0.164	57.411	0.718
Ireland	0.006	3.503	0.439	0.006	3.503	0.438
Denmark	0.016	5.14	0.759	0.015	5.14	0.758
Greece	0.015	10.123	0.375	0.015	10.123	0.376
Spain	0.082	38.959	0.531	0.082	38.959	0.532
Portugal	0.015	9.377	0.402	0.015	9.377	0.402
Austria	0.024	7.718	0.786	0.024	7.718	0.786
Switzerland	0.025	6.796	0.943	0.026	6.796	0.944
Finland	0.015	4.986	0.764	0.015	4.986	0.765
Iceland	0.001	0.256	0.773	0.001	0.256	0.773
Norway	0.012	4.241	0.702	0.012	4.241	0.703
Sweden	0.027	8.559	0.797	0.027	8.559	0.796
Turkey	0.056	56.473	0.251	0.056	56.473	0.251
Australia	0.051	17.066	0.753	0.051	17.066	0.754
New Zealand	0.008	3.379	0.598	0.008	3.379	0.598
Japan	0.419	123.54	0.853	0.420	123.54	0.854
Canada	0.095	26.62	0.895	0.095	26.62	0.895
United States	1.000	251.394	1.000	1.000	251.394	1.000

Table 1 shows that the CKS method works very well. In fact the results obtained are quite close to those obtained using the Geary-Khamis method. Such a result is to be expected since the methods are employed to aggregate data from commodity groups which are already aggregated. For example, the international comparison work normally is based on 200 or more basic headings whereas the illustration here is based only on 11 categories (twelve categories less one for exports).

Table 2: International Prices from GK and CKS methods

Commodity Group	GK International Prices	CKS International Prices
Food, Beverages and Tobacco	0.86911	0.85944
Clothing and Footwear	0.86925	0.87807
Gross Rent, Fuel and Power	0.87399	0.89717
House Furnishing and Operations	0.89869	0.88692
Medical Care	0.87692	0.8652
Transport and Communications	0.87322	0.87067
Recreation and Education	0.87504	0.87419
Other Expenditure	0.87349	0.88261
Construction	0.9096	0.89725
Producer durables	0.82943	0.83213
Government	1.25124	1.25635

Results in Table 2 show that the international price structures derived under the two methods are essentially very similar. The government has prices above average but all the other commodity groups have international price levels below unity.

The main point to note about the results in Tables 1 and 2 is that they adequately demonstrate the feasibility of the CKS method and thus possible to construct versions of the Kurabayashi-Sakuma method that satisfy the property of commensurability.

6. Conclusions

The main objectives of the present paper are to demonstrate that it is possible to specify variants of the Kurabayashi-Sakuma (1981) method for international comparisons which satisfy the property of commensurability. Previous versions of the KS method failed to satisfy this property, thus limiting its applications to international comparisons. The new method, termed the CKS method, is examined in considerable detail in Section 4. A number of important properties are established. It is shown that it is possible to provide analytical expressions for the volumes as well as purchasing power parities of currencies when the number of countries included is equal to 2. In addition, the paper has also succeeded in providing an answer to the conjecture made by Prasada Rao (1997) that Geary-Khamis method is the only method that satisfies additive consistency in the framework of Rao-Khamis where the eigen values involved in equations that provide solutions for the parities and volumes must be equal to unity. In this paper, a method which is different from the Geary-Khamis method is shown to satisfy the additive consistency property, thus providing an answer in the negative to the Rao conjecture. On the basis of this result, it appears that it may be feasible to construct several other variants which will have the same additive consistency property.

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