Increasingly Unequal Yet Similar 21st Century Constituent Canadian Income Distributions: Introducing and Applying New Gini Based Polarization, Segmentation and Ambiguity Indicators

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January 2018

Summary.

Inequity across its many constituencies has long been a focus of the Canadian political agenda, its income distribution, a mixture of many different constituent Aboriginal–Non-Aboriginal, Male–Female, Urban–Rural sub distributions, is a case in point. Social Justice arguments with respect to incomes suggest that, given common circumstance and effort, different constituencies should exhibit similarity in their respective income distributions (unfortunately increasing similarity also increases the potential for ambiguity or lack of unanimity among wellbeing indicators in a given class). Assuming such commonalities, measuring equality among constituencies is a matter of capturing the degree to which constituent distributions are not dissimilar which is in turn related to notions of polarization. In exploiting the non-decomposability property of the Gini coefficient, this study of the evolution of constituent Canadian Income distributions introduces three new tools for measuring the extent of segmentation, polarization and the potential for ambiguity in societal Income Wellbeing orderings. The results reinforce important distinctions between inequality and polarization, revealing increasing inequality coincident with diminishing segmentation and polarization in 21st Century Canadian Incomes, suggesting some advancement of the avowed Equal Opportunity agenda. There also appears to be increasing potential for ambiguity or conflict among wellbeing indicators.

1. Introduction

Concerns about emerging and disappearing classes (Atkinson and Brandolini 2013, Banerjee and Duflo 2008, Quah 1997) have fostered interest in societal trends in inequality, polarization and segmentation. While Gini's Inequality and Transvariation measures (Gini 1912, 1916) have been foundational in this area, they are not without some limitation when applied to a collection of distributions. The Transvariation measure is confined to comparing just 2 distributions and, when contemplating inequality within and between a collection of K subgroups in a society, the Gini coefficient is not subgroup decomposable (Bourguignon 1979), so that the extent of inequality cannot be readily attributed to inequalities within and between subgroups. However, here this property of the Gini is used to advantage by exploiting the fact that when subgroups are completely segmented, Gini is subgroup decomposable (Mookherjee and Shorrocks 1982, Yitzahki 1979, 1994). Under complete segmentation (i.e. non-intersecting subgroup distributions or maximal multi-group Transvariation) the societal Gini is a weighted sum of within subgroup Gini's plus a between subgroup Gini, all of which are readily computed when subgroups are identified. Thus the difference between the overall Gini and what it would be under perfect segmentation presents a measure of the extent to which a society is not completely segmented. Basically the residual is a weighted sum of measures of the extent to which subgroup distributions overlap. Noting that the extent and nature of distributional overlap presents difficulties for wellbeing index construction (it engenders ambiguity or conflict among wellbeing indicators in a given class) an index of the potential for ambiguity in a collection of sub-distributions can also be derived. These relationships are exploited in developing new segmentation, polarization and ambiguity indices to facilitate examination of the evolution of subpopulation income distributions in Canada.

Treating the Canadian Income Distribution as a mixture of many constituent distributions (for example Aboriginal–Non-Aboriginal, Male–Female and Urban–Rural), understanding the progress (or not) toward equality of Income wellbeing across constituencies becomes a matter of understanding how the changing shape of the overall distribution depends upon the changing anatomy of its constituent distributions. There are many instruments for describing the nature of distributional anatomy, measures of location, dispersion etc. combinations of which turn out to provide complete, though often ambiguous[1] (i.e.

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[1] The ambiguity issue is best illustrated in the debate over whether mean or median incomes should be employed as a measure of wellbeing (Stiglitz, Sen and Fitoussi 2010). Both measures are in the class of monotonic non-decreasing wellbeing functions but frquently yield conflicting orderings when employed in comparing a collection of

conflicting), orderings of Income Wellbeing states. As an alternative, Stochastic Dominance criteria have been employed to provide an unambiguous but incomplete, i.e. partial, ordering of states. Recently employed to study equality of opportunity issues (Lefranc, Pistolesi and Trannoy, 2008, 2009), a problem with this technique is its partial nature, it is non-informative as to how much better one state is than another and, with respect to equality of opportunity, it can only inform as to whether the optimal state has or has not been achieved, it doesn't facilitate measurement of progress towards the optimal state.

An alternative way of construing the issue is that a given difference in constituent group average incomes has less import when the constituent income distributions have high variance (distributions overlap a lot) than when their respective variances are low (distributions overlap little). This notion is fundamental to conceptually distinguishing Polarization from Inequality, a distinction which has not always been considered important (Zhang and Kanbur 2001). Founded upon notions of within group association and between group alienation, Esteban and Ray (1994) and Duclos, Esteban and Ray (2004) formulated a family of Polarization indices for an overall distribution on the basis of the extent to which its many latent or unidentified sub-distributions were separate or segmented and individually concentrated. This family is indexed by a polarization intensity parameter $\alpha > 0$ which has as a special case the Gini coefficient when $\alpha = 0$ (in effect higher values of α intensify the importance of nodes in the overall calculus).

Sorting out the degree of segmentation between two groups is a relatively straightforward problem (Anderson 2004), sorting out the extent of segmentation in many groups is more complex but it can be facilitated by utilizing the aforementioned lack of subgroup decomposability of the Gini. When it is decomposable (i.e. the subgroups are segmented), it can be written as a simple linear function of subgroup Gini coefficients, their population shares and relative mean income differences, in essence a weighted sum of the subgroup and between group Gini coefficients. It follows that the difference between the value of the overall Gini and this simple linear function can be used to examine the extent to which constituencies are segmented or not, which in turn contributes to the development of a Gini related many group polarization index.

Here these ideas are employed to study the degree of segmentation and polarization of identified constituent distributions of an overall distribution in the context of a decomposition of the Gini coefficient. The architecture of Gini-based segmentation and polarization indices is developed in Section

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societies. They would not conflict if the societies respective income distributions obeyed a strict first order dominance ordering.

2. They are related to the Absolute Gini and associated Lorenz and Generalized Lorenz curves as measures of wellbeing together with the introduction of an Ambiguity Index for complete orderings in a given class in Section 3. After the Canadian background for interest in these issues is outlined in section 4, the measures are employed in examining the degree and progress of income inequality and segmentation and index ambiguity of 10 Aboriginal-non-Aboriginal gender- and geographic-based constituencies in Canada in the first decade of the 21st Century in Section 5. This in turn facilitates reflection upon progress with the Equal Opportunity imperative in the conclusions in Section 6. The results reinforce the inequality – polarization distinction, indicating that amidst the almost ubiquitous increase in inequality over the period, the changing anatomy of the constituent income distributions has resulted in constituencies sharing more in common, i.e. becoming less polarized and more alike, indicating some progress toward an equal opportunity goal over the period but with increasingly ambiguous wellbeing measures.

2. The Gini Coefficient and Its Implicit Segmentation and Polarization Indices

Suppose a society has K constituent subgroups labelled $k = 1,..,K$, following Mookherjee and Shorrocks (1982) its Gini coefficient is the sum of 3 components, a weighted sum of subgroup Gini coefficients, a weighted sum of subgroup absolute relative mean differences (in essence a between group GINI) and a component, NSF, which is a measure of the extent to which the subgroups are not segmented (see Appendix for derivations). Letting the kth subgroup Gini be G_k, the relative subgroup size be w_k and its mean income μ_k and noting that the overall mean $\mu = \sum_{k=1}^{K} w_k \mu_k$, NSF may be written as:

$$NSF = GINI - \sum_{k=1}^{K}(w_k)^2 \frac{\mu_k}{\mu} G_k - \sum_{k=1}^{K}\sum_{h=1}^{K} \frac{w_k w_h}{2\mu}|\mu_k - \mu_h| \tag{1}$$

In the following, discussion is pursued in terms of continuous distributions where $f(x)$ is considered a mixture of K subgroup distributions $f_k(x)$, $k = 1,..K$, with compact support on R$^+$ such that:

$$f(x) = \sum_{k=1}^{K} w_k f_k(x)$$

where $f_k(x)$ is such that $E_{f_k(x)}(x) = \mu_k$; $\infty > V_{f_k(x)}(x) = \sigma_k^2 > 0$ and $\sum_{k=1}^{K} w_k = 1$ so $E_{f(x)}(x) = \mu$. For convenience let $\mu_k > \mu_j \Leftrightarrow k > j$. The Gini coefficient is given by:

$$GINI = \frac{1}{E(x)} \int_0^\infty \int_0^\infty f(y) f(x) |x - y| dx dy \qquad (2)$$

where $E(x) = \mu = \sum_{k=1}^K w_k \mu_k$.

From (2) it follows (see Appendix) that:

$$GINI = \sum_{k=1}^K w_k^2 \frac{\mu_k}{\mu} G_k + \frac{1}{\mu} \sum_{k=2}^K \sum_{j=1}^k w_k w_j |\mu_k - \mu_j|$$

$$+ \frac{2}{\mu} \sum_{k=2}^K \sum_{j=1}^{k-1} w_k w_j \int_0^\infty f_k(y) \int_y^\infty f_j(x) (x - y) dx dy$$

$$= \sum_{k=1}^K w_k^2 \frac{\mu_k}{\mu} G_k + \frac{1}{\mu} \sum_{k=2}^K \sum_{j=1}^k w_k w_j |\mu_k - \mu_j| + NSF$$

$$(3)$$

where $NSF = \frac{2}{\mu} \sum_{k=2}^K \sum_{j=1}^{k-1} w_k w_j \int_0^\infty f_k(y) \int_y^\infty f_j(x) (x - y) dx dy$.

$GINI$ is thus a weighted sum of subgroup $GINI$s plus a weighted sum of subgroup "dominating mean differences" plus a component which is a weighted sum of the extent to which there are individuals in lower group j who overlap with, i.e. have greater incomes than, individuals in upper group k weighted by the extent to which they have more. In essence, $GINI$ is a linear function of within and between group Gini coefficients plus a term measuring the extent to which subgroups overlap or are not segmented.

Considering NSF, first note that when subgroups k and j are perfectly segmented (so that $f_k(x) = 0$ for all $f_j(x) > 0$ and $f_j(x) = 0$ for all $f_k(x) > 0$), the corresponding term in the component vanishes. To see this, for any $j \neq k$, consider the corresponding term in NSF and observe that if $f_k(y) > 0$ and $f_j(y) = 0$ for $y \in Y^*$, $f_k(y) = 0$ and $f_j(y) > 0$ for $y \in Y^{**}$, and $f_k(y) = 0$ and $f_j(y) = 0$ for $y \in Y^{***}$ and $Y^* \cup Y^{**} \cup Y^{***} \equiv R^+$ then:

$$\int\limits_{0}^{\infty} f_k(y) \int\limits_{y}^{\infty} f_j(x)\,(x-y)dxdy$$

$$= \int\limits_{y\in Y^*}^{\infty} f_k(y) \int\limits_{y}^{\infty} f_j(x)\,(x-y)dxdy + \int\limits_{y\in Y^{**}}^{\infty} f_k(y) \int\limits_{y}^{\infty} f_j(x)\,(x-y)dxdy$$

$$+ \int\limits_{y\in Y^{***}}^{\infty} f_k(y) \int\limits_{y}^{\infty} f_j(x)\,(x-y)dxdy = 0$$

In the particular case where this is true for all $j \neq k$, observe the Mookherjee and Shorrocks (1982) result:

$$GINI = \sum_{k=1}^{K} w_k^2 \frac{\mu_k}{\mu} GINI_k + \frac{2}{\mu}\sum_{k=2}^{K}\sum_{j=1}^{k} w_k w_j \left|\mu_k - \mu_j\right|$$

Noting that in general all three components of $GINI$ are non-negative and that $0 \leq NSF \leq GINI$, then $0 \leq NSF/GINI \leq 1$, and thus SI, a segmentation index, may be written as:

$$SI = 1 - \frac{NSF}{GINI} \tag{4}$$

where $0 \leq SI \leq 1$ provides an index of segmentation, a measure of the degree to which constituent groups are segmented. Furthermore, the analysis can be done with respect to particular groups, so the extent to which the poor or the rich are segmented from the rest of society may be readily analyzed. Considering just the poor group ($j = 1$) observe the component:

$$NSF_{poor} = \frac{2}{\mu}\sum_{k=2}^{K} w_k w_l \int\limits_{0}^{\infty} f_k(y) \int\limits_{y}^{\infty} f_l(x)(x-y)dxdy$$

This is twice a weighted sum of the (expected) average value of the excess of incomes of people in the poor group over those of people in the non-poor groups normalized by average income which is of interest in contemplating the "isolation" of the poor. Similarly, an index of the segmentation of the "rich" group ($k = K$) can be obtained as:

$$NSF_{rich} = \frac{2}{\mu}\sum_{j=1}^{K-1} \int\limits_{0}^{\infty} f_K(y) \int\limits_{y}^{\infty} f_j(x)(x-y)dxdy$$

which is twice a weighted sum of the (expected) average value of the excess of incomes of people in the poor group over those of people in the richest group normalized by average income which is of interest in contemplating the "isolation" of the rich. Clearly NSF_{poor} or NSF_{rich} could be inserted in place of NSF in (4) to obtain an index of the segmentation of the poor or rich respectively.

2.1. A Gini Based Polarization Index with Many Subgroups

Conceptually polarization is based upon the concepts of between group alienation and within group association. A Gini based Bi-Polarization measure (Wolfson, 1994) concerned itself with just 2 groups, essentially measuring the difference between the combined empirical distribution and one which has all of the population concentrated at the center. In the present context there are many (i.e. more than 2) groups and interest focusses on the extent to which the various pairings in such a collection are polarizing from each other as a collection. To capture this, a general polarization index covering many, possibly latent, groups was developed for continuous distributions in Duclos, Esteban and Ray (2004), where $g(x)$ is the probability distribution function, may be written as:

$$P_\alpha(g) = S \int_0^\infty g(x) \int_0^\infty g(y)^{1+\alpha} |y - x| dy dx \tag{5}$$

Here S is a standardising factor and α is the polarization sensitivity factor, which is confined to [0.25,1] and, if α is set to 0 and S is suitably chosen, (5) yields the continuous distribution version of Gini in (2). $P_\alpha(g)$ can be interpreted as the expected value of all possible rectangles formed under the distribution with height $g(y)^{1+\alpha}$ and base $|y - x|$ where $g(y)$ reflects the association component (larger $g(y)$ reflects more association) and $|y - x|$ reflects the alienation factor.

To develop this index, Duclos, Esteban and Ray (2004) work with a domain that is the union of one or more basic symmetric unimodal densities $f(y)$ with compact support which are subject to "slides" and "squeezes". Slides, which move basic densities apart and squeezes, which concentrate densities around their location, are deemed polarizing. When f undergoes a slide, the basic density is effectively re-centered in a way that preserves its shape. When $f(x)$ is subject to a positive slide (adding a constant $\theta > 0$ to each value of x), the distribution of the transformed variable $f^\theta(x)$ will First Order Dominate (FOD) the original distribution since $F^\theta(x) \leq F(x)$ for all x with strict inequality somewhere. A squeeze transforms a basic density by concentrating mass around the mean so that, for $\lambda > 0$, the new "squeezed" density is given by:

$$f^{\lambda}(x) = \frac{1}{\lambda} f\left(\frac{x - (1 - \lambda)\mu}{\lambda}\right)$$

For $0 < \lambda < \lambda' < 1$, f^{λ}, $f^{\lambda'}$ can be shown to be proper densities such that $E(X|f^{\lambda}) = E(X|f^{\lambda'}) = E(X|f)$ and $f^{\lambda'}$ Second Order Dominates f^{λ}, which in turn Second Order Dominates f so that:

$$\int_0^\infty f^{\lambda'}(y) \int_y^\infty f(x)(x - y)dxdy \leq \int_0^\infty f^{\lambda}(y) \int_y^\infty f_j(x)(x - y)dxdy$$

which is to say polarizing squeezes reduce the non-segmentation factor. However, squeezes will not alter the Between Group Gini factor which only depends upon between group means. Note that while polarizing slides can be associated with increasing between group inequalities, polarizing squeezes cannot, so a sufficient condition for establishing polarization (convergence) between groups is a combination of increased (decreased) between group inequality and segmentation.

To study the connection between segmentation and polarization, attention is focused on any component pair in the decomposition $f_i(x)$ and $f_j(x)$ where, conveniently assuming that $\mu_i = \int x f_i(x)dx > \int x f_j(x)dx = \mu_j$, the corresponding component in the NSF sum is given by:

$$w_i w_j \int_0^\infty f_i(y) \int_y^\infty f_j(x)(x - y)dxdy$$

and the corresponding component in the between group inequality Gini is $w_i w_j (\mu_i - \mu_j)$, the weighted sum of the difference in means. Clearly the Between Group Inequality component will enlarge under the θ transformation of $f_i(x)$ increasing the alienation factor. Furthermore, since First Order Dominance implies Second Order Dominance, for $0 < \theta < \theta'$ observe that, for the corresponding component of the segmentation factor:

$$\int_0^\infty f_i^{\theta'}(y) \int_y^\infty f_j(x)(x - y)dxdy \leq \int_0^\infty f_i^{\theta}(y) \int_y^\infty f_j(x)(x - y)dxdy$$

which is to say a polarizing slide increases the between group inequality factor and reduces the non-segmentation factor. As the expected value under $f_i(y)$ of the partial moment of x above y, the NSF component is a measure of the extent to which agents in the lower income distribution f_j have higher incomes than agents in the higher distribution f_i. It follows that a sufficient (though not necessary)

condition for a polarizing change is a non-decreasing change in the between group factor standardized by the overall Gini combined with a non-decreasing change in segmentation. This would be captured by GBP, a Gini Based Polarization Index, which is a weighted sum of the between group Gini coefficient (BGINI) and the segmentation index. Here the weighted geometric mean is chosen which, for $0 < \gamma < 1$, may be written as:

$$GBP = \left(\frac{BGINI}{GINI}\right)^{\gamma} SI^{1-\gamma}$$ (6)

In the following, $\gamma = 0.5$.

Within the context of the whole population, increasing the number of subgroups increases the likelihood of overlap (decreasing segmentation) and increases between group inequality. Intuitively, increasing the number of groups on a common support (for example considering males and females or urban and rural populations of the same group as separate constituencies) increases the chance of overlap i.e. increases the size of the non-segmentation factor and thus decreases the chance of segmentation. Similarly, it will increase the between group inequality and thus *a fortiori* reduce the within group inequality component (since overall inequality remains unaffected). To understand this, consider the hth constituency to be a mixture of two sub constituencies with distributions $f_{hl}(x)$ and $f_{hu}(x)$ and relative sizes w_{hl} and $(1 - w_{hl})$ where $0 < w_{hl} < 1$, then considering them separately will simply add a non-negative term to the non-segmentation factor. For convenience suppose that $\mu_{h-1} < \mu_{hl} < \mu_h < \mu_{hu} < \mu_{h+1}$ so there is no change in ordering of distributions and, noting that $f_h(x) = w_{hl}f_{hl}(x) + (1 - w_{hl})f_{hu}(x)$, NSF_h, the sum of terms in the non-segmentation factor involving $f_h(x)$ may be written as:

$$NSF_h = \frac{2}{\mu}\left\{w_h \sum_{j=1}^{h-1} w_j \int_0^\infty f_h(y) \int_y^\infty f_j(x)(x-y)dxdy + \sum_{k=2}^{K} w_k w_h \int_0^\infty f_k(y) \int_y^\infty f_h(x)(x-y)dxdy\right\}$$

which may be written:

$$\frac{2}{\mu}\left\{w_h \sum_{j=1}^{h-1} w_j \int_0^\infty w_{hl}f_{hl}(y) \int_y^\infty f_j(x)(x-y)dxdy + \sum_{k=2}^{K} w_k w_h \int_0^\infty f_k(y) \int_y^\infty w_{hl}f_{hl}(x)(x-y)dxdy\right\}$$

$$+\frac{2}{\mu}\left\{w_h \sum_{j=1}^{h-1} w_j \int_0^\infty (1-w_h^*)f_{hu}(y) \int_y^\infty f_j(x)(x-y)dxdy + \sum_{k=2}^{K} w_k w_h \int_0^\infty f_k(y) \int_y^\infty (1-w_h^*)f_{hu}(x)(x-y)dxdy\right\}$$

When considered as separate constituencies,

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$$NFS_{h+} = \frac{2}{\mu}\left\{ w_h \sum_{j=1}^{h-1} w_j \int_0^\infty w_{hl}f_{hl}(y) \int_y^\infty f_j(x)(x-y)dxdy + \sum_{k=2}^{K} w_k w_h \int_0^\infty f_k(y) \int_y^\infty w_{hl}f_{hl}(x)(x-y)dxdy \right\}$$

$$+ \frac{2}{\mu}\left\{ w_h \sum_{j=1}^{h-1} w_j \int_0^\infty (1-w_{hl})f_{hu}(y) \int_y^\infty f_j(x)(x-y)dxdy \right.$$

$$\left. + \sum_{k=2}^{K} w_k w_h \int_0^\infty f_k(y) \int_y^\infty (1-w_{hl})f_{hu}(x)(x-y)dxdy \right\}$$

$$+ \frac{2}{\mu}\left\{ w_h^2 w_{hl}(1-w_{hl}) \int_0^\infty f_{hu}(y) \int_y^\infty f_{hl}(x)(x-y)dxdy \right\}$$

$$= NSF_h + \frac{2}{\mu}\left\{ w_h^2 w_{hl}(1-w_{hl}) \int_0^\infty f_{hu}(y) \int_y^\infty f_{hl}(x)(x-y)dxdy \right\} \geq NSF_h$$

To see the effect on between group inequality, consider in a similar fashion BGI_h, the between group inequality factor components associated with h, which may be written as:

$$BGI_h = \frac{2}{\mu}\left\{ w_h \sum_{j=1}^{h} w_j |\mu_h - \mu_j| + w_h \sum_{k=h+1}^{K} w_k |\mu_k - \mu_h| \right\}$$

$$= \frac{2}{\mu}\left\{ w_{hl}w_h \sum_{j=1}^{h} w_j |\mu_{hl} - \mu_j| + w_h \sum_{k=h+1}^{K} w_k |\mu_k - \mu_{hl}| + (1-w_{hl})w_h \sum_{j=1}^{h} w_j |\mu_{hu} - \mu_j| \right.$$

$$\left. + w_h \sum_{k=h+1}^{K} w_k |\mu_k - \mu_{hu}| \right\}$$

when considered as separate constituencies,

$$BGI_{h+} = \frac{2}{\mu}\left\{ w_{hl}w_h \sum_{j=1}^{h} w_j |\mu_{hl} - \mu_j| + w_h \sum_{k=h+1}^{K} w_k |\mu_k - \mu_{hl}| + (1-w_{hl})w_h \sum_{j=1}^{h} w_j |\mu_{hu} - \mu_j| + \right.$$

$$\left. w_h \sum_{k=h+1}^{K} w_k |\mu_k - \mu_{hu}| \right\}$$

$$+ \frac{2}{\mu}(1-w_{hl})w_{hl}w_h^2 |\mu_{hu} - \mu_{hl}| > BGI_h$$

3. Absolute Gini, Generalized Lorenz Dominance and Wellbeing Measurement

Gini and the foregoing associated decomposition measures are all relative measures, considering distances between individuals or groups relative to an overall mean location measure. They are closely

related to Lorenz Curve analysis through the relationship between the Gini coefficient and the Lorenz curve (essentially $GINI = 2 \int (p - L(p)) dp$ where $L(p)$ is the Lorenz function relating L, the proportion of aggregate average income received, to p, the corresponding bottom proportion of the income ordered population). When Lorenz curves do not intersect the corresponding Gini coefficients represent an unambiguous ordering of relative to mean constituency inequality levels. (Atkinson 1970)[2] demonstrated equivalency of Lorenz curve comparisons and second order dominance or wellbeing comparisons of distributions with common means, in essence all constituencies are compared as if they enjoy the same average level of income, the only thing differentiating them is their relative inequality levels.

Patently, in the current context, average incomes vary across constituencies, to circumvent this problem Shorrocks (1983) proposed Generalized Lorenz Dominance comparisons, essentially comparing the Generalized Lorenz curves $GL_k(p)$ $k = 1, .., K$, (here $GL_k(p) = \mu_k L_k(p)$ when μ is mean income). This turns out to be equivalent to standard second order dominance comparisons of income distributions when the underlying wellbeing function is assumed monotonic, non-decreasing and concave in income. In this paradigm income distribution $f_A(x)$ is unambiguously preferred to distribution $f_B(x)$ under all such wellbeing functions when their corresponding Generalized Lorenz curves $GL_A(p)$ and $GL_B(p)$ obey the following relationship:

$$GL_A(p) - GL_B(p) \geq 0 \ \forall \ p \text{ with strict inequality somewhere.}$$

Closely associated with the Gini coefficient is $AGINI$, the Absolute Gini which has a corresponding relationship with the Generalized Lorenz curve (Hey and Lambert 1980, Weymark 2003, Yitzhaki 1979). Simply put, $AGINI = \mu * GINI$, thus, recalling Gini's relationship to the Lorenz curve, $AGINI$ may be seen to be twice the area between the Generalized Lorenz curve ($\mu * Lorenz$) and $\mu * 45°$ line. To see the relationship with the decompositions from (3) observe:

$$GINI = \sum_{k=1}^{K} w_k^2 \frac{AGINI_k}{\mu} + \frac{1}{\mu} \sum_{k=2}^{K} \sum_{j=1}^{k} w_k w_j |\mu_k - \mu_j| + \frac{2}{\mu} \sum_{k=2}^{K} \sum_{j=1}^{k-1} w_k w_j \int_0^\infty f_k(y) \int_y^\infty f_j(x)(x - y) dx dy$$

and

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[2] It is interesting to note that, in his seminal paper, Atkinson (1970) was somewhat skeptical of the usefulness of dominance comparisons because of the many Lorenz curve intersections he had observed in practice.

$$AGINI = \sum_{k=1}^{K} w_k^2 AGINI_k + \sum_{k=2}^{K} \sum_{j=1}^{k} w_k w_j |\mu_k - \mu_j| + 2 \sum_{k=2}^{K} \sum_{j=1}^{k-1} w_k w_j \int_0^\infty f_k(y) \int_y^\infty f_j(x)(x-y)dxdy$$

where $AGINI_k = 2\mu_k \int_0^1 (p - L_k(p))dp$, $L_k(p) = \int_0^{F_k^{-1}(p)} f_k(x)dx$ and $GL_k(p) = \mu_k L_k(p)$, $k = 1, \dots K$. Along with Atkinson's Equally Distributed Income Level, the Absolute Gini has long been a part of the standard toolbox for measuring wellbeing (Atkinson 1970, 1983, Sen 1973) in the context of wellbeing functions assumed monotonic, non-decreasing and concave in incomes.

Generalized Lorenz Dominance is a two-way comparison, here the income distributions of many constituencies are being compared simultaneously. A useful tool for this purpose is the Generalized Lorenz Transvariation (GLT) (Leo 2017), defined in terms of the upper and lower envelopes of the collection of K Generalized Lorenz Curves $GL_k(p)$, $k = 1, \dots, K$, this may be written as:

$$GLT = \int_0^1 \max(GL_1(p), GL_2(p), \dots, GL_k(p)) - \min(GL_1(p), GL_2(p), \dots, GL_k(p)) \, dp$$

GLT corresponds to a measure of the potential variation in a collection of Generalized Lorenz curves. It is analogous to an extension of Gini's Transvariation (Gini 1916, 1959, Anderson, Linton and Thomas 2017) for a collection of distributions and is the distributional analogue of the range of a collection of numbers, i.e. it is like the potential range of a collection of Generalized Lorenz curves.

To compare many distributions, Anderson, Post and Whang (2017)[3] proposed "Utopia-Dystopia" indices, which, for a given order of dominance, facilitate a complete ordering of many distributions,[4] GLT is used in computing analogous indices, $UDIGL$, for Generalized Lorenz curves of the form:

$$UDIGL_k = \frac{\int_0^1 \left[GL_k(p) - \min_k(GL_1(p), GL_2(p), \dots, GL_k(p)) \right] dp}{GLT}$$

Like their distributional counterparts, these indices obey many of the usual axioms (Anderson and Leo 2017) relating to wellbeing and inequality indices and reflect wellbeing in the class of monotonic non-decreasing, concave wellbeing functions. More importantly they provide a complete, though not

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[3] See also Anderson and Leo (2017) and Leshno and Levy (2016).
[4] Whereas in the Gini decomposition the influence of a constituency was weighted by its relative size, here GLC comparisons are made without weighting, thus having a representative agent flavour.

necessarily unambiguous[5], ordering of the underlying distributions, hence the need for a measure of ambiguity.

If a particular curve uniquely dominated all other curves its index would be 1 (i.e. Utopian), if it was dominated by all other curves its index would be 0 (i.e. Dystopian). Thus ambiguity arises when curves intersect in which case the dominance relationship is contradicted, if the region of contradiction is small one could claim "almost dominance", an analysis which is closely related to the notion of Almost Stochastic Dominance in a two way comparison (Leshno and Levy 2002) translated to Generalized Lorenz curves (Zheng 2016).[6] Leshno and Levy's idea was to compare the contradiction area between the two curves with their overall transvariation, if it was small in some pre-specified manner then "almost dominance" could be claimed.[7] An alternative interpretation is that it is a measure of the degree of ambiguity in the ordering of the pair. Here, in the context of many curves that are subject to comparison, the contradiction area between two curves will be compared to the overall Transvariation of the collection of curves. The intuition is that the relevant comparator for the contradiction area is the degree of distributional variation in the collection of distributions. If interest centers on ambiguity, then this would be an appropriate measure of the degree of ambiguity and an aggregation of such measures would correspond to a measure of the degree of ambiguity in the system – a measure of the extent to which the curves are not unambiguously different.

3.1. An Ambiguity Measure

Letting I(z) be an indicator function returning 1 when $z > 0$ and 0 otherwise, and where $\int (GL_a(p) - GL_b(p))dp > 0$, the operational index $\theta(f_a(x), f_b(x))$ in terms of Generalized Lorenz curves employed by Leshno and Levy (2002) is given by:

$$\theta\big(f_a(x), f_b(x)\big) = \frac{\int_0^1 I(GL_b(p) - GL_a(p))(GL_a(p) - GL_b(p))dp}{\int_0^1 |GL_a(p) - GL_b(p)|dp}$$

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[5] If there were no intersections in the Generalized Lorenz curves the ordering would be unambiguous as would any index in the class of monotonic, non-decreasing, and concave wellbeing indices.
[6] For inference in this environment see Leo (2017).
[7] Their idea, in the context of portfolio choice, was based upon the notion that some pathological risk averse preferences, for example preferring a certain $1 to a 99% chance of $100000, should be ruled out. In the present context, societal preferences that would see a society prepared to give up a large amount of income for a very small reduction in inequality are being ruled out.

Extending this to many distributions consider a collection of K distributions $f_k(x)$ with corresponding Generalized Lorenz curves $GL_k(p), k = 1, .., K$, where, for convenience and without loss of generality, $i > j \implies \mu_i \geq \mu_j$ for $i, j \in 1, .., K$. Note this implies $\int \left(GL_j(p) - GL_i(p) \right) dp = \mu_i - \mu_j \geq 0$ for $i > j$. Define $GLT(K)$, the Generalized Lorenz analogue of Gini's distributional Transvariation (Gini 1916, Dagum 1968, 2017, Anderson, Linton and Thomas 2017, Leo 2017) for a collection of distributions as:

$$GLT(K) = \int_0^1 \{\max\left(GL_1(p), GL_2(p), ... GL_K(p)\right) - \min(GL_1(p), GL_2(p), ... GL_K(p))\} dp$$

Then an ambiguity index $AI(K)$ can be written as:

$$AI(K) = \frac{\gamma}{GLT(K)} \sum_{j=2}^{K} \sum_{i=1}^{j} \int_0^1 I(GL_i(p) - GL_j(p))(GL_j(p) - GL_i(p)) dp$$

For some choice of $\gamma > 0$, $AI(K)$ measures magnitude of the Lorenz Dominance Transgressions in the collection relative to the Lorenz Transvariation. Natural values for γ are 1, whereby $AI(K)$ corresponds to the cumulated Lorenz Dominance transgressions relative to $GLT(K)$, $\frac{2}{K(K-1)}$, whereby $AI(K)$ corresponds to the average Generalized Lorenz Dominance transgression value relative to $GLT(K)$ over all possible pairwise comparisons and $\frac{1}{K^*}$ where K^* is the number of instances in which $I(GL_i(p) - GL_j(p))$ is non zero whereby $AI(K)$ corresponds to the average Generalized Lorenz Dominance transgression value relative to $GLT(K)$ over all pairwise comparisons that exhibited transgressions. In each of these cases when the collection of curves do not intersect at all, $AI(K) = 0$ and the respective AGini's are unambiguously ordered (indeed any index from the corresponding class would yield the same ordering), when they intersect there is potential ambiguity in the ordering and $AI(K) > 0$.

Here the third variant of γ will be used with the interpretation that $AI(K)$ is the average value of the transgression area when there is one. Thinking about it from an inferential perspective, if one wished to test the hypothesis that any ordering in the class was unambiguous and a critical value was based upon an average value of 1% of overall transvariation for all transgressions, then a version of the central limit

theorem would give us $AI(K) \sim_a N(0.01, \frac{0.0099}{K^*})$ yielding an upper tailed test of size α with a critical value

$$C = 0.01 + \sqrt{\frac{0.0099}{K^*}} Z_{1-\alpha} \,.$$

4. The Canadian Background.

The plight of Canadian Aboriginal communities has long been on the agenda across the Canadian political spectrum. In arguing how essential it was that Canada close the gaps between Aboriginal and Non-Aboriginal peoples, a former Canadian prime minister, Paul Martin, wrote: "the descendants of the people who first occupied this land deserve an equal chance to work for and to enjoy the benefits of our collective prosperity" (foreword in Weinstein, 2007, p. vi). These words echoed those of Prime Minister Mulroney during the opening statement to the First Ministers' Conference on the Rights of Aboriginal People in 1985: "I could read you the litany of social indicators on the disparities suffered by Aboriginal peoples…symptoms of an underlying problem which we must address" (Mulroney, 1985). Jean Chrétien also expressed similar concerns, asking his Cabinet to "find new and better ways to close the gap in life chances between Aboriginal and non-Aboriginal Canadians" (Chrétien, 2002). Gender equality has also been an important challenge in Canada for well over a century. Agnes Macphail, the first female elected to the Parliament of Canada, said in 1925, "I want for myself what I want for other women, absolute equality" (Crowley, 1990). Having embedded such equalities in the Charter of Rights and Freedoms Pierre Elliot Trudeau was lauded by Prime Minister Jean Chrétien who noted that Trudeau "came to this House of Commons to build a country…that affords all of its citizens an equal opportunity to succeed in life; whatever their background or beliefs" (Chrétien, 2000). In essence, these entreaties reflect a social and economic justice imperative that sees Equality of Opportunity as a good thing, inequalities that are the result of choice are not "bad," whereas inequalities that are the result of involuntary circumstance are, and clearly, Aboriginal identity and gender are seldom a matter of choice.[8]

That Aboriginal peoples in Canada have, on average, lower incomes than non-Aboriginals is well documented, as is the male-female earnings gap[9]. For example, Pendakur and Pendakur (2011) show that,

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[8] The Equality of Opportunity paradigm is premised on the proposition that, for given levels of skill and effort, inequality of income is a consequence of involuntary circumstance. Making the heroic assumption that skill and effort are commonly distributed across gender and Aboriginal constituencies, differences in the income distributions of constituencies then reflect inequalities in opportunity between the different constituencies.

[9] In this paper, we focus on earnings gaps, but we recognize that there are many other gaps between Aboriginal and non-Aboriginal people, for example, in education, labour force participation, well-being, political representation,

in comparison with non-minority native-born workers with similar characteristics, Aboriginal women in Canada faced income and earnings gaps of 10 to 20 per cent between 1995 and 2005 while Aboriginal men faced gaps of 20 to 50 per cent.[10] During this same time period, the gender pay ratio of full-time full-year workers aged 25-54 years averaged 0.70 per cent (Moyser, 2017). The perceived inequalities these gaps represent are cause for concern and, as has been argued, the nation faces a moral imperative to close them. The first decade of the 21st century saw a variety of government policies directed at improving the lot of Aboriginal peoples in pursuit of this Equal Opportunity imperative. The Affordable Housing Initiative (2001-2007) aimed at constructing and renovating affordable housing units and the Urban Aboriginal Strategy for improving the socio-economic status of urban Aboriginal people (Bonesteel 2006), the Aboriginal Head Start in Urban and Northern Communities (Kay-Raining Bird 2011), the Aboriginal Head Start on Reserve, and the First Nations and Inuit Child Care Initiative (ITK 2014) are all examples. From the perspective of gender equity, the latter half of the 20th century saw the introduction of the Charter of Rights and Freedoms guaranteeing equal status to women and men, arguably in pursuit of Equal Opportunity. In addition, at the turn of the century, the Federal Government implemented "two five-year plans for achieving equality for women: The Federal Plan for Gender Equality (1995-2000) and The Agenda for Gender Equality (2000-2005)" (Status of Women Canada, 2005, p. 14). It is thus of interest in the Canadian context to see if inequalities between men and women and Aboriginal and non-Aboriginal peoples have changed over the first decade of the 21st Century in the extent to which they are segmented or overlap.

5. Inequality and Segmentation in Canada in the 21st Century

5.1 Data Sources

Data from Statistics Canada's 2001 Census and 2011 National Household Survey (NHS) public use microdata files are employed to construct various Gini coefficients for the 2 observation years 2000 and 2010. Both the 2001 Census and the 2011 NHS provide detailed data on the social and demographic characteristics of the Canadian population. The 2001 Census was composed of two parts: the long-form and the short-form, with fewer questions, the short-form census required less time to complete than the

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incarceration, suicide, mental health, and physical health, among others (Feir and Hancock 2016, NAEDB 2012, NAEDB 2015). There are also many other gaps between women and men, for example, in time devoted to childrearing, caring for elders, and housework, as well as part-time work, violence, and political and corporate representation, among others (Status of Women Canada, 2017).

[10] That there are also significant disparities between the First Nations, Inuit and Metis Aboriginal identity groups that are recognized within Canada (NAEDB, 2012; NAEDB, 2015), features somewhat less in public discourse.

long-form census. In 2001 the short-form census was delivered to 100% of households, while the long-form census was distributed to one-fifth of Canadian households. Both surveys were mandatory. The long-form census had a response rate of about 94% (Penny, 2013). In 2011, the short-form census was delivered as usual, but the 2011 National Household Survey (NHS) replaced the long-form census. Unlike the previous long-form census, the 2011 NHS was voluntary and distributed to 33% of households. The response rate, unsurprisingly, dropped precipitously to 69%.

This change in survey methodology could have important implications for analysis if non-responses are correlated with income. However, the census is arguably still one of the best data sources for studying Aboriginal populations, since it is "by far the most complete survey of Indigenous peoples in Canada living both on and off reserve" (Feir and Hancock 2016, p. 354). In fact, for the past several decades, "100% of households on reserves have been intended to receive the long-form" (Feir and Hancock 2016, p. 354). Although the census is, in many ways, "the richest and most complete source of demographic data for the" Aboriginal population of Canada (Feir and Hancock 2016, p. 354), it is important to note that there are unique additional challenges when using data on Aboriginal peoples from the Censuses. First, some reserves in Canada are not enumerated. Second, the ethnic origin question has changed its structure a number of times. Finally, there is the issue of intra-generational ethnic mobility. Each of these could potentially cause some exogenous variation in the size and characteristics of the Aboriginal population. However, other data sources on the Aboriginal population suffer from similar challenges, and sometimes these challenges are magnified due to the exclusion of the on-reserve population (Feir and Hancock 2016).

5.2. Summary Statistics

Table 1 reports summary statistics of the basic data employed in the analysis and Table 2 reports Gini coefficients for all constituencies in both periods. Inequality in the Canadian income distribution has increased significantly over the decade from 0.4624 to 0.4927 and casual inspection suggests that inequality has increased in virtually all constituencies. Indeed, closer analysis[11] in Table 3 suggests that, with the exception of Rural Non-Aboriginal and Metis females, the statistically significant increase in

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[11] Inference was performed using Giles (2004) standard errors for Gini coefficients, an asymptotic normality assumption and independence of all subgroups within and between years. Both Modarres and Gastwirth (2006) and Davidson (2009) have noted that Giles (2004) standard errors should be considered an upper bound for the true standard errors but the Gini differences are so large relative to these standard errors that more sophisticated computation was deemed unnecessary.

inequality as measured by the Gini over the decade prevails in all constituencies. There also appear to be significant differences with regard to Gender as highlighted in Table 4.

Table 1: Summary Statistics, Total Income, Aboriginal and Non-Aboriginal, 2000 and 2010

		2000			2010		
		Income	Standard Deviation	N	Income	Standard Deviation	N
Urban Non-Aboriginal	Male	37,715	34,091.9	184,051	53,209	79,679.3	229,760
	Female	24,386	20,938.7	192,574	35,714	37,798.0	242,273
Rural Non-Aboriginal	Male	32,342	26,836.2	110,478	45,919	47,656.3	99,252
	Female	19,718	17,001.2	110,723	30,015	27,391.5	100,706
North American Indian	Male	19,609	20,024.1	4,640	29,255	38,665.9	6,655
	Female	15,546	14,260.7	5,076	24,516	25,479.6	7,306
Metis	Male	26,796	24,153.0	2,371	42,622	50,645.3	3,915
	Female	17,770	15,916.9	2,297	29,359	27,207.0	4,196
Inuit	Male	22,052	20,710.9	344	33,414	36,323.1	429
	Female	19,379	18,557.5	344	31,417	33,663.3	482
Total Population	Male	35,350	31,517.9	301,884	50,359	70,551.3	340,539
	Female	22,514	19,634.3	311,014	33,715	34,800.8	355,478

Table 2: Constituency Gini Coefficients, 2000 and 2010

Constituency	2000	2010
Urban non-Aboriginal, female	0.4475	0.4748
Urban non-Aboriginal, male	0.4504	0.5205
Rural non-Aboriginal, female	0.4436	0.4393
Rural non-Aboriginal, male	0.4240	0.4508
North American Indian, female	0.4752	0.4827
North American Indian, male	0.5169	0.5499
Metis, female	0.4604	0.4504
Metis, male	0.4673	0.4902
Inuit, female	0.4916	0.5137
Inuit, male	0.4920	0.5216
Total Population Gini, female	0.4507	0.4676
Total Population Gini, male	0.4451	0.5031
Total Population Gini	0.4624	0.4967

In 2000, with the exception of Rural Non-Aboriginal people and the Inuit, male groups experienced significantly more inequality than female groups, for Rural Non-Aboriginal people, females experience significantly more inequality than males and there was no significant difference between the genders in the Inuit constituencies. 2010 saw males experiencing significantly more inequality than females in all

groups except for the Inuit where the difference was not significant. Indeed, the difference-in-differences results in Table 5 appear to indicate that again, with the exception of the Inuit, the gender "inequality" gap appears to be widening.

Table 3: Subgroup Year-by-Year Differences 2010-2000 H_0: $GINI_{2010} - GINI_{2000} \leq 0$

	Difference (DIF)	Standard Error (SE)	$P(Z > \frac{DIF}{SE})$
Urban non-Aboriginal female	0.0273	0.0003	2.2251e-308
Urban non-Aboriginal male	0.0701	0.0003	2.2251e-308
Rural non-Aboriginal female	-0.0043	0.0003	1.0000
Rural non-Aboriginal male	0.0268	0.0003	2.2251e-308
North American Indian female	0.0075	0.0013	1.2397e-08
North American Indian male	0.0330	0.0013	9.9564e-147
Metis female	-0.0100	0.0020	1.0000
Metis male	0.0229	0.0019	4.5853e-33
Inuit female	0.0221	0.0051	7.5970e-06
Inuit male	0.0296	0.0050	1.2747e-09
Total Population, female	0.0168	0.0002	0.0000
Total Population, male	0.0580	0.0002	0.0000

Table 4: Gender Differences (Female-Male), 2000 and 2010

Constituency	2000			2010		
	Difference (DIF)	Standard Error (SE)	$P(Z < \frac{DIF}{SE})$	Difference (DIF)	Standard Error (SE)	$P(Z < \frac{DIF}{SE})$
Urban non-Aboriginal	-0.0029	0.0003	0.0000	-0.0457	0.0003	0.0000
Rural non-Aboriginal	0.0196	0.0003	1.0000	-0.0115	0.0003	0.0000
North American Indian	-0.0417	0.0014	0.0000	-0.0672	0.0014	0.0000
Metis	-0.0069	0.0022	0.0008	-0.0398	0.0022	0.0000
Inuit	-0.0004	0.0054	0.4707	-0.0079	0.0054	0.0734
Total Population	0.0057	0.0002	1.0000	-0.0355	0.0002	0.0000

Table 5: Differences in Gender Differences (Female-Male), 2010-2000

	Difference (DIF)	Standard Error (SE)	$P(Z < \frac{DIF}{SE})$
Urban non-Aboriginal	-0.0428	0.0004	0.0000
Rural non-Aboriginal	-0.0311	0.0004	0.0000
North American Indian	-0.0255	0.0020	1.5587e-37
Metis	-0.0329	0.0031	1.3767e-26
Inuit	-0.0075	0.0077	0.1650
Total Population	-0.0412	0.0003	0.0000

5.3. Decomposition of the Gini

With regard to the decomposition of the Gini index, Table 6 reports Overall Gini, Segmentation and Polarization indices together with the Within and Between Group inequality and Segmentation factors. Observe that, while all subgroup Gini's are increasing over the decade, the between group segmentation index is diminishing (note that this diminishing segmentation disappears when genders are combined, suggesting that the observed convergence has much to do with the convergence of the gender based distributions). Also, consistent with diminished polarization, the Between Group Gini coefficient is diminishing significantly over the period in all groupings[12] since the null hypothesis of a reduction of between group inequality is never rejected over the various groupings. While within group inequality increased in non-Aboriginal subgroups, it decreased for Aboriginal groups. Excluding the urban non-Aboriginal population diminishes the overall Gini consistent with the idea that rural non-Aboriginal communities are more comparable with Aboriginal communities and, as is to be expected, it does increase the segmentation index (recall reduction in the number of groups reduces the opportunity for overlap). The Gini Based Polarization index is decreasing over the period for all comparisons, all of which leads to the conclusion that while Canadian society was becoming more unequal, its constituencies were becoming less polarized, reinforcing the crucial distinction between inequality and polarization.

Table 6: Gini Decomposition

	Overall (10 groups)		Aboriginal (6 groups)		Non-Aboriginal (4 groups)		Rural Only (8 groups)		Overall Gender (2 groups)	
	2000	2010	2000	2010	2000	2010	2000	2010	2000	2010
Population Gini	0.4624	0.4967	0.4964	0.5097	0.4608	0.4956	0.4551	0.4643	0.4624	0.4967
Non-Segmentation Factor	0.2114	0.2407	0.2740	0.2903	0.2088	0.2355	0.1352	0.1682	0.1281	0.1541
Segmentation Index (SI)	0.5428	0.5153	0.4481	0.4303	0.5475	0.5248	0.7029	0.6378	0.7229	0.6897
Gini Based Polarization Index	0.3989	0.3537	0.3028	0.2904	0.3815	0.3464	0.4432	0.3921	0.4171	0.3713
Between Group Inequality	0.1355	0.1206	0.1016	0.0999	0.1320	0.1165	0.1272	0.1119	0.1113	0.0993
Within Group Inequality	0.1154	0.1354	0.1209	0.1194	0.1203	0.1436	0.1927	0.1843	0.2230	0.2432
Combined Gender Distribution Segmentation	0.6173	0.6363	0.6081	0.6130	0.6266	0.6566	0.9218	0.8592	0.4624	0.4967
Z Statistic for Between Increase[13]		-0.9102		-3.3831		-2.2806		-2.3358		-12.1060

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[12] Since it is a linear function of independent sample means, it is possible to calculate its standard error, and, assuming 2000 and 2010 are independently sampled, the standard error of the difference.

[13] For inference purposes since, with $w_0 = w_{K+1} = 0$, the between group Gini component may be written in dominating mean differences form (where $i > j \Rightarrow \mu_i > \mu_j$) as follows:

5.4 Absolute Gini, Generalized Lorenz Dominance and Relative Wellbeing

The Generalized Lorenz curves for all males and all females are shown in Figures 1 and 2 followed by "Tukeyized"[14] versions of these curves in Figures 3 and 4. These Figures nicely illustrate visually the changing anatomy of male-female income distributions over the period as well as the role of the Leshno-Levy Ambiguity analysis. From Figures 1 and 3, since the 2000 male Generalized Lorenz Curve is everywhere at least as high as the corresponding female curve, the ambiguity index would be 0, the male distribution strictly Second Order dominating the female distribution that year and hence all monotonic, non-decreasing, concave indices of economic wellbeing would yield an unambiguous i.e. identical ranking. This is not the case in 2010 (see Figure 2, and more clearly in Figure 4), where the Generalized Lorenz Curves clearly intersect, the ambiguity index will be greater than 0 and second order rankings are ambiguous suggesting a slight possibility of conflicting or contradictory rankings.

The Generalized Lorenz curves of the 10 constituencies are displayed in Figures 5 and 6 together with the upper and lower envelopes $\max GL$ and $\min GL$. As may be seen, the curves intersect in several places precluding an unambiguous ordering of states and the extent of intersections appear to be more common in 2010 suggestive of more ambiguity in 2010 than 2000. The corresponding Utopia-Dystopia indices together with the Absolute Gini and Atkinson Wellbeing[15] indices (perhaps the most popular indices in the class of monotonically, non-decreasing inequality averse wellbeing measures) and their ranks for the 10 constituencies are reported in Table 7 and, as predicted, there are several ranking conflicts, with more instances of conflicts in 2010 (9 conflicting comparisons, 1 uniformity comparison) than in 2000 (7

$$\frac{1}{\mu} \sum_{i=2}^{K} \sum_{j=1}^{i-1} (\mu_i - \mu_j) w_i w_j = \frac{1}{\mu} \sum_{i=1}^{K} \mu_i w_i \left(\sum_{j=0}^{i-1} w_j - \sum_{j=i+1}^{K+1} w_j \right)$$

Assuming μ to be a known constant and independent sampling of constituencies, subgroup means and variances can be replaced by their sample equivalents and the variance of this component written as:

$$V = \frac{1}{\mu^2} \sum_{i=1}^{K} \frac{\sigma_i^2}{n_i} \left(w_i \left(\sum_{j=0}^{i-1} w_j - \sum_{j=i+1}^{K+1} w_j \right) \right)^2$$

where n_i is the sample size drawn from constituency i and $w_i = \frac{n_i}{\Sigma_j n_j}$.

[14] Tukey (1977) proposed "Rootgrams" to emphasize differences in particular regions of a distribution (see Anderson. Linton and Thomas (2017) for an application to Generalized Transvariation analysis).

[15] This instrument is the Atkinson (1970) Index with an inequality aversion parameter of 1 (in effect a mean income less geometric mean income index) used by The United Nations Development Program (UNDP 2016).

conflicting comparisons, 3 uniformity comparisons). Notice also there is no conflict in the overall male-female rankings in either year but diagram 4 suggests there would only be a small chance of such an event.

Figure 1: Generalized Lorenz Curves (GLCs), Male-Female, 2000   Figure 2: Generalized Lorenz Curves (GLCs), Male-Female, 2010
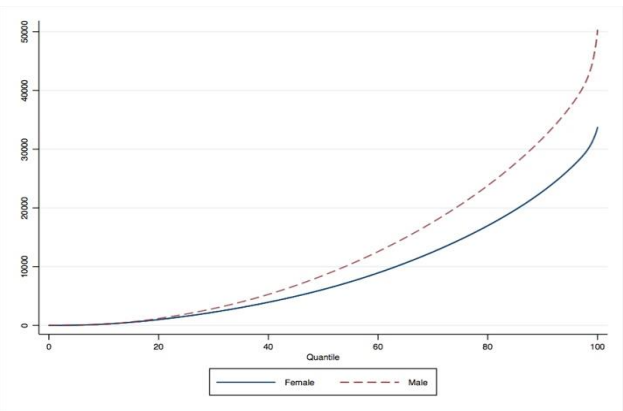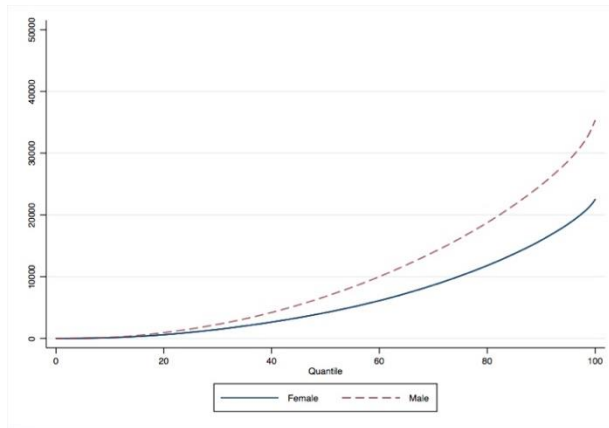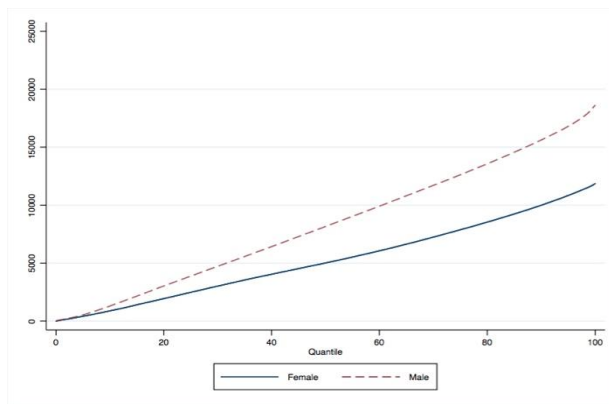

Figure 3: GLCs, Male-Female, Weighted, 2000   *Figure 4: GLCs, Male-Female, Weighted, 2010*
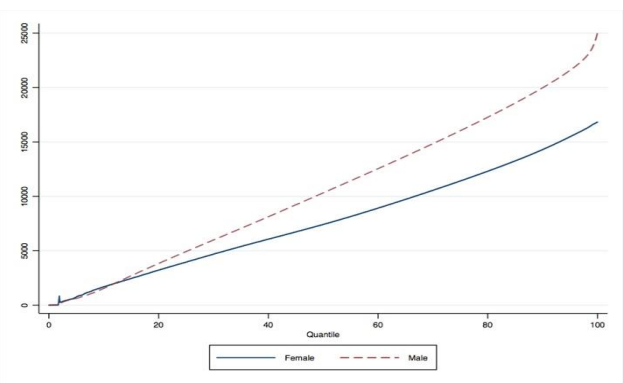

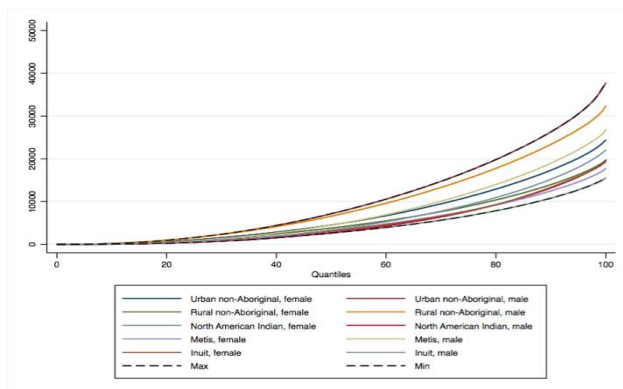Figure 5: Generalized Lorenz Curves (GLCs), 2000   Figure 6: Generalized Lorenz Curves (GLCs), 2010
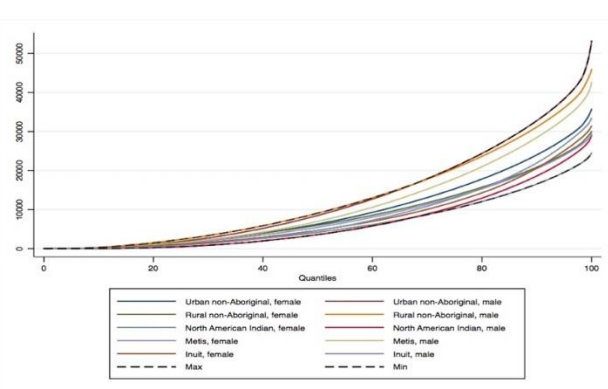
Table 7: Utopia-Dystopia, Absolute Gini and Atkinson's Wellbeing Indices, for $U'(x) > 0$, $U''(x) \leq 0$

Constituency	2000						2010					
	Utopia-Dystopia Index	Rank	Absolute Gini	Rank	Atkinson's Measure	Rank	Utopia-Dystopia Index	Rank	Absolute Gini	Rank	Atkinson's Measure	Rank
Urban non-Aboriginal, female [UNAF]	0.4240	4	10,983	4	9,597	6	0.4735	4	18,589	4	16,580	6
Urban non-Aboriginal, male [UNAM]	0.9995	1	16,877	1	15,235	1	0.9611	1	25,264	1	28,369	1
Rural non-Aboriginal, female [RNAF]	0.2256	6	8,360	8	7,406	9	0.3344	5	13,531	9	11,791	10
Rural non-Aboriginal, male [RNAM]	0.8330	2	14,347	2	11,772	3	0.9390	2	20,172	2	19,268	5
North American Indian, female [NAIF]	0.0026	10	8,036	10	7,402	10	0.0358	10	13,481	10	12,865	9
North American Indian, male [NAIM]	0.1070	9	9,318	7	10,592	4	0.0711	9	14,122	8	19,790	3
Metis, female [METF]	0.1160	8	8,304	9	7,418	8	0.2844	6	14,392	7	13,147	8
Metis, male [METM]	0.4877	3	12,337	3	12,045	2	0.6878	3	19,197	3	22,294	2
Inuit, female [INUF]	0.1370	7	9,534	6	8,408	7	0.2230	8	16,387	6	16,065	7
Inuit, male [INUM]	0.2441	5	10,841	5	10,301	5	0.2742	7	17,165	5	19,387	4
Total population, female	0.0000	2	10,148	2	8,841	2	0.0003	2	15,764	2	15,094	2
Total population, male	1.0000	1	15,733	1	13,986	1	0.9997	1	25,335	1	25,551	1

Spearman's Rank Correlation Coefficients (Standard Error 0.2108)					
2000			2010		
UD-AG	UD-AT	AG-AT	UD-AG	UD-AT	AG-AT
0.94	0.76	0.89	0.84	0.45	0.73

With regard to the Utopia-Dystopia relative wellbeing index, the Urban Non-Aboriginal male distribution is very close to Utopian and the North American Indian Female distribution is close to Dystopian in 2000. They retain their respective positions in 2010 however their gap (the observed range of the index) has narrowed from 0.9969 to 0.9032. Over the period Rural non-Aboriginal Females and Metis Females succeeded in elevating their status at the expense of Inuit Males. Obviously, the contradictions in the corresponding Index rankings suggest some ambiguity in the ranking process.

Table 8 reports the ratio of the dominance contradiction area to the full transvariation of all constituent distributions as the relevant comparator for each pairwise comparison and the average value of all such contradictions. In this comparison, reflecting the visual impression of diagrams 5 and 6, there were 11 contradictions in 2000 with an average value of 0.0075 and 16 contradictions in 2010 with an average value of 0.0135. Average areas were proportionately larger in 2010 than in 2000 implying that there was more overlap or commonality in 2010 than in 2000. If one were to set an acceptable average value of contradiction area at 0.0005 the critical value for a 5% upper tailed test would be 0.0116 in 2000 so one would fail to reject the hypothesis of "No Ambiguity" in the rankings, on the other hand in 2010 the critical value would be 0.0097 and the hypothesis of "No Ambiguity" would be rejected. It is also interesting to note from Table 7 that none of the rank correlation coefficients for the various pairings of indices were significantly different from 1 at usual levels of significance in 2000, whereas the UD-AT index coefficient Z-score was 2.609 in 2010 (a rank correlation statistic of 1 is consistent with an unambiguous ranking). All of which reinforces the Gini decomposition evidence for greater commonality amidst increasing inequality of incomes. Alternatively put the Household Income Wellbeing ordering in 2010 was more ambiguous than in 2000 because of the diminishing segmentation in Canadian Society over the preceding period.

Given a suitable income deflator one can combine the years 2000 and 2010 and make intertemporal wellbeing comparisons. Using Statistics Canada's consumer price index for a standard basket of goods, a scaling factor for 2010 incomes of 0.81888412 is obtained, subsequently yielding the 2000-2010 league table in Table 9. According to the Utopia-Dystopia index all constituencies did better in 2010 than in 2000. Urban Non-Aboriginal males did better in 2010 than any other constituency at any time. Urban non-Aboriginal males did better in 2000 than any other constituency in either 2000 or 2010, that is to say no other constituency had caught up to their 2000 level by 2010. With the exception of Urban Non-Aboriginal males, Rural non-Aboriginal males did better than all other constituencies at all times. Metis males in 2010 come next, though there is quite a drop in the index here, indeed the dominance relationships reveal Metis males are second order dominated by all non-Aboriginal Males in all years, with Urban Non-Aboriginal females in 2010 being the first to feature in the top 10. As for the ambiguity index in this case, with a value of 0.0128 and a critical value of 0.00353, the hypothesis of no ambiguity has to be rejected, indeed there are only 2 out of 20 instances of unanimity among the indices (2010 Non Aboriginal Males and 2000 North American Indian Females) and again the rank correlation coefficient for the Utopia-Dystopia-Atkinson measure pairing is significantly lower than 1, consistent with the idea of some ambiguity inherent in the ranking.

$\int \left(GL_B(x) - GL_A(x)\right)^+ dx / \int (max\ GL(x) - min\ GL(x)) dx$

Panel A: 2000
(Average Value of Leshno-Levy Instances Relative to Transvariation: 0.00748)

		Dominated								
		RNAM	METM	UNAF	INUM	RNAF	INUF	METF	NAIM	NAIF
Dominating	UNAM	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	RNAM		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	METM			0.0104	0.0000	0.0022	0.0002	0.0000	0.0000	0.0000
	UNAF				0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	INUM					0.0291	0.0028	0.0026	0.0000	0.0000
	RNAF						0.0000	0.0000	0.0000	0.0000
	INUF							0.0070	0.0004	0.0000
	METF								0.0245	0.0000
	NAIM									0.0026

Male-Female Total Population Transvariation: 3,631,589.5
Leshno-Levy Instances Relative to Transvariation: 0.0000

Panel B: 2010 Leshno-Levy Proportions
(Average Value of Leshno-Levy Instances Relative to Transvariation: 0.0135)

		Dominated								
		RNAM	METM	UNAF	RNAF	METF	INUM	INUF	NAIM	NAIF
Dominating	UNAM	0.0389	0.0000	0.0002	0.0014	0.0002	0.0000	0.0000	0.0000	0.0000
	RNAM		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	METM			0.0015	0.0047	0.0006	0.0000	0.0001	0.0000	0.0000
	UNAF				0.0036	0.0000	0.0000	0.0000	0.0000	0.0000
	RNAF					0.0000	0.0382	0.0107	0.0000	0.0000
	METF						0.0485	0.0165	0.0000	0.0000
	INUM							0.0125	0.0000	0.0020
	INUF								0.0000	0.0000
	NAIM									0.0358

Male-Female Total Population Transvariation: 3,547,839.3
Leshno-Levy Instances Relative to Transvariation: 0.0003

Table 9: League Table, 2000 and 2010

Constituency	Year	Utopia-Dystopia Index (UD)	Rank	Absolute Gini (AG)	Rank	Atkinson's Measure (AT)	Rank
Urban non-Aboriginal male	2010	0.9658	1	20,688	1	23,231	1
Urban non-Aboriginal male	2000	0.9527	2	16,877	2	15,235	6
Rural non-Aboriginal male	2010	0.9469	3	16,519	3	15,779	5
Rural non-Aboriginal male	2000	0.7943	4	14,347	6	11,772	10
Metis male	2010	0.7315	5	15,720	4	18,257	2
Urban non-Aboriginal female	2010	0.5477	6	15,223	5	13,577	7
Metis male	2000	0.4657	7	12,337	9	12,045	9
Rural non-Aboriginal female	2010	0.4285	8	11,080	12	9,655	15
Urban non-Aboriginal female	2000	0.4051	9	10,983	14	9,597	16
Metis female	2010	0.3856	10	11,785	10	10,765	11
Inuit male	2010	0.3768	11	14,056	7	15,876	4
Inuit female	2010	0.3329	12	13,419	8	13,155	8
Inuit male	2000	0.2339	13	10,841	15	10,301	14
Rural non-Aboriginal female	2000	0.2163	14	8,360	18	7,406	19
North American Indian male	2010	0.2026	15	11,564	11	16,206	3
North American Indian female	2010	0.1724	16	11,040	13	10,535	13
Inuit female	2000	0.1320	17	9,534	16	8,408	17
Metis female	2000	0.1120	18	8,304	19	7,418	18
North American Indian male	2000	0.1035	19	9,318	17	10,592	12
North American Indian female	2000	0.0041	20	8,036	20	7,402	20

Spearman's Rank Correlation Statistics (Standard Error 0.1451)

UD-AG	UD-AT	AG-AT
0.90	0.65	0.88

6. Conclusions

Inequality of life chances between various constituencies has long been on the political agenda in Canada. Under albeit strong assumptions, measuring the extent of "inequality of opportunity" is a matter of measuring the extent to which constituent income distributions are unequal. Highlighting the inequality – polarization distinction, this task is complicated by the fact that a collection of distributions can at once become more equal and more polarized. Thus, it is possible that groups in a society can become less (more) equal and yet at the same time have more (less) in common, raising the possibility of reduced (increased) disaffection among groups experiencing increasing (diminishing) inequality. Re-interpretation of a well-known sub-group decomposition of the Gini coefficient has facilitated analysis of these aspects and enabled the introduction of multi-constituency segmentation and polarization indices. Extension to the Absolute Gini and the related Generalized Lorenz Curves has provided a variety of measures for ordering constituent groups in a wellbeing sense as well as providing a measure of the degree of potential

"ambiguity" in any ordering instrument drawn from the class of monotonic, non-decreasing, and concave wellbeing indicators.

These techniques were employed in a study of the evolution of Aboriginal–Non-Aboriginal, Male–Female and Urban–Rural constituencies in Canada which revealed that, in the first decade of the 21st century, while overall inequality was increasing significantly, whatever the collection of groups, the extent of segmentation, between group inequality and ultimately polarization is diminishing. That is to say the society is less segmented and less polarized since subgroups are becoming more alike in their distributions. Overall the population is becoming more unequal, largely the result of increased inequality within constituencies. At the same time, there was greater commonality in incomes between the various constituencies leading one to conclude that attempts at addressing the inequalities of life chances across the various Canadian constituencies has not been entirely unsuccessful, assuming no major changes in the underlying constituencies through intra-generational ethnic mobility. As a sidebar, with respect to aggregate male–female distributions, it was concluded that while the ordering of any measure from the class of monotonic, non-decreasing, concave wellbeing indicators was unambiguous in 2000 it could not be deemed so in the 2010 data set, reinforcing the idea that it was harder to discriminate between constituent distributions in 2010 than in 2000.

6. Appendix

Suppose that, in a society of N individuals with incomes y_i $i = 1, \dots, N$, there are K ordered groups indexed $k = 1, \dots, K$ identified by their size N_k and ordered by their mean income μ_k so that $\sum_k N_k = N$ and $\mu_h < \mu_k \Leftrightarrow h < k$. It is known from which group the ith individual comes by writing $i \in N_k$. When two groups N_h and N_k are segmented, where $h < k$, there is no one in N_h that has an income higher than anyone in N_k. From Mookherjee and Shorrocks (1982):

$$
\begin{aligned}
GINI &= \frac{1}{2N^2\mu}\sum_{i=1}^{N}\sum_{j=1}^{N}|y_i - y_j| = \frac{1}{2N^2\mu}\sum_{k=1}^{K}\left(\sum_{i\in N_k}\sum_{j\in N_k}|y_i - y_j| + \sum_{i\in N_k}\sum_{j\notin N_k}|y_i - y_j|\right) \\
&= \sum_{k=1}^{K}\left(\frac{N_k}{N}\right)^2\frac{\mu_k}{\mu}G_k + \frac{1}{2N^2\mu}\sum_{k=1}^{K}\sum_{i\in N_k}\sum_{j\notin N_k}|y_i - y_j|
\end{aligned}
\tag{A1}
$$

With complete group segmentation between groups k and h:

$$
\sum_{i\in N_k}\sum_{j\in N_h}|y_i - y_j| = N_k N_h |\mu_k - \mu_h|
$$

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So with all groups segmented (A1) becomes:

$$\sum_{k=1}^{K}\left(\frac{N_k}{N}\right)^2\frac{\mu_k}{\mu}G_k + \sum_{k=1}^{K}\sum_{h=1}^{K}\frac{N_k N_h}{2N^2\mu}|\mu_k - \mu_h|$$

Thus, in general, (A1) may be written as:

$$\sum_{k=1}^{K}\left(\frac{N_k}{N}\right)^2\frac{\mu_k}{\mu}G_k + \sum_{k=1}^{K}\sum_{h=1}^{K}\frac{N_k N_h}{2N^2\mu}|\mu_k - \mu_h|$$
$$+\left\{\frac{1}{2N^2\mu}\sum_{k=1}^{K}\sum_{i\in N_k}\sum_{j\notin N_k}|y_i - y_j| - \sum_{k=1}^{K}\sum_{h=1}^{K}\frac{N_k N_h}{2N^2\mu}|\mu_k - \mu_h|\right\}$$

(A2)

where the non-segmentation factor (NSF) is:

$$\left\{\frac{1}{2N^2\mu}\sum_{k=1}^{K}\sum_{i\in N_k}\sum_{j\notin N_k}|y_i - y_j| - \sum_{k=1}^{K}\sum_{h=1}^{K}\frac{N_k N_h}{2N^2\mu}|\mu_k - \mu_h|\right\}$$

So that:

$$GINI = \sum_{k=1}^{K}(w_k)^2\frac{\mu_k}{\mu}G_k + \sum_{k=1}^{K}\sum_{h=1}^{K}\frac{w_k w_h}{2\mu}|\mu_k - \mu_h| + NSF$$

(A3)

To derive the segmentation version of $GINI$ in the context of continuous distributions note that:

$$G = \frac{1}{E(x)}\int_{0}^{\infty}\int_{0}^{\infty}f(y)f(x)|x-y|dxdy$$

$$=\frac{1}{\sum_{k=1}^{K}w_k\mu_k}\int_{0}^{\infty}\int_{0}^{\infty}\sum_{k=1}^{K}w_k f_k(y)\sum_{k=1}^{K}w_k f_k(x)\,|x-y|dxdy$$

$$=\frac{1}{\sum_{k=1}^{K}w_k\mu_k}\int_{0}^{\infty}\int_{0}^{\infty}\left[\sum_{k=1}^{K}w_k^2 f_k(y)f_k(x)|x-y| + 2\sum_{k=2}^{K}w_k f_k(y)\sum_{j=1}^{k-1}w_j f_j(x)\,|x-y|\right]dxdy$$

$$=\frac{1}{\mu}\int_{0}^{\infty}\int_{0}^{\infty}\sum_{k=1}^{K}w_k^2\frac{\mu_k}{\mu_k}f_k(y)f_k(x)|x-y|\,dxdy + \frac{2}{\mu}\int_{0}^{\infty}\int_{0}^{\infty}\sum_{k=2}^{K}w_k f_k(y)\sum_{j=1}^{k-1}w_j f_j(x)\,|x-y|dxdy$$

$$=\sum_{k=1}^{K}w_k^2\frac{\mu_k}{\mu}GINI_k + \frac{2}{\mu}\int_{0}^{\infty}\int_{0}^{\infty}\sum_{k=2}^{K}w_k f_k(y)\sum_{j=1}^{k-1}w_j f_j(x)|x-y|dxdy$$

where

$$GINI_k = \frac{1}{\mu_k} \int_0^\infty \int_0^\infty f_k(y) f_k(x) |x - y| dxdy$$

From the second component, consider a typical term:

$$\int_0^\infty \int_0^\infty w_k f_k(y) w_j f_j(x) |x - y| dxdy$$

$$= w_k w_j \int_0^\infty f_k(y) \int_0^\infty f_j(x) |x - y| dxdy$$

$$= w_k w_j \int_0^\infty f_k(y) \left[-\int_0^y f_j(x)(x - y)dx + \int_y^\infty f_j(x)(x - y)dx \right] dy$$

$$= w_k w_j \int_0^\infty f_k(y) \left[-\int_0^\infty f_j(x)(x - y)dx + 2\int_y^\infty f_j(x)(x - y)dx \right] dy$$

$$= w_k w_j \int_0^\infty f_k(y) \left[-(\mu_j - y) + 2\int_y^\infty f_j(x)(x - y)dx \right] dy$$

$$= w_k w_j \left[|\mu_k - \mu_j| + \int_0^\infty f_k(y) \int_y^\infty f_j(x)(x - y)dxdy \right]$$

Note $\int_0^\infty f_k(y) \int_y^\infty f_j(x)(x - y)dxdy > 0$, so that:

$$GINI = \sum_{k=1}^K w_k^2 \frac{\mu_k}{\mu} GINI_k + \frac{2}{\mu} \sum_{k=2}^K \sum_{j=1}^k w_k w_j |\mu_k - \mu_j|$$
$$+ \frac{2}{\mu} \sum_{k=2}^K \sum_{j=1}^{k-1} w_k w_j \int_0^\infty f_k(y) \int_y^\infty f_j(x)(x - y)dxdy \qquad \text{(A4)}$$

For completeness suppose all groups have identical distributions so that $f_j(y) = f_k(y)$, $GINI_k = GINI_j = GINI$ for all j, k and all y, then $\mu_k = \mu$, and (A4) becomes:

$$GINI = \sum_{k=1}^K w_k^2 \frac{\mu_k}{\mu} GINI_k + \frac{2}{\mu} \sum_{k=2}^K \sum_{j=1}^{k-1} w_k w_j \int_0^\infty f(y) \int_y^\infty f(x)(x - y)dxdy$$

$$= \sum_{k=1}^{K} w_k^2 \frac{\mu_k}{\mu} GINI_k + \frac{1}{\mu} \sum_{k=1}^{K} \sum_{\substack{j=1 \\ j \neq k}}^{K} w_k w_j \int_0^{\infty} f(y) \int_y^{\infty} f(x)(x-y) dx dy$$

Since $GINI_k = GINI_j = GINI$ and $\mu_k = \mu_j = \mu$ for all j, k, we have:

$$GINI = \sum_{k=1}^{K} w_k^2 \, GINI + \sum_{k=1}^{K} \sum_{\substack{j=1 \\ j \neq k}}^{K} w_k w_j \, GINI \tag{A5}$$

Multi-group overlap measures do not account for distances between groups (mutually exclusive groups with large distances between groups may be considered more "segmented" than mutually exclusive groups with small distances between groups). However, components of the $GINI$ coefficient can be shown to do so. We may think of overlap measures as first order comparisons and $GINI$-type measures as second order comparisons each conveying different pieces of information.

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