

Propagation of Quantity Shocks in Who-to-whom Networks

Celestino Giron

(European Central Bank)

Marta Rodriguez Vives

(European Central Bank)

Antonio Matas

(European Central Bank).

Paper prepared for the 35th IARIW General Conference

Copenhagen, Denmark, August 20-25, 2018

Session 7E-1: Financial Accounts

FRIDAY, AUGUST 24, 14:00-17:30

Propagation of quantity shocks in who-to-whom Networks¹

Celestino Girón²

Abstract

We examine the use of who-to-whom (w-t-w) matrices to study the local propagation dynamics of quantity shocks in investment and financing. To that aim, we propose a decomposition of shocks into n-order effects on the basis of an "inverse of Leontief" representation of the w-t-w matrices. We further propose an eigenvector decomposition of the effects to provide an analytical description of the propagation process. This reveals the deep connection between the propagation role of economic agents/sectors and their centrality in the w-t-w network. We also provide an introduction to the use of the Leontief representation in look-through algorithms.

Keywords: financial accounts, who-to-whom matrices, inverse of Leontief, financial networks, eigenvector centrality, shock propagation, look-through, Perron-Frobenius

JEL classification: C67, E16, E50

¹ The paper is based on preliminary work discussed at the SAFE/ECB Workshop "Making Use of Financial Accounts" (Frankfurt am Main, 17 July 2017) and at the Banca d'Italia Conference 'How Financial Systems Work: Evidence from Financial Accounts' (Rome, 30 November, 1 December 2017). The paper has enormously benefited from the comments received in the two fora.

² In collaboration with Antonio Matas-Mir and Marta Rodriguez-Vives

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1. Introduction

Who-to-whom (w-t-w) matrices extend the standard financial accounts³ presentation of balance-sheets by also tracking counterparty information for both assets and liabilities. For instance, the holdings of debt securities by non-financial corporations, which are presented within the assets of the sector in the standard balance-sheet presentation, are also broken down by sector of the issuer of the debt in the w-t-w accounts. A similar breakdown applies to each recorded liability, so that e.g. loans received by non-financial corporations are broken down by sector of the lender. This applies to all sectors in the system, yielding one matrix of creditor/debtor relationships for each financial instrument⁴.

W-t-w matrices embed information on indirect intersector financing/investment patterns and on indirect exposures and risks. Applying appropriate tools, indirect exposures and financing dependences between two sectors, say A and B, resulting from A's holdings of liabilities of a third sector C which in turns holds assets on B, can be quantified. Likewise, the algebraic structure of the matrices conveys information on how assets and liabilities are distributed across the economy via direct and indirect links, which can be used to characterise the implicit network of financial interrelationships.

The algebraic properties of the w-t-w matrices can also be seen as providing a characterisation of the processes of propagation across sectors of socks affecting outstanding amounts of assets and liabilities. Such characterisation can only be considered as "local", implying that no temporal path can be derived from it, but rather an analytical, non-time-related decomposition of the shock on the basis of the current set of w-t-w interrelationships.

This paper explores this relationship between the matrix structure and the propagation of shocks by using a representation similar to the inverse of Leontief of the Input/ Output analysis⁵. The resulting description of the direct and indirect propagation effects is further decomposed into contributions by the eigenvectors of the "diffusion matrix" (closely related to the inverse of Leontief). This approach provides a handle for dimensionality reduction in the description of propagation, and a link between the propagation features and the eigenvector centrality score of the various sectors in the network of which the diffusion matrix is the matrix representation.

As indicated, the analytical decomposition proposed does not aim at describing the propagation process across time. Time dynamics would be severely affected by the accommodation of the w-t-w algebraic structure to the shocks, an issue that it is outside the scope of this paper.

The annex to this paper also describes the use of the inverse of Leontief representation to estimate indirect exposures in a w-t-w framework. This approach

³ A basic methodological reference for financial accounts and sector accounts is the European System of Accounts (Eurostat, 2010).

⁴ See w-t-w tables for the euro area in European Central Bank (2018a).

⁵ See Tsujimura, Mizoshita (2003) for an earlier research on the use of the Leontief framework in the flow of funds context.

can be used to prepare look-through representations of assets and liabilities. An immediate application to households' portfolios is suggested.

The rest of the paper is organized as follows. Section 2 introduces a simplified w-t-w model with a numerical example and its representation by means of the inverse of Leontief. The example will serve to illustrate our approach throughout the paper. Section 3 describes how the Leontief representation can be used to express the local dynamics of quantity shocks. Section 4 proposes an eigenvector decomposition of such dynamics and section 5 discusses its links with eigenvector centrality. Section 7 concludes. The annex proposes the use of the inverse of Leontief approach for look-through estimates.

2. Debt diffusion matrices

Let us assume a simplified economy with four sectors, the non-financial sector (SN in Figure 1), a commercial banking sector (S12K), a central bank (S121) and a government (S13). We distinguish two kinds of assets, debt, presented on a w-t-w basis in Figure 1, and other assets encompassing equity and non-financial assets.

Figure 1 represents this economy in a tabular form, where elements $z_{i,j}$ denote debt assets held by sector *i* that are liabilities of sector *j*, *i,j* indexed from 1 to 4 as ordinal numerals of sectors SN, S12K, S121, S13. Elements n_i are the assets other than debt held by sector *i*, and t_i the total assets held by sector *i*.

Without loss of generality, we further assume that government does not hold any asset $(z_{4,j} = 0)$, but does issue debt that we distinctly denote g_i $(g_i = z_{i,4}, i \neq$ 4), meaning government debt held by sector *i*. This specific presentation for government is also done for the sake of convenience, as the rest of the paper will use as an example of quantity shock changes in government debt holdings by the other sectors. Total assets held by sector *i* meets the equality $t_i = \sum_{j=1 \text{ to } 3} z_{i,j} + g_i + n_i$.

A simplified w-t-w framework

Figure

		Deb				
Assets	SN	S12K	S121	S13	Other assets	TOTAL ASSETS
of,						
SN	<i>Z</i> _{1,1}	<i>Z</i> _{1,2}	Z _{1,3}	g_1	n_1	t_1
S12K	Z _{2,1}	Z _{2,2}	Z _{2,3}	<i>g</i> ₂	<i>n</i> ₂	t ₂
S121	Z _{3,1}	Z _{3,2}	Z _{3,3}	g_3	<i>n</i> ₃	t ₃

SN: Non-financial sectors; S12K: Commercial banks; S121: Central bank; S13: Government.

A matrix algebra presentation of the latest equality is:

 $z_{i,j}$: debt liabilities of sector *j* held by sector *i*; g_i : debt issued by government and held by sector *i*; n_i : assets other than debt held sector *i*; t_i : total assets held by sector *i*

v

$$\boldsymbol{t} = \boldsymbol{Z} * \boldsymbol{1} + (\boldsymbol{n} + \boldsymbol{g})$$

Where **t** is the vector
$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$
, **Z** the matrix $\begin{pmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{pmatrix}$, **1** a 3x1 unitary ector and $\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, $\boldsymbol{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$.

Let us now define the elements $a_{i,j} = \frac{z_{i,j}}{t_j}$, representing the (stock of) financing provided by sector *i* to sector *j* per unit of (stock of) investment of *j*. We call the matrix $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$ the diffusion matrix, its elements representing how the investment of the various sectors react to the investment of others (in order to finance them).

Basic manipulation of (1) using the diffusion matrix gives us t = A * t + (n + g)and the expression

$$t = [I - A]^{-1} * (n + g)$$
 (1)⁶

(1) resembles the input-output model, $[I - A]^{-1}$ being the inverse of Leontief and the diffusion matrix **A** corresponding to the technical coefficient matrix in that model (Leontief, 1941). Whereas Leontief deals with input per unit of output, we consider here financing per unit of investment, but the overall logic behind the two representations is the same one.

The elements in the diffusion matrix in our model have interesting interpretations in terms of well-known financial ratios. Thus, $a_{1,1}$ and $a_{2,1}$ are the ratios of financing from non-banks and banks, respectively, to the investment of the non-financial sectors (also assuming that $a_{3,1} = 0$, i.e. that the central bank does not directly finance the non-financial sectors). The ratios $a_{1,2}$, $a_{2,2}$ and $a_{3,2}$ represent the mix of financing sources for bank investment, indicating how banks resort to public funding (typically deposits), interbank financing and central bank leverage ratio, reflecting the capital position of the banking system. $a_{1,3}$ and $a_{2,3}$ express, as ratios to total central bank investment, the recourse to general public financing (banknotes) and commercial banking financing (central bank reserves) by the central bank.

In the rest of the paper we will work with a specific diffusion matrix representing a stylised economy:

$$\boldsymbol{A} = \begin{pmatrix} 0.1 & 0.6 & 0.3 \\ 0.5 & 0.25 & 0.7 \\ 0 & 0.05 & 0 \end{pmatrix}$$

We will also examine a concrete quantity shock, the purchase by the central bank of government debt previously held by commercial banks, i.e. a case of

⁶ As the sum of elements in the columns of **A** are smaller than 1, then all eigenvalues have module lower than 1 and $[I - A]^{-1}$ exists.

quantitative easing. In terms of the vector \boldsymbol{g} above, we have that our shock in unitary terms is:

$$\Delta \boldsymbol{g} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

And applying (1) we obtain:

$$\Delta \boldsymbol{t} = [\boldsymbol{I} - \boldsymbol{A}]^{-1} * \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.13 & 1.83 & 1.92 \\ 1.49 & 2.68 & 2.32 \\ 0.07 & 0.13 & 1.12 \end{pmatrix} * \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.09 \\ -0.36 \\ 0.98 \end{pmatrix}$$

We see that once the w-t-w structure is considered, as done via the inverse of Leontief, the effects of the shock are more complicated than just a reduction and an increase in the investment by commercial banks and central banks offsetting each other. Central bank investment increases almost one-to-one as expected, but a perhaps unexpected small positive effect is exerted on the investment of the nonfinancial sector, and the investment of commercial banks does not decay as much as would immediately follow from the selling of government debt.

Part of the explanation for the latter lies in the fact that banks would have received central bank reserves as an immediate counterpart for the debt selling. However, the portfolio allocation forces embedded in the w-t-w structure causes the bank reserve increases to not totally offset the decline in government debt, and a genuine reduction in total bank assets by 36% of the initial quantity shock occurs. The central bank would have financed the acquisition of government debt precisely with the reserves acquired by the banks, but as the latter is lower than the former part of the financing of the debt acquisition is ultimately provided by the non-financial sector (via banknotes). From the point of view of the non-financial sectors, this would have been in substance a portfolio shift (into banknotes) away from deposits with banks, mirroring the overall decrease in bank financing needs due to the overall reduction in bank investment.

These are the individual changes in investment and financing derived from the shock, and other than the shock itself:

$$\Delta \mathbf{Z} = \begin{pmatrix} 0.009 & -0.214 & 0.295 \\ 0.045 & -0.089 & 0.688 \\ 0 & -0.018 & 0 \end{pmatrix}, \Delta z_{i,j} = a_{i,j} * \Delta t_j$$

Note that, overall, the ultimate financing of the government debt would come from the non-financial sector before and after the shock. However, the shock shifts such financing from transiting via deposits invested by banks into government debt, to travel partially via banknotes, and partially still via deposits, but now invested in central bank reserves which are in turn invested in government debt. The latter is a sort of third-order investment channel for non-financial sector to finance government debt.

More in general, the changes in investment and financing triggered by shocks are governed by the set of direct and indirect relationships embedded in the w-t-w diffusion matrix, including intricate investment/ financing paths of any order, even beyond the third-order one referred to above for our example. In the next section

we propose a decomposition of the shocks that separates these individual *n*-order effects.

3. Shock dynamics

On the basis of the power series representation of the inverse of Leontief, the total change in investment produced by a shock can be expressed as follows:

$$\Delta t = [I - A]^{-1} * \Delta g = \Delta g + A * \Delta g + A^2 * \Delta g + A^3 * \Delta g \dots + A^{n-1} * \Delta g + \cdots$$
(2)

The shock effect gets decomposed into (i) the shock itself -the vector Δg indicating the original changes in investment-, (ii) the investment effort needed to finance such original investment change – the vector $\mathbf{A} * \Delta g$ –, (iii) the investment effort needed to finance the investment effort needed to finance the original investment change –the vector $\mathbf{A}^2 * \Delta g$ – and so on into infinite n-order investment efforts.

Figure 2 presents the decomposition into the first 15 orders in the case of our example.



The shock itself -or first order effect- consists in a reduction of investment by banks and an increase by the central bank. The second-order effect reflects a reduction of investment by the non-financial sectors and the central bank as banks have reduced their financing requirements. However, the second-order effect is positive for banks as a result of increasing investment in central bank reserves. In more detail, this latter effect for banks is derived as

$$(a_{2,1} \quad a_{2,2} \quad a_{2,3}) * \Delta \boldsymbol{g} = (0.5 \quad 0.25 \quad 0.7) * \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = -0.25 + 0.7 = 0.45$$

where it can be seen that the positive result derives from the fact that the dependency of the central bank on reserves (70%) is larger than the dependency of banks on interbank financing (25%). Although theoretically possible, it is difficult to imagine an economy where this ranking of dependencies does not hold.

The third-order effect for banks turn again negative as a result of the reduction in non-financial sector investment and its corresponding financing needs in the previous second-order round of effects:

$$(a_{2,1} \quad a_{2,2} \quad a_{2,3}) * (\boldsymbol{A} * \Delta \boldsymbol{g}) = (0.5 \quad 0.25 \quad 0.7) * \begin{pmatrix} -0.3 \\ 0.45 \\ -0.05 \end{pmatrix} = -0.073$$

Key for this result is the relatively high dependency of the non-financial sectors on bank financing (50%) compared with the recourse to interbank financing by banks (25%). However, the fact that the second-order effects are more pronounced for banks than for the other two sectors dampens the effect of this differential in ratios and brings the overall result close to cero.

The non-financial sectors experience in turn again a positive third-order effect as a result of the previous positive second-order effect in bank investment:

$$(a_{1,1} \quad a_{1,2} \quad a_{1,3}) * (\boldsymbol{A} * \Delta \boldsymbol{g}) = (0.1 \quad 0.6 \quad 0.3) * \begin{pmatrix} -0.3 \\ 0.45 \\ -0.05 \end{pmatrix} = 0.225$$

The high dependency of banks on deposits and the like (60%), and the fact that the second-order positive effect for banks is large are behind this result.

The alternation of rounds where first the effect on banks is negative and the effect on the non-financial sector is positive, and then the reverse happens, would in principle repeat itself as one sector catches up the behaviour of the other in the previous round. However, this would end up vanishing due to the dampening effect mentioned for banks when discussing the third-order effect, as the successive odd rounds deliver bank effects closer and closer to zero to eventually become positive. This happens as soon as in the fifth round of effects in our example.

It is important to note that the succession of rounds here described does not imply a temporal sequence. They result from the power series representation of the inverse of Leontief, which has no implication in terms of time dynamics. The n-order effects are to be considered as taking place simultaneously, and the dynamics provided to be qualified as "local" (no-time related).

It can nevertheless be argued that a simultaneous adjustment is not possible in the real world, and that the decomposition contains *some* information on a temporal adjustment path. Without discussing whether this is true or not, it must be stated that the temporal dynamics would likely be dominated by how the w-t-w structure adapts to the shock, and on how such adaptation evolves in time. None of these aspects are subject of this paper.

The Leontief decomposition provides a tool for better understanding the effects of the shocks in investment and financing by sector. However, the resulting

description of the propagation process is still complicated and does not clearly reveal the forces behind the different reactions of the sectors to the shocks. For instance, it is not evident why the n-order effects (n> 1, indirect effects) in the central bank are so small, nor there is a clear structural feature that can be pointed out as causing this and that could be used for making comparisons across economies.

In the next section we take a step further to tackle this and we express the sector propagation paths as linear combinations of eigenvectors of the diffusion matrix. This would enable us to better characterise the dynamics, and to summarise it with fewer information.

4. Analytical representation of shock propagation

The n-order effects are the result of applying the diffusion matrix **A** as an operator on the vector containing the (n-1)-order effects, as follows:

$$\boldsymbol{A} \ast (\boldsymbol{A}^{n-2} \ast \Delta \boldsymbol{g})$$

Where the vector $A^{n-2} * \Delta g$ contains the (n-1)-order effects.

Let us call V, E the matrix of eigenvectors and diagonal matrix of eigenvalues of **A** -which we assume as diagonalizable, as in our example- respectively, so that $A = V * E * V^{-1}$ and $A^n = V * E^n * V^{-1}$ (see for instance Meyer, 2000). This allows the following representation of the n-effects:

$$A^{n-1} * \Delta g = V * E^{n-1} * (V^{-1} * \Delta g) \quad (3)$$

The vector $(\mathbf{V}^{-1} * \Delta \mathbf{g}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ contains the components of the shock vector $\Delta \mathbf{g}$ expressed in the base of eigenvectors. As $\mathbf{V} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3) = \begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix}$

and $\mathbf{E} = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_2 \end{pmatrix}$ in our 3x3 example, (3) can also be expressed as follows:

$$\boldsymbol{A^{n-1}} * \Delta \boldsymbol{g} = \rho_1^{n-1} * c_1 * \begin{pmatrix} v_{1,1} \\ v_{2,1} \\ v_{3,1} \end{pmatrix} + \rho_2^{n-1} * c_2 * \begin{pmatrix} v_{1,2} \\ v_{2,2} \\ v_{3,2} \end{pmatrix} + \rho_3^{n-1} * c_3 * \begin{pmatrix} v_{1,3} \\ v_{2,3} \\ v_{3,3} \end{pmatrix} (3.a)$$

The individual n-order effects being expressed as a linear combination of the eigenvectors of the diffusion matrix, weighted by the interaction of the corresponding eigenvalues with the components of the shock in the base formed by the eigenvectors. Figure 3 presents the eigenvectors of our numerical example.



SN: Non-financial sectors; S12K: Commercial banks; S121: Central bank v_i : eigenvector i; ρ_i : eigenvalue i

This presentation would allow us better understanding the features that govern the propagation effects and link them to network centrality, as well as perform dimensionality reduction to simplify the presentation of the shock dynamics.

Figure 4 shows the decomposition of the effects on the banking sector (S12K) for n>1 (indirect effects).

Eigenvector decomposition of (n>1)-order effects. Banks (S12K).



Equation for eigenvalue 0.76 (blue line) $\rho_1^{n-1} * c_1 * v_{2,1} = 0.76^{n-1} * 0.16 * 0.73$

Equation for eigenvalue -0.40 (red line) $\rho_2^{n-1} * c_2 * v_{2,2} = -0.40^{n-1} * 1.34 * -0.66$

Equation for eigenvalue -0.01 (green line) $\rho_3^{n-1} * c_3 * v_{2,3} = -0.01^{n-1} * -1.44 * 0.16$

The dynamics after the shock is decomposed into a persistent positive subeffect (blue line in Figure 4), and two sign-oscillating sub-effects (red, green) inducing the alternation of positive and negative effects described in section 3. The nature of the signs as oscillating or not depends on the sign of the corresponding eigenvalue, those with positive value (0.76 in our case) delivering a constant sign contribution which depends on the sign of the product of the component of the shock in the eigenbase (0.16 in our example) and the sector component in the eigenvector associated to the eigenvalue (0.73, delivering an overll positive sign path).

The size of the sub-effects depends on the corresponding module⁷ of the eigenvalues, the components of the eigenvectors and the components of the shock. The persistence of the n-order effects depends on the module of the eigenvalue. Thus, the sub-effects linked to the eigenvalue -0.01 (blue line in Figure 4) are extremely small and disappearing fast, to the extent that they can be totally ignored for characterising the shock dynamics.

Moreover, the sub-effects linked to -0.40 (red line) start being the largest contributions to the n-order effects due to the large impact of the module of the eigenvector and shock components, but soon diminish to be clearly outweighed by the sub-effects linked to the first, more persistent, larger-module eigenvalue (0.76, blue line). The different persistence of the sub-effects can be seen in Figure 5 showing the accumulation of effects.

Accumulated (n>1)-sub-effects in eigenvector decomposition. Banks (S12K).



N: Non-financial sectors; S12K: Commercial banks; S121: Central bank; S13: Government.

As indicated, the third eigenvalue can be totally disregarded, and the second eigenvalue only contributes to the first few n-order effects in a sizeable manner. Actually, the overall dynamics after the negative shock is clearly dominated by the

⁷ Note that eigenvalues, eigenvectors and shocks in the base of eigenvectors are in general complex numbers if we allow for diffusion matrices that are diagonalizable in the complex plane.

positive sub-effects of the first eigenvalue. This also provides an explanation to why the final total effect of the shock is not as negative as the shock itself.

Similar roles are played by the various eigenvalues for the other two sectors, for which a similar simplification of the dynamics can be constructed. More in general, higher order non-defective diffusion matrices can be reduced to the dynamics caused by the few eigenvalues with the highest module⁸. Moreover, the most persistent dynamics is always provided by the eigenvalue with the largest module, whose associated eigenvector, and the eigenvalue itself, have some interesting properties that will be discussed in the next section for certain types of diffusion matrices.

We can use the decomposition to try and answer the question we put forward above on the very low (n>1)-order effects for the central bank. The matrix of the absolute value of the eigenvectors is:

$$abs(\mathbf{V}) = \begin{pmatrix} 0.68 & 0.75 & 0.77 \\ 0.73 & 0.66 & 0.16 \\ 0.05 & 0.08 & 0.61 \end{pmatrix}$$

The third row, (0.05 0.08 0.61), corresponds to the central bank and contains the (absolute value of the) coefficients that are multiplicated by powers of the eigenvalues to yield the paths of the sub-effects for that sector. We observe that the values corresponding to the eigenvalues of higher modules are extremely low (0.05 for eigenvalue 0.76, 0.08 for -0.40), while the larger value, 0.61, is associated to an eigenvalue of very low module, -0.01. That means that the persistent sub-effects stemming from the large-module eigenvalues have a small size (small component in the eigenvectors), and large size sub-effects (the large component, 0.61) disappear fast because the corresponding eigenvalue is very small⁹.

The specific central bank sub-effect paths followed would be different (in terms of the relative amplitude of the sub-effects) depending on the precise shock and its corresponding components in the base of eigenvectors, but the features described in the previous paragraph would still apply. They are a consequence of the algebraical characteristics of the diffusion matrix and the role of the sector in the network of relationships embedded in the w-t-w structure. Section 5 looks into this, in particular in relation with the eigenvector associated to the largest-module eigenvalue.

⁸ This induces a dimensionality reduction similar to Principal Components Analysis (see for instance Abdi and Williams, 2010).

⁹ Note that the choice of the specific components of the eigenvectors is arbitrary, and that components that are a factor of those here used for a given eigenvalue constitute a suitable eigenvector for such eigenvalue as well. That doesn't mean that the size of the sub-effects are arbitrary, as the components of the shocks in the base of eigenvectors would also be different and get divided by the same factor by which the eigenvectors have been multiplied.

5. Shock propagation and network centrality

The w-t-w data can be seen as a network of interrelationships in which the nodes – the elements interlinked in the network- are institutional sectors and the edges –the links between nodes- are asset/liability links. The edges in the network would be "weighted" by the amounts involved in every asset/liability relationship¹⁰.

In network analysis, eigenvector centrality is a measure of the influence of the various nodes in the network. It consists in an array of scores for nodes that satisfy the principle that higher scores are assigned to nodes that are highly connected to nodes that in turn have high scores themselves. It is therefore a metrics of a recursive nature capturing second, third and higher orders of influence in the network. The concept is therefore particularly well suited to emphasising the importance of indirect links in measuring interconnectedness¹¹.

The "eigenvector centrality" scores correspond to the components of an eigenvector associated to a specific eigenvalue of the weight matrix (the matrix representation of the network). The specific eigenvalue, called Perron-Fobrenius eigenvalue, is the one of maximum module, which can be proved to be real and positive for non-negative "irreducible matrices", i.e. for matrices that can be associated to strongly connected directed networks/graphs (graphs that present direct or indirect connections between any pair of nodes), and unique for a subsector of such matrices, called "primitive matrices", whose associated graphs present paths between nodes of lengths that are coprimes (it is a sufficient condition for a irreducible matrix to be primitive to have a positive value in its diagonal). Moreover, it is guaranteed for irreducible matrices that an eigenvector can be chosen associated to the Perron-Fobrenius eigenvalue with all components strictly positive (Perron-Frobenius Theorem)¹²¹³.

Turning to the diffusion matrix \mathbf{A} of our example, the most persistent path for banks (and also for the other two sectors), identified in blue in Figures 4 and 5, is nothing but the one associated to the Perron-Frobenius eigenvalue as just defined for eigenvector centrality for the network associated to \mathbf{A} (which is a primitive matrix). The associated eigenvector components are both the size of the persistent

¹³ The specific characteristic of a Perron-Fobrenius eigenvector that makes it suitable for measuring centrality is the following property: for any vector d, it exists a Perron-Fobrenius eigenvector, αv , such as

$$\lim_{p\to\infty}\left(\frac{w}{\rho}\right)^p \boldsymbol{d} = \alpha \boldsymbol{\nu},$$

Where ρ , W are the Perron-Fobrenius eigenvalue and the weight matrix (matrix representing the network edge weights) respectively. In particular for d = 1, the unitary vector, it results that

 $\alpha * v_i = \sum_{j=1}^n \left(\lim_{p \to \infty} \left(\frac{w}{\rho} \right)^p \right)_{i,j} \text{for all } v_i \text{ components of the eigenvector.}$

Put it somewhat less formally, it exists an integer sufficiently large q, such as $\alpha * \rho^p * v_i \approx \sum_{j=1}^n (W^p)_{i,j}$ for any p > q, i.e. the components (centrality scores) in v are approximately distributed as the row sums of the (sufficiently large) power of the weight matrix.

¹⁰ For network representations of w-t-w data, see for instance Castrén and Rancan (2013) and Antoun de Almeida (2015). Also see w-t-w graphs for the euro area in European Central Bank (2018b).

¹¹ See an example of the use of this concept of centrality with w-t-w data in Girón and Matas-Mir (2017).

¹² For a reference, see for instance Newman (2010).

propagation sub-effect for each sector and the centrality score for each sector. They are essentially the same thing.

We know from the Perron-Fobrenius Theorem that such maximum-module eigenvalue is a unique positive real number for matrices with certain characteristics. As a consequence, the sing of the associated most-persistent sub-effect is constant, not oscillating for all sectors and all diffusion matrices of the same kind as **A** (non-defective primitive matrices), not only a specific characteristic of our concrete example. Moreover, we also know from the Theorem that the components of the associated eigenvectors are all of the same sign (usually taken positive). The specific sing of the Perron-Frobenius sub-effects by sector entirely depends then on the sign of the shock components in the basis of eigenvectors.

We can claim that the relative size by sector of the most persistent sub-effect is fundamentally nothing but the expression of the relative centrality of the sectors (the components in the Perron-Frobenius eigenvector). Being the most persistent, these Perron-Frobenius sub-effects dominate the indirect (n>1)-order effects of the shocks. We can therefore also claim that the accumulated size of indirect effects responds to the sector centrality.

Coming back to the question in section 3 on the small size of the indirect effects for the central bank in our example, we can now say that this is due to its low centrality in the w-t-w network, and that the structural parameter that indicates this is its (low) score in eigenvector centrality¹⁴.

6. Summary and further developments

In this paper we have illustrated, on the basis of a simplified example, the use of a Leontief representation for w-t-w financial matrices. This serves in particular to study the propagation dynamics of quantity shocks in investment and financing by making use of a Leontief power series expansion.

Assuming certain regularities in the w-t-w algebraic structure, the dynamics is decomposed into contributions by the eigenvectors of the diffusion matrix, the matrix containing the ratios of financing by counterpart sector per unit of investment by sector measured in stock terms. The dynamics of the direct and indirect effects emerge as linear combinations of declining trajectories associated to the eigenvalues of the matrix, weighted on the components of the shocks in the base of eigenvectors. The trajectory weights change from sector to sector on the basis of the components of the eigenvectors associated to the corresponding eigenvalues/trajectories.

The decomposition in sub-effects allows for a reduction of dimensionality as only the few largest eigenvalues do have a sizeable contribution to the overall dynamics. Moreover, beyond the few first indirect effects, the dynamics is

¹⁴ Note that this should not be understood as a cause-effect relationship by virtue of which an independent feature regarding centrality causes the size of the indirect effects. Actually the definition itself of eigenvector centrality is making reference to the algebraic structure of the matrix behind the network, and therefore implicitly referring to the decomposition we proposed in section 4. Centrality and size of the indirect propagation effects are actually two ways of referring to the same thing.

dominated by the Perron-Frobenius eigenvalue, the one with largest module and inducing the most persistent trajectory. The components of the eigenvector of this eigenvalue are therefore key to understanding the distribution of indirect effects across sectors. These components are a well-known network centrality metrics: eigenvector centrality.

The proposed framework can be used to decompose effects caused by quantity shocks of any nature. The shock used as a way of example in this paper is described as a case of central bank quantitative easing affecting the volume of assets held by the relevant sectors, but any other kind of shock affecting the distribution of stock value can be considered, including price shocks. Given that the dynamics provided are of local nature -i.e. they are not describing temporal adjustment paths- it might be argued that the representation is more appropriate to characterise price shocks, as their propagation can be thought as close to instantaneous. The Leontief approach would then provide an analytical tool for combining sector account data (w-t-w matrices) and macroprudential analysis of risk dependencies and price shock propagation.

Future research should tackle the lack of temporal dimension in the description of the propagation process. While large structural changes in the w-t-w relationships would be difficult to model, very short-term changes might be studied on the basis of the literature on eigenvalue perturbations. Thus, bounds can be established for the changes in eigenvalues -and therefore in the associated trajectories- exerted by perturbations in line with the Bauer-Fike Theorem (see for instance Wei et al., 2006), from which boundaries for time paths might be derived. Moreover, Bauer-Fike bounds depend on the eigenvector structure of the (diffusion) matrix, which would allow making statements regarding the relative stability of the propagation features of comparable diffusion matrices.

Beyond the use to characterise propagation effects, the Leontief representation can be used to extract other relevant information from the w-t-w data. The annex to this note proposes, for example, a usage for look-trough algorithms. Once the specific Leontief representation is implemented, and provided that the supporting matrices meet the regularities indicated in this paper, the same techniques here used for propagation can also be used in those other contexts, allowing as well for eigenvector decomposition, dimensionality reduction and linkages with network centrality.

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Annex: "look through" algorithms

Look through analysis aims at unveiling indirect exposures of investors to final debtors. Most of the literature focuses on indirect exposures via institutional investors¹⁵, but also via all financial institutions¹⁶. The look-trough analysis typically lies on available detailed granular data, but it might also require the use of iterative algorithms to cover for the absence of such data for some instruments or sectors. The algorithms make use of shares of holdings as resulting from w-t-w macro data, assuming that such holdings can be proportionally applied to calculate indirect exposures. This annex proposes a Leontief kind of representation of w-t-w data that can serve to generalise such algorithms.

Let us call $b_{i,j} = \frac{z_{i,j}}{t_i}$ to the ratios of assets of sector *i* to sector *j* ($z_{i,j}$) to the total assets of sector *i* (t_i) for a given (set of) asset(s). The matrix $\mathbf{B} = \begin{pmatrix} b_{i,j} \end{pmatrix}$ -where some $b_{i,j}$ are set to zero when a given indirect exposure is disregarded (because no sufficient risk pass-through exists)- and the vectors $\mathbf{z}_{i,r} = (z_{i,1} \cdots z_{i,k})$, of total asset by counterpart sector, can be used to calculate the first-order indirect exposures of sector *i* as $\mathbf{z}_{i,r} * \mathbf{B}$.

The sum of direct and first-order indirect exposures would be $(z_{i,.} + z_{i,.} * B) - z_{i,.}^{\circ}(B * 1)'$, the term $z_{i,.}^{\circ}(B * 1)'$ capturing direct exposures that have been reallocated as first-order indirect exposures (1 being a unitary vector, and ° being the Hadamard operator).

An example helps understand the mechanics of this calculation. Let us assume a five-sector economy where only one sector gives rise to indirect exposures (say the financial sector). Then, for instance:

	/0.00	0.00	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
$B_1 =$	0.00	0.00	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
	\0.20	0.20	0.20	0.15	0.25/

To calculate direct and first-order indirect exposures of a sector (say households) with a total asset vector $\mathbf{z}_{i,r} = (163 \ 20 \ 25 \ 24 \ 769)^{17}$ we apply the expression above to get:

¹⁵ Gadsby and Girón (2009), Marionnet (2009), Cardillo and Coletta (2017)

¹⁶ Boutillier et al.(2007), Pavot, (2017)

¹⁷ This example is taken from a numerical illustration in Pavot (2017)

$$\left(z_{i_{*}} + z_{i_{*}} * B_{1} \right) - z_{i_{*}}^{\circ} (B_{1} * 1)' = z_{i_{*}} * (I + B_{1}) - z_{i_{*}}^{\circ} (B_{1} * 1)' =$$

$$= (163 \ 20 \ 25 \ 24 \ 769) * \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.20 & 0.20 & 0.20 & 0.15 & 1.25 \end{pmatrix}$$

$$- (163 \ 20 \ 25 \ 24 \ 769)^{\circ} \left[\begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.20 & 0.20 & 0.15 & 0.25 \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right]' =$$

$$= (317 \ 174 \ 179 \ 139 \ 961) - (0 \ 0 \ 0 \ 0 \ 769) =$$

$$= (317 \ 174 \ 179 \ 139 \ 192)$$

A more general pass-through matrix (with more than one sector inducing indirect exposures) could be:

 $\boldsymbol{B}_{2} = \begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.50 \\ 0.70 & 0.00 & 0.10 & 0.00 & 0.00 \\ 0.30 & 0.00 & 0.10 & 0.00 & 0.50 \\ 0.30 & 0.00 & 0.00 & 0.00 & 0.30 \\ 0.20 & 0.20 & 0.10 & 0.15 & 0.25 \end{pmatrix}$

In which case ...

$$(\mathbf{z}_{i,} + \mathbf{z}_{i,} * \mathbf{B}_2) - \mathbf{z}_{i,} \circ (\mathbf{B}_2 * \mathbf{1})' = (346 \ 174 \ 106 \ 139 \ 1063) - (82 \ 16 \ 23 \ 14 \ 692)$$

= (264 \ 158 \ 84 \ 125 \ 370)

The second-order exposures respond to the expression $(z_{i,.} * B) * B$, i.e. the first-order indirect exposures of the first-order indirect exposures, and the sum of direct, first-order and second-order exposures is $(z_{i,.} + z_{i,.} * B + z_{i,.} * B^2) - (z_{i,.} + z_{i,.} * B)^{\circ}(B * 1)'$, where $(z_{i,.} + z_{i,.} * B)^{\circ}(B * 1)'$ are the sum of direct exposures reallocated as first-order indirect exposures and of the first-order indirect exposures in turn reallocated as second-order indirect exposures.

In our first example we have:

$$(z_{i_{r}} + z_{i_{r}} * B_{1} + z_{i_{r}} * B_{1}^{2}) - (z_{i_{r}} + z_{i_{r}} * B_{1})^{\circ}(B_{1} * 1)' = (z_{i_{r}} + z_{i_{r}} * B_{1}) + ((z_{i_{r}} * B_{1}) * B_{1}) - z_{i_{r}}^{\circ}(B_{1} * 1)' - ((z_{i_{r}} * B_{1})^{\circ}(B_{1} * 1)') =$$

 $(317 \ 174 \ 179 \ 139 \ 961) + (154 \ 154 \ 154 \ 115 \ 192) \begin{pmatrix} 0.00 \ 0.00$

 $(39 \ 39 \ 39 \ 29 \ 48) - (0 \ 0 \ 0 \ 769) - (0 \ 0 \ 0 \ 0 \ 192) =$

(355 212 217 168 1009) - (0 0 0 0 961) = (355 212 217 168 48)

And in the second:

$$(\mathbf{z}_{i_{\nu}} + \mathbf{z}_{i_{\nu}} * \mathbf{B}_{2} + \mathbf{z}_{i_{\nu}} * \mathbf{B}_{2}^{2}) - (\mathbf{z}_{i_{\nu}} + \mathbf{z}_{i_{\nu}} * \mathbf{B}_{2})^{\circ} (\mathbf{B}_{2} * \mathbf{1})' = (571 \ 233 \ 159 \ 183 \ 1302) - (173 \ 139 \ 96 \ 84 \ 956) = (398 \ 94 \ 64 \ 100 \ 346)$$

In general, n-order exposures will be of the form $z_{i,} * B^n$, and the sum of all exposures up to n-order

$$(z_{i,.} + z_{i,.} * B + z_{i,.} * B^{2} + \dots + z_{i,.} * B^{n}) - (z_{i,.} + z_{i,.} * B + \dots + z_{i,.} * B^{n-1})^{\circ}(B * 1)'$$

When $n \to \infty$, the two sum members above contain the power series expansion of the inverse of Leontief in **B**, i.e.

$$\mathbf{z}_{i,}(I-B)^{-1} - \left[\mathbf{z}_{i,}(I-B)^{-1}\right]^{\circ}(B*1)'$$
(4)

Applying (4) with B_1 we obtain the final portfolio allocation (368 225 230 177 0), while for B_2 we get (539 87 31 135 209).

(4), with an appropriate choice of the elements in **B**, provides a generalisation of look-trough algorithms for asset allocation in accordance with ultimate exposures.

Note that a similar generalisation can be derived for liabilities allocation vis-àvis ultimate investors through the expression:

$$(I - C)^{-1} * z_{,j} - [(I - C)^{-1} * z_{,j}]^{\circ} (1' * C)'$$
 (5)

Where $\boldsymbol{C} = (c_{i,j}), c_{i,j} = \frac{z_{i,j}}{d_j}, \boldsymbol{z}_{,j} = \begin{pmatrix} z_{1,j} \\ \vdots \\ z_{k,j} \end{pmatrix}, d_j$ is the total liabilities of sector *j*.